Rendiconti di Matematica, Serie VII
Volume 19, Roma (1999), 123-136

Some tensorfields having some universal properties

O. AMICI – B. CASCIARO – M. FRANCAVIGLIA

RIASSUNTO: *Si considerano in dettaglio alcuni operatori tensoriali che si ottengono in modo naturale dallo studio della variazione seconda delle Lagrangiane definite mediante il tensore di curvatura. Di tali operatori si danno alcune utili proprietá.*

ABSTRACT: *In this paper we consider in detail a number of tensor operators which arise from calculations concerning the second variation of curvature invariants. A few properties are listed which are relevant for practical calculations.*

– Introduction

In view of possible applications to a number of physically relevant problems in gravitational theories (such as, e.g., stability, inflationary models, higher-derivative gravity, quantization and singularity issues) we have recently considered the general structure of first and second order variations of curvature quadratic invariants defined on a manifold M endowed with a pseudo-Riemannian metric g and a linear (torsionless) connection Γ. See [3] and references quoted therein.

The general setting for calculating second variations was established in [6], where the notion of generalized Jacobi equation was discussed. An interesting application to Jacobi fields along geodesics of a Riemannian manifold (M, g) was given in [7], while a general theory of *curvature*

KEY WORDS AND PHRASES: *Second variation – Curvature – Tensorfields.*
A.M.S. CLASSIFICATION: 53C80

structures for variational principles was discussed in [1], which is a continuation of [6]; a short review may be found in [8]. The general results found therein were later applied to curvature Lagrangians of the non-linear type $f(R)$, $f(\|\,\mathrm{Ric}\,\|^2)$ and $f(\|\,\mathrm{Riem}\,\|^2)$, where R, Ric and Riem are respectively the scalar curvature, the Ricci tensor and the Riemann tensor of (M, g, Γ) and the norms are standard (see [2], [3] for details and [4] for some relevant mathematical tricks which make the variational calculations involving functions rather than scalar densities easier).

The general formulae which in [2] and [3] define the second variation and the relevant Jacobi equations for this family of Lagrangians show a number of common features which are in fact due to the natural symmetries of curvature tensors and are in turn expressible by means of a few natural tensor operators in (M, g, Γ).

The aim of this paper is to establish a number of properties which encode the aforementioned relations, with the explicit purpose of simplifying all calculations involving these curvature invariants. In particular, our results will find applications in the general theory of conservation laws for non-linear gravitational Lagrangians (see [5]).

1 – Notations and the main results

Let M be a C^∞-differentiable n-dimensional manifold. We shall denote by $T^h_k(M)$ the bundle of tensors of type (h, k) over M, for any $h, k \geq 0$, with the standard conventions. The space of its sections $\mathcal{I}^h_k(M)$ is the module of tensorfields of type (h, k); we set in particular $\mathcal{X}(M) = \mathcal{I}^1_0(M)$, $\Omega^1(M) = \mathcal{I}^0_1(M)$ and $\mathcal{F}(M) = \mathcal{I}^0_0(M)$ (the Lie algebra of vectorfields, the space of 1-forms and the ring of differentiable functions on M, respectively). As usual, symmetrization on two (or more) indices will be denoted by round brackets and skew-symmetrization by square brackets. We also denote by $\mathrm{tr}_p : \mathcal{I}^{p+h}_{p+k}(M) \to \mathcal{I}^h_k(M)$ (for any triple of integers p, h, k) the trace map obtained by ordered contraction of the first p covariant indices with the first p contravariant ones. Analogously, by $\mathrm{tr}^\dagger_p : \mathcal{I}^{h+p}_{k+p}(M) \to \mathcal{I}^h_k(M)$ we denote the trace map obtained by ordered contraction of the last p covariant indices with the last p contravariant ones. For the sake of simplicity we also set $\mathrm{tr}_1 = \mathrm{tr}$ and $\mathrm{tr}^\dagger_1 = \mathrm{tr}^\dagger$.

Let $\mathcal{C}_S(M)$ be the bundle whose sections $\Gamma(\mathcal{C}_S(M))$ are symmetric linear connections. Since $\mathcal{C}_S(M)$ is an affine bundle, any symmetric linear connection $\tilde{\Gamma}$ determines a morphism $h_{\tilde{\Gamma}} : \Gamma(\mathcal{C}_S(M)) \to \mathcal{I}^1_2(M)$ defined by $h_{\tilde{\Gamma}}(\Gamma) = \Gamma - \tilde{\Gamma}$, for each symmetric linear connection Γ. As usual, we denote by $J^k\mathcal{C}_S(M)$ the bundle of jets of order k over $\mathcal{C}_S(M)$. Notation follows [11].

We first define a linear morphism $\Psi : \mathcal{I}^0_3(M) \to \mathcal{I}^0_3(M)$ by setting:

$$\Psi(t)(X,Y,Z) = \frac{1}{2}(-t(X,Y,Z) + t(Y,Z,X) + t(Z,X,Y)) \quad \forall t \in \mathcal{I}^0_3(M), \forall X,Y,Z \in \mathcal{X}(M)\,. \tag{1.1}$$

In any local chart (U, x^μ) we have in components

$$[\Psi(t)]_{\alpha\beta\gamma} = \frac{1}{2}(-t_{\alpha\beta\gamma} + t_{\beta\gamma\alpha} + t_{\gamma\alpha\beta}), \tag{1.1$'$}$$

and we shall use the following notation:

$$[\Psi(t)]_{\alpha\beta\gamma} \equiv t_{\{\alpha\beta\gamma\}} \equiv \frac{1}{2}(-t_{\alpha\beta\gamma} + t_{\beta\gamma\alpha} + t_{\gamma\alpha\beta}). \tag{1.1$''$}$$

There is a "dual morphism" $\Psi^* : \mathcal{I}^3_0(M) \to \mathcal{I}^3_0(M)$ defined by:

$$\Psi^*(s)(\theta,\sigma,\rho) = \frac{1}{2}(-s(\theta,\sigma,\rho) + s(\sigma,\rho,\theta) + s(\rho,\theta,\sigma)) \quad \forall s \in \mathcal{I}^3_0(M), \forall \theta,\sigma,\rho \in \Omega^1(M), \tag{1.2}$$

with obvious coordinate expressions. Again we set the notation:

$$[\Psi^*(s)]^{\alpha\beta\gamma} \equiv s^{\{\alpha\beta\gamma\}}. \tag{1.2$'$}$$

Here "duality" means that the following holds:

$$\mathrm{tr}_3(\Psi(t) \otimes s) = \mathrm{tr}_3(t \otimes \Psi^*(s)) \quad \forall t \in \mathcal{I}^0_3(M), \forall s \in \mathcal{I}^3_0(M), \tag{1.3}$$

i.e.

$$t_{\{\alpha\beta\gamma\}} s^{\alpha\beta\gamma} = t_{\alpha\beta\gamma} s^{\{\alpha\beta\gamma\}}. \tag{1.3$'$}$$

We also define the "lower symmetrization operator" $\phi : \mathcal{I}^h_{k+2}(M) \to \mathcal{I}^h_{k+2}(M)$ by setting:

$$\operatorname{tr}_2^\dagger(\phi(t) \otimes X \otimes Y) = \frac{1}{2}[\operatorname{tr}_2^\dagger(t \otimes X \otimes Y) + \operatorname{tr}_2^\dagger(t \otimes Y \otimes X)], \quad \forall t \in \mathcal{I}^h_{k+2}(M), \forall X, Y \in \mathcal{X}(M); \tag{1.4}$$

in components this amounts in fact to symmetrize with respect to the last two lower indices.

Let then $\Delta \in \mathcal{I}^1_1(M)$ be the Kronecker (unit) tensor (i.e., the tensor having components δ^μ_ν in each coordinate system). We define tensorfields $B \in \mathcal{I}^3_3(M)$ and $F \in \mathcal{I}^4_4(M)$ by setting:

$$B = \Psi^*(\Delta \otimes \phi(\Delta \otimes \Delta)), \tag{1.5}$$

and

$$F = \Delta \otimes B. \tag{1.6}$$

The tensorfields B and F define two linear morphisms, which by an abuse of notation will be denoted again by $B : \mathcal{I}^0_3(M) \to \mathcal{I}^0_3(M)$ and $F : \mathcal{I}^2_0(M) \times \mathcal{I}^3_0(M) \to \mathcal{I}^1_2(M)$, in the following way:

$$B(z) = \operatorname{tr}_3^\dagger(B \otimes z), \quad \forall z \in \mathcal{I}^0_3(M), \tag{1.7}$$

$$F(m, z) = \operatorname{tr}_2(m \otimes \operatorname{tr}_3^\dagger(F \otimes z)), \quad \forall z \in \mathcal{I}^0_3(M), \quad \forall m \in \mathcal{I}^2_0(M). \tag{1.8}$$

The local components of B and F are respectively given by:

$$B^{\rho\mu\nu}_{\alpha\beta\gamma} = \delta^{\{\rho}_{\alpha}\delta^{\mu}_{(\beta}\delta^{\nu\}}_{\gamma)}, \tag{1.7'}$$

and

$$F^{\lambda\rho\mu\nu}_{\sigma\alpha\beta\gamma} = \delta^\lambda_\sigma B^{\rho\mu\nu}_{\alpha\beta\gamma}. \tag{1.8'}$$

The operators (1.7) and (1.8) are related to the morphism Ψ defined in (1.1) by:

$$B(z) = \Psi(\phi(z)), \qquad B(z)_{\alpha\beta\gamma} = z_{\{\alpha(\beta\gamma)\}}, \tag{1.9}$$

$$F(m, z) = \operatorname{tr}_2(m \otimes B(z)), \quad F(m, z)^\lambda_{\alpha\beta} = m^{\lambda\gamma} z_{\{\gamma(\alpha\beta)\}}. \tag{1.10}$$

The tensorfield B (and F) will be respectively called the *generator of the Christoffel symbols of the first kind* (respectively *of the second kind*). In fact, the following holds:

PROPOSITION 1. *Let Γ be any linear connection and ∇ its covariant derivative. Let g be any pseudo-Riemannian metric on M. Then the Levi-Civita connection $\Gamma_{LC}(g)$ of g is related to Γ by $\Gamma_{LC}(g) = \Gamma + \Pi$, being $\Pi = F(g^*, \nabla g)$; or, in other words, $h_\Gamma(\Gamma_{LC}(g)) = F(g^*, \nabla g)$, being g^* the contravariant metric dual to g. Moreover B and F are parallel:*

$$\nabla B = 0 \quad , \quad \nabla F = 0. \tag{1.11}$$

The above morphisms and tensorfields find useful applications in a number of investigations concerning curvature invariants of a Riemannian manifold (M, g). In fact, they have been used in [1] (modulo inessential cyclic permutations) to calculate the first and second variation of the scalar curvature $r(g)$, of the squared norm of the Ricci tensor $\text{Ric}(g)$ and of the squared norm of the Riemann tensor $\text{Riem}(g)$ of any pseudo-Riemannian metric g on M. More precisely, following [1] let us set

$$\begin{aligned} \mathcal{B}_{\alpha\beta\gamma} &\equiv \frac{1}{2}(z_{\alpha\beta\gamma} - z_{\beta\gamma\alpha} + z_{\gamma\alpha\beta}), \\ \mathcal{F}^{\alpha}_{\beta\gamma} &\equiv m^{\alpha\rho}\mathcal{B}_{\beta\rho\gamma}, \end{aligned} \tag{1.12}$$

for each $m \in \mathcal{I}^2_0(M)$ and $z \in \mathcal{I}^0_3(M)$. Then we have:

$$[B(z)]_{\alpha\beta\gamma} = \mathcal{B}_{\gamma\alpha\beta} \quad , \quad [F(m,z)]^{\alpha}_{\beta\gamma} = \mathcal{F}^{\alpha}_{(\beta\gamma)}. \tag{1.13}$$

We shall be mainly interested here in the case in which the tensorfield z is symmetric with respect to the last two indices; under this further hypothesis equation (1.13) simplifies to

$$[F(m,z)]^{a}_{\beta\gamma} = \mathcal{F}^{a}_{\beta\gamma}.$$

Define now a linear morphism $\mathcal{Q} : \mathcal{I}^0_3(M) \to \mathcal{I}^0_3(M)$ by setting:

$$\mathcal{Q}(t)(X,Y,Z) = t(X,Y,Z) + t(Y,Z,X) + t(Z,X,Y), \tag{1.14}$$

and let $\mathcal{Q}(M) = ker(\mathcal{Q})$. The elements of $\mathcal{Q}(M)$ will be called "Jacobi structures"; if $t \in \mathcal{Q}(M)$, the identity $\mathcal{Q}(t) = 0$ will be called "Jacobi identity". From [1] we know that the module $\mathcal{I}_3^0(M)$ admits the following direct sum splitting $\mathcal{I}_3^0(M) = \mathcal{S}^3(M) \oplus_M \Omega^3(M) \oplus_M \mathcal{Q}(M)$, where $\mathcal{S}^3(M)$ and $\Omega^3(M)$ are the bundles of symmetric tensorfields and of three-forms, respectively. The projectors onto $\mathcal{S}^3(M)$ and $\Omega^3(M)$ are the standard symmetrization and the standard alternation operators, respectively; while the projector $Q : \mathcal{I}_3^0(M) \to \mathcal{Q}(M)$ is defined by:

$$3Q = \Delta \otimes \Delta \otimes \Delta - 2\Psi. \tag{1.15}$$

Finally, a tensorfield $H \in \mathcal{I}_{k+3}^h(M)$, with k, h arbitrary, is called a *generalized curvature structure* iff there exists a suitable contraction $\mathcal{C}$ over $h + k$ indices such that $\mathcal{C}(H \otimes t) \in \mathcal{Q}(M)$ for any $t \in \mathcal{I}_k^h(M)$. In this case we will call "first Bianchi identity" the corresponding Jacobi identity. With this definition, we see that the tensorfield B defined by (1.6) is a generalized curvature structure. This notion extends the discussion of [12].

Now we consider the tensorfields $C \in \mathcal{I}_5^5(M)$ and $E \in \mathcal{I}_7^7(M)$ defined by:

$$\begin{aligned} \mathrm{tr}_2^\dagger(C \otimes X \otimes Y) = \Delta \otimes \{&\mathrm{tr}[X \otimes \mathrm{tr}^\dagger(F \otimes Y)] + \\ &- \mathrm{tr}[Y \otimes \mathrm{tr}^\dagger(F \otimes X)]\}, \end{aligned} \tag{1.16}$$

and

$$\begin{aligned} &\mathrm{tr}_3[Z \otimes V \otimes W \otimes \mathrm{tr}_2^\dagger(E \otimes X \otimes Y)] = \\ &\quad = \Delta \otimes \Big\{\mathrm{tr}[V \otimes \mathrm{tr}^\dagger(B \otimes X)] \otimes \mathrm{tr}_2[Z \otimes W \otimes \mathrm{tr}^\dagger(B \otimes Y)] + \\ &\qquad - \mathrm{tr}[V \otimes \mathrm{tr}^\dagger(B \otimes Y)] \otimes \mathrm{tr}_2[Z \otimes W \otimes \mathrm{tr}^\dagger(B \otimes X)]\Big\}, \end{aligned} \tag{1.17}$$

for any $X, Y, Z, V, W \in \mathcal{X}(M)$. These tensorfields determine two linear morphisms which, by an abuse of notation, will be again denoted by C and E. More precisely we define $C : \mathcal{I}_0^2(M) \times \mathcal{I}_4^0(M) \to \mathcal{I}_3^1(M)$ and $E : (\mathcal{I}_0^2(M))^2 \times (\mathcal{I}_3^0(M))^2 \to \mathcal{I}_3^1(M)$ by setting:

$$C(m, p) \equiv \mathrm{tr}_2[m \otimes \mathrm{tr}_4^\dagger(C \otimes p)], \tag{1.18}$$

and

$$E(m,\bar m,z,\bar z)\equiv \mathrm{tr}_4[m\otimes\bar m\otimes \mathrm{tr}_6^{\dagger}(E\otimes z\otimes\bar z)], \tag{1.19}$$

for each $m,\bar m\in\mathcal{I}_0^2(M)$, $z,\bar z\in\mathcal{I}_3^0(M)$ and $p\in\mathcal{I}_4^0(M)$. Using (1.19), from the tensorfield E we can define a new tensorfield $D\in\mathcal{I}_7^7(M)$ by means of the linear morphism $D:(\mathcal{I}_0^2(M))^2\times(\mathcal{I}_3^0(M))^2\to\mathcal{I}_3^1(M)$ defined by:

$$\begin{aligned}4D(m,\bar m,z,\bar z)&\equiv 4\mathrm{tr}_4[m\otimes\bar m\otimes\mathrm{tr}_6^{\dagger}(D\otimes z\otimes\bar z)]=\\&=E(m,\bar m,z,\bar z)+E(\bar m,m,z,\bar z)+\\&\quad+E(m,\bar m,\bar z,z)+E(\bar m,m,\bar z,z),\end{aligned} \tag{1.20}$$

for any $m,\bar m\in\mathcal{I}_0^2(M)$ and $z,\bar z\in\mathcal{I}_3^0(M)$. It then follows immediately:

$$\begin{aligned}{}[C(m,p)]^{\lambda}_{\beta\gamma\eta}&=m^{\lambda\alpha}[p_{\gamma\{\alpha(\beta\eta)\}}-p_{\eta\{\alpha(\beta\gamma)\}}],\\ [E(m,m,z,z)]^{\lambda}_{\beta\gamma\eta}&=m^{\varepsilon\alpha}m^{\zeta\lambda}[z_{\{\varepsilon(\zeta\eta)\}}z_{\{\alpha(\beta\gamma)\}}-z_{\{\varepsilon(\zeta\gamma)\}}z_{\{\alpha(\beta\eta)\}}],\\ D(m,z)&\equiv D(m,m,z,z)=E(m,m,z,z).\end{aligned} \tag{1.21}$$

Using the above mappings we finally construct a new map $\mathcal{R}:\mathcal{I}_0^2(M)\times\mathcal{I}_3^0(M)\times\mathcal{I}_4^0(M)\to\mathcal{I}_3^1(M)$ by setting:

$$\mathcal{R}(m,z,p)=C(m,p)+D(m,z), \tag{1.22}$$

for each $m\in\mathcal{I}_0^2(M)$, $z\in\mathcal{I}_3^0(M)$ and $p\in\mathcal{I}_4^0(M)$. The mapping $\mathcal{R}$ so defined will be called the *generator of the Riemannian curvatures* (*of pseudo-Riemannian metrics on* M). In fact the following holds:

PROPOSITION 2. *For any linear connection* Γ *and any pseudo-Riemannian metric* g *on* M *we have:*

$$\mathrm{Riem}(g)=\mathrm{Riem}(\Gamma)+\mathcal{R}(g^*,\nabla g,\nabla\nabla g), \tag{1.23}$$

and

$$\nabla C=0\quad,\quad\nabla D=0, \tag{1.24}$$

The parallel tensorfields C *and* D *are generalized curvature structures.*

This suggests us to define a further mapping:

$$\mathcal{R}\mathrm{iem} : \Gamma(J^1\mathcal{C}_S(M)) \times \mathcal{I}^2_0(M) \times \mathcal{I}^0_3(M) \times \mathcal{I}^0_4(M) \to \mathcal{I}^1_3(M),$$

by setting

$$\mathcal{R}\mathrm{iem}(j^1\Gamma, g^*, \nabla g, \nabla\nabla g) = \mathrm{Riem}(\Gamma) + \mathcal{R}(g^*, \nabla g, \nabla\nabla g), \tag{1.25}$$

for any $\Gamma \in \Gamma(\mathcal{C}_S(M))$ and any pseudo-Riemannian metric g on M. Then, the following identity holds:

$$\mathcal{R}\mathrm{iem}(j^1\Gamma, g^*, \nabla g, \nabla\nabla g) = \mathrm{Riem}(g). \tag{1.26}$$

An analogous construction holds for the Ricci tensor. In fact, the tensorfields defined by (1.16), (1.17) and (1.20) determine tensorfields $\hat{C} \in \mathcal{I}^4_4(M)$, $\hat{E} \in \mathcal{I}^6_6(M)$ and $\hat{D} \in \mathcal{I}^6_6(M)$ by setting:

$$\begin{aligned} \mathrm{tr}_2[m \otimes \mathrm{tr}^\dagger_4(\hat{C} \otimes p)] &= -\,\mathrm{tr}^\dagger[C(m,p)], \\ \mathrm{tr}_4[m \otimes \bar{m} \otimes \mathrm{tr}^\dagger_6(\hat{E} \otimes z \otimes \bar{z})] &= -\,\mathrm{tr}^\dagger[E(m,\bar{m},z,\bar{z})], \\ \mathrm{tr}_4[m \otimes \bar{m} \otimes \mathrm{tr}^\dagger_6(\hat{D} \otimes z \otimes \bar{z})] &= -\,\mathrm{tr}^\dagger[D(m,\bar{m},z,\bar{z})], \end{aligned} \tag{1.27}$$

for any $m, \bar{m} \in \mathcal{I}^2_0(M)$, $z, \bar{z} \in \mathcal{I}^0_3(M)$ and $p \in \mathcal{I}^0_4(M)$. With the standard abuse of notation these tensorfields define in turn linear operators $\hat{C}$: $\mathcal{I}^2_0(M) \times \mathcal{I}^0_4(M) \to \mathcal{I}^0_2(M)$ and $\hat{E}, \hat{D} : (\mathcal{I}^2_0(M))^2 \times (\mathcal{I}^0_3(M))^2 \to \mathcal{I}^0_2(M)$ by:

$$\begin{aligned} \hat{C}(m,p) \equiv& \mathrm{tr}_2[m \otimes \mathrm{tr}^\dagger_4(\hat{C} \otimes p)], \\ \hat{E}(m,\bar{m},z,\bar{z}) \equiv& \mathrm{tr}_4[m \otimes \bar{m} \otimes \mathrm{tr}^\dagger_6(\hat{E} \otimes z \otimes \bar{z})], \\ \hat{D}(m,\bar{m},z,\bar{z}) \equiv& \mathrm{tr}_4[m \otimes \bar{m} \otimes \mathrm{tr}^\dagger_6(\hat{D} \otimes z \otimes \bar{z})]. \end{aligned} \tag{1.28}$$

for each $m, \bar{m} \in \mathcal{I}^2_0(M)$, $z, \bar{z} \in \mathcal{I}^0_3(M)$ and $p \in \mathcal{I}^0_4(M)$. It follows immediately:

$$\begin{aligned} [\hat{C}(m,p)]_{\beta\gamma} =& m^{\eta\alpha}[p_{\eta\{\alpha(\beta\gamma)\}} - p_{\gamma\{\alpha(\beta\eta)\}}], \\ [\hat{E}(m,m,z,z)]^{\beta\gamma} =& m^{\alpha\varepsilon}m^{\zeta\eta}[z_{\{\varepsilon(\zeta\gamma)\}}z_{\{\alpha(\beta\eta)\}} - z_{\{\varepsilon(\zeta\eta)\}}z_{\{\alpha(\beta\gamma)\}}], \\ \hat{D}(m,z) \equiv& \hat{D}(m,m,z,z) = \hat{E}(m,m,z,z). \end{aligned} \tag{1.29}$$

Finally, we define a map $\widehat{\mathcal{R}} : \mathcal{I}_0^2(M) \times \mathcal{I}_3^0(M) \times \mathcal{I}_4^0(M) \to \mathcal{I}_2^0(M)$ by setting:

$$\widehat{\mathcal{R}}(m, z, p) = \hat{C}(m, p) + \hat{D}(m, z), \tag{1.30}$$

for each $m \in \mathcal{I}_0^2(M)$, $z \in \mathcal{I}_3^0(M)$ and $p \in \mathcal{I}_4^0(M)$. The mapping $\widehat{\mathcal{R}}$ will be called the *generator of the Ricci tensors* (*of pseudo-Riemannian metrics on* M). In fact, the following holds:

Proposition 3. *For any linear connection* Γ *and for any pseudo-Riemannian metric* g *on* M *we have:*

$$\mathrm{Ric}(g) = \mathrm{Ric}(\Gamma) + \hat{\mathcal{R}}(g^*, \nabla g, \nabla\nabla g), \tag{1.31}$$

and

$$\nabla \hat{C} = 0 \quad , \quad \nabla \hat{D} = 0. \tag{1.32}$$

As for the Riemannian curvature, we define the new operator

$$\mathcal{R}\mathrm{ic} : \Gamma(J^1\mathcal{C}_S(M)) \times \mathcal{I}_0^2(M) \times \mathcal{I}_3^0(M) \times \mathcal{I}_4^0(M) \to \mathcal{I}_3^1(M),$$

by setting

$$\mathcal{R}\mathrm{ic}(j^1\Gamma, g^*, \nabla g, \nabla\nabla g) = \mathrm{Ric}(\Gamma) + \widehat{\mathcal{R}}(g^*, \nabla g, \nabla\nabla g), \tag{1.33}$$

for any $\Gamma \in \Gamma(\mathcal{C}_S(M))$ and any pseudo-Riemannian metric g on M. Then, the following identity holds:

$$\mathcal{R}\mathrm{ic}(j^1\Gamma, g^*, \nabla g, \nabla\nabla g) = \mathrm{Ric}(g). \tag{1.34}$$

Now we consider the first-order deformation $\mathcal{R}_{(1)}$ of $\mathcal{R}$, as introduced in [2], and the Hessian mapping of $\mathcal{R}$, given respectively by:

$$\mathcal{R}_{(1)}(x, \bar{x}) = C(\bar{m}, p) + C(m, \bar{p}) + 2D(\bar{m}, m, z, z) + 2D(m, m, \bar{z}, z), \tag{1.35}$$

and

$$\begin{aligned}\mathcal{H}\text{ess}(\mathcal{R})_x(\bar{x},\check{x}) = C(\check{m},\bar{p}) + C(\bar{m},\check{p}) + 2D(\bar{m},\check{m},z,z)+ \\ + 4D(\bar{m},m,\check{z},z) + 4D(\check{m},m,\bar{z},z)+ \\ + 2D(m,m,\check{z},\bar{z}),\end{aligned} \tag{1.36}$$

for each $x = (m, z, p)$, $\bar{x} = (\bar{m}, \bar{z}, \bar{p})$ and $\check{x} = (\check{m}, \check{z}, \check{p})$ belonging to the module $\mathcal{I}_2^0(M) \times \mathcal{I}_3^0(M) \times \mathcal{I}_4^0(M)$.

Let g be any pseudo-Riemannian metric, Γ be any (symmetric) connection on M and $\bar{q}$ and $\check{q}$ be two symmetric twice-covariant tensorfields. We denote by $\bar{q}^*, \check{q}^* \in \mathcal{I}_0^2(M)$ the tensorfields whose local components are $\bar{q}^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\bar{q}_{\alpha\beta}$ and $\check{q}^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\check{q}_{\alpha\beta}$, where $\bar{q}_{\alpha\beta}$ and $\check{q}_{\alpha\beta}$ are the local components of $\bar{q}$ and $\check{q}$, respectively. Finally, we set $x = (g^*, \tilde{\nabla} g, \tilde{\nabla}\tilde{\nabla} g)$, $\bar{x} = (\bar{q}^*, \tilde{\nabla}\bar{q}, \tilde{\nabla}\tilde{\nabla}\bar{q})$ and $\check{x} = (\check{q}^*, \tilde{\nabla}\check{q}, \tilde{\nabla}\tilde{\nabla}\check{q})$. Under these assumptions it is easy to prove the following statements:

PROPOSITION 4. *The following properties hold*

$$\mathcal{R}_{(1)}(x,\bar{x})^{\lambda}_{\beta\gamma\mu} = 2\nabla_{[\gamma}\bar{\phi}^{\lambda}_{\mu]\beta}, \tag{1.37}$$

and

$$\mathcal{H}\text{ess}(\mathcal{R})_x(\bar{x},\check{x})^{\lambda}_{\beta\gamma\mu} = 2\check{\phi}_{[\gamma}(\bar{\phi}^{\lambda}_{\mu]\beta}) - 2g^{\lambda\rho}[\nabla_{[\gamma}(\check{\phi}^{\varepsilon}_{\mu]\beta}\bar{q}_{\rho\varepsilon}) + \nabla_{[\gamma}(\bar{\phi}^{\varepsilon}_{\mu]\beta}\check{q}_{\rho\varepsilon})], \tag{1.38}$$

where $\bar{\phi} = F(g^*, \tilde{\nabla}\bar{q})$, $\check{\phi} = F(g^*, \tilde{\nabla}\check{q})$ *and*

$$\check{\phi}_{\gamma}(\bar{\phi}^{\lambda}_{\mu\beta}) = \check{\phi}^{\lambda}_{\sigma\gamma}\bar{\phi}^{\sigma}_{\mu\beta} - \check{\phi}^{\sigma}_{\mu\gamma}\bar{\phi}^{\lambda}_{\sigma\beta} - \check{\phi}^{\sigma}_{\beta\gamma}\bar{\phi}^{\lambda}_{\mu\sigma}. \tag{1.39}$$

PROPOSITION 5. *Let* g_s*, with* $s \in]-a, a[$ *and* $a > 0$ *be a homotopic variation of* g *and* $\bar{q} = \delta g_s$ *be the first variation of* g_s*. Then* $\bar{\phi} \equiv F(g^*, \tilde{\nabla}\bar{q}) = \delta\Gamma_{LC}(g)$ *is the first variation of the Levi-Civita connection* $\Gamma_{LC}(g)$ *of* g *and we have* $\delta\,\mathrm{Riem}(g_\varepsilon) = \mathcal{R}_{(1)}(x,\bar{x})$.

The left hand side of equation (1.37) is well known (see, e.g., [11]), via Proposition 5. Equation (1.38) has an analogous meaning with respect to the second variation (see [1]).

The corresponding formulae for the Ricci tensors are given by:

$$\widehat{\mathcal{R}}_{(1)}(x,\bar{x}) = \hat{C}(\bar{m},p)+\hat{C}(m,\bar{p})+2\hat{D}(\bar{m},m,z,z)+2\hat{D}(m,m,\bar{z},z). \tag{1.40}$$

and

$$\begin{aligned}\mathcal{H}\mathrm{ess}(\widehat{\mathcal{R}})_x(\bar{x},\check{x}) = {} & \hat{C}(\check{m},\bar{p}) + \hat{C}(\bar{m},\check{p}) + 2\hat{D}(\bar{m},\check{m},z,z)+ \\ & + 4\hat{D}(\bar{m},m,\check{z},z) + 4\hat{D}(\check{m},m,\bar{z},z)+ \\ & + 2\hat{D}(m,m,\check{z},\bar{z}).\end{aligned} \tag{1.41}$$

Finally, we have:

Proposition 6. *Under the same assumptions as in Propositions 4 and 5 the following hold:*

$$\widehat{\mathcal{R}}_{(1)}(x,\bar{x})_{\beta\mu} = 2\nabla_{[\lambda}\bar{\phi}^{\lambda}_{\mu]\beta}, \tag{1.42}$$

and

$$\mathrm{Hess}(\widehat{\mathcal{R}})_x(\bar{x},\check{x})_{\beta\mu} = 2\check{\phi}_{[\lambda}(\bar{\phi}^{\lambda}_{\mu]\beta})-2g^{\lambda\rho}[\nabla_{[\lambda}(\check{\phi}^{\varepsilon}_{\mu]\beta}\bar{q}_{\rho\varepsilon})+\nabla_{[\lambda}(\bar{\phi}^{\varepsilon}_{\mu]\beta}\check{q}_{\rho\varepsilon})]. \tag{1.43}$$

Results analogous to those of Proposition 5 can be easily obtained from Proposition 6.

As a final remark, let us recall that on the domain of any sufficiently regular chart of M one can consider the linear (local) connection Γ induced by the standard flat connection of $\mathbb{R}^n$. This amounts to set the connection coefficients to be zero in the given coordinates. With this (non-covariant) procedure one easily recovers the results corresponding to the above Propositions in terms of natural coordinates on jet-bundles.

2 – Conclusions

Let us first remark that, as it is well known, the curvature tensorfield of a complete Riemannian manifold imposes strong conditions on the "shape" of the manifold itself. Many of these conditions are related to the properties of the geodesics of the manifold and hence to the variational problem defined by the Riemannian metric itself. In [1] we proved that all first order variational problems (even those generically related to field theories not involving metric structures or connections) define in fact, through an appropriate form of the second variation, a suitable "curvature tensorfield". This last one verifies suitable generalized Bianchi identities and has the same relation with the Hessian mapping as the curvature tensorfield of the Riemannian metric has with respect to the classical "Jacobi equation for geodesic deviation". In the case of variational problems generically related to field theories it is more difficult (and perhaps even impossible) to find a unique definition of "completeness" among the various "compactness conditions" for the relevant operators involved, especially in view of the fact that these conditions are generally given by means of inequalities. In any case, a relation among compactness of the differential operators ensuing from the variational problem, curvature of the variational problem and "shape" of the configuration manifold of the theory seems to be more than reasonable in many cases. Because of this, we suggest to call *non trivial* (resp. *trivial*) the curvature of the variational problem if such a relation exists (resp., does not exist). The notion of curvature can be easily extended also to variational problems of any order. Our construction, together with the results of [1], [2] and [3], shows that the curvature of the variational problems related to curvature invariants is trivial, since it is essentially determined by the Kronecker tensor and since the relevant curvature structures are essentially determined by tensor products of covariantly constant tensors obtained out of the (constant) Kronecker tensor Δ.

The concluding remarks above emphasize the fact that the mathematical construction which associates to a pseudo-Riemannian metric its Levi-Civita connection, the curvature and the Ricci tensor, together with their variations, has a power which is astonishingly deep and unsuspected. Interpreting in fact these structures in view of the results of [1], the notion of "generalized curvature" becomes more clear, since the curvature of the Levi-Civita connection of any pseudo-Riemannian metric coincides with the curvature of the variational problem generating the geodesics of the metric itself. In [1] it was shown that this last curvature can be calculated by using any "background" alternative connection (which might be useful to introduce for contingent reasons related with the possible physical applications sought for; see, e.g., [10]).

The tensorfields introduced in this paper produce a strong simplification in all calculations concerning variational problems based on Lagrangian functions depending on the curvature invariants (see [2] and [3]). A further simplification seems to be produced by the use of these general structures when investigating conservation laws for this family of Lagrangians. Work in this direction is in progress and we aim to address it in a future publication ([5]).

REFERENCES

[1] O. Amici – B. Casciaro – M. Francaviglia: *Covariant second variation for first order Lagrangians on fibered manifolds II: generalized curvature and Bianchi identities*, Rend. Mat. Univ. Roma, **16** (1996), 637-669.

[2] O. Amici– B. Casciaro – M. Francaviglia: *Second Variation and Generalized Jacobi Equations for Curvature Invariants*, Atti Acc. Peloritana, to appear, 1997.

[3] O. Amici– B. Casciaro – M. Francaviglia: *The Second Variation for Non-Linear Gravitational Lagrangians*, Atti Acc. Scienze di Torino (1997) to appear.

[4] O. Amici – B. Casciaro – M. Francaviglia: *Relations between Variational Derivatives of Functions and Scalar Densities*, Rend. Circ. Mat. Palermo (1997), to appear.

[5] O. Amici– B. Casciaro – M. Francaviglia: *Conservation Laws for non-Linear Gravitational Lagrangians*, in preparation.

[6] B. Casciaro – M. Francaviglia: *Covariant second variation for first order Lagrangians on fibered manifolds I: generalized Jacobi fields*, Rend. Mat. Univ. Roma, **16** (1996), 233-246.

[7] B. Casciaro – M. Francaviglia: *A New Variational Characterization of Jacobi Fields along Geodesics*, Ann. Mat. Pura Appl. (1997), to appear.

[8] B. Casciaro – M. Francaviglia: *Generalized Jacobi Equations and the Curvature of Variational Principles*, in: Proceedings WCNA-96, Athens, July 1996, to appear, 1997.

[9] B. Casciaro – M. Francaviglia – V. Tapia: *On the Variational Characterization of Generalized Jacobi Equations,*, in: "Differential Geometry and Its Applications", Proceedings Brno 1995; J. Janiska *et al.* eds. (1996), pp. 353-372.

[10] M. Ferraris – M. Francaviglia: *First order Lagrangians, energy-density and superpotentials in general relativity*, Journ. Gen. Rel. Grav., **22** (1990), 965-985.

[11] M. Francaviglia: *Relativistic Theories (the Variational Structure)*, Lectures at the 13th Summer School in Math. Phys., Ravello, 1988. Quaderni del CNR, 1990

[12] S. Kobayashi and K. Nomizu: *Foundations of Differential Geometry I, II*, Interscience, New York, 1969

Lavoro pervenuto alla redazione il 28 ottobre 1997
ed accettato per la pubblicazione il 25 novembre 1998.
Bozze licenziate il 22 marzo 1999

Indirizzo degli Autori:

Oriella Amici – Biagio Casciaro – Dipartimento di Matematica – Università di Bari – Via Orabona 5 – 70125 Bari, Italy

Mauro Francaviglia – Dipartimento di Matematica – Università di Torino – Via C. Alberto 10 – 10123 Torino, Italy