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Non trivial solutions of non-linear partial differential inequations and order cut-off

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RIASSUNTO: *In questo lavoro otteniamo delle condizioni necessarie sulla dimensione dello spazio affinché una classe di disequazioni variazionali ammetta una soluzione non triviale.*

Dimostriamo che una soluzione non triviale $\gamma(\gamma : \mathbb{R}^N \to \mathbb{R}^k)$ di questo tipo di disequazioni esiste solo se la dimensione N è sufficientemente grande rispetto all'ordine minimo dell'operatore a derivate parziali considerato. Riconosciamo inoltre che la proprietà di cut-off è individuata dal numero di variabili che compaiono effettivamente nell'operatore.

Introduciamo infine alcuni modelli provenienti dalla teoria delle decisioni statistiche nei quali intervengono le disequazioni ed i fenomeni di tipo cut-off considerati.

ABSTRACT: *We derive necessary conditions on the space dimension such that a class of partial differential inequations admit a non trivial solution.*

We show that a nontrivial solution $\gamma(\gamma : \mathbb{R}^N \to \mathbb{R}^k)$ of this type of inequations may exist only if the dimension N is sufficiently large with respect to the minimal order of the partial differential operator which is investigated. Furthermore, we prove that the cut-off property is actually governed by the number of variables which genuinely occur in the operator.

We briefly introduce a few motivations in statistical decision theory that lead to such inequations and dimension cut-off phenomenon.

KEY WORDS AND PHRASES: *Partial differential inequations – Non trivial solutions – Dimension cut-off.*
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1 – Introduction

The goal of this paper is to study nonlinear partial differential inequalities of the form

$$\mathcal{R}\gamma \le 0 \text{ in } \mathcal{D}'(\mathbb{R}^N) \tag{1}$$

where

$$\mathcal{R}\gamma = \sum_{i=1}^{k} \sum_{\substack{(\alpha_1,\cdots,\alpha_N)\\ |\alpha|\le M}} a^i_{\alpha_1\cdots\alpha_N} \frac{\partial^{|\alpha|}\gamma_i}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}} + \|\gamma\|^q \,. \tag{2}$$

More precisely, we investigate a space dimension phenomenon connected with the existence of a nontrivial solution to (1) in the space $(L^q_{\rm loc}(\mathbb{R}^N))^k$.

In (2), N, k and M are non-negative integers, $\gamma = (\gamma_1, \ldots, \gamma_k)$ is a function from $\mathbb{R}^N$ into $\mathbb{R}^k$, $(\alpha_1, \ldots, \alpha_N)$ is a non null multi-index (i.e. an N-tuple of non-negative integers) such that its length satisfies $|\alpha| = \alpha_1 + \cdots + \alpha_N \le M$, $a^i_{\alpha_1\cdots\alpha_N}$ is a constant (all these constants being not simultaneously null) and q is a real number such that $q > 1$.

Let us stress that the structure of the operator $\mathcal{R}$ given by (2) is quite general (in particular $\mathcal{R}$ is not assumed to be elliptic). Moreover γ may be a vector field. Such a feature takes its origin in Statistics where a few statistical problems are connected with partial differential inequalities involving various operators. Let us just mention two basic examples and postpone to an appendix (see Section 5) the development of the underlying statistical motivations and analysis.

The first example is given by $k = N$ and

$$\mathcal{R}\gamma = 2\text{div } \gamma + \|\gamma\|^2 \tag{3}$$

where a statistical analysis (see Section 5) suggest that in this case, the inequation (1) has no nontrivial solution when the space dimension N is less than or equal to 2.

By contrast, it is easy to check, when $N \ge 3$, that $\gamma(x) = \frac{-c}{\|x\|^2}$ leads to $\mathcal{R}\gamma(x) < 0$ for any $x \ne 0$ and for any constant c such that $0 < c < 2(N-2)$.

The second example is given by $k = 1$ and

$$\mathcal{R}\gamma = 2\Delta\gamma + \gamma^2 \,. \tag{4}$$

Here the statistical analysis leads to conjecture that the inequation (1) has no nontrivial solution when the space dimension N is less than or equal to 4. By contrast, an example of a function γ satisfying $\mathcal{R}\gamma(x) \leq 0$ is $\gamma(x) = \frac{d}{\|x\|^2}$ for any $x \neq 0$ anf for a constant d such that $0 < d < 4(N-4)$ and $N \geq 5$.

Although the structures of $\mathcal{R}$ in examples (3) and (4) are strongly different, the above mentioned results prompt to conjecture that in both case the inequality (1) exhibits the same dimension cut-off phenomenon: there exists a critical value N_c of the dimension N such that, for $N \leq N_c$, the only solution γ is $\gamma \equiv 0$ while, for $N > N_c$, nontrivial solutions exist.

As a consequence of the study of (1) under the general form (2), we indeed prove that $N_c = 2$ in case (3) and $N_c = 4$ in case (4).

Similar results have been obtained independently of the present work, and under the label "Nonlinear Liouville theorems", by H. Berestycki, I. Capuzzo Dolcetta and L. Nirenberg in the case where the principal part of $\mathcal{R}$ is elliptic (see [1]). In the same framework, a few results concerning the case where the coefficients of $\mathcal{R}$ depend on the space variable x are given by I. Birindelli, I. Capuzzo Dolcetta and A. Cutri in [3].

The paper is organized as follows. In Section 2, we first derive a necessary condition on the dimension N (depending on q and on the lowest order of partial derivation occuring in $\mathcal{R}$) for the inequation (1) to admit a non trivial solution. Then this result is improved through taking into account the number $P(\leq N)$ of variables that really appear in the derivatives in the expression (2). Section 3 yields applications of the results of Section 2 to the dimension cut-off problems mentioned above. We give, in Section 4, some concluding remarks. Finally, Section 5 is an appendix which details the statistical background.

2 – Space dimension and non-trivial solutions to $\mathcal{R}\gamma \leq 0$

This section is devoted to the derivation of necessary conditions so that the inequation $\mathcal{R}\gamma \leq 0$ may admit non-trivial solutions. Indeed the setting is intricately linked to the functional space in which a function γ satisfying the inequation $\mathcal{R}\gamma \leq 0$ is to Ly. In this paper, we address the above inequation in the usual space of distributions which is commonly used as for as partial differential equations or inequations are concerned.

In this framework, in view of the expression (2) of $\mathcal{R}$, defining the nonlinear term $\|\gamma\|^q$ as a distribution prompts to seek γ in the space $(L^q_{\rm loc}(\mathbb{R}^N))^k$ in which case the linear part of $\mathcal{R}$ is perfectly defined.

In the following, it will be convenient to denote by m the minimum of the length of the multi-indexes such that $|\alpha| \le M$ and $a^i_{\alpha_1\cdots\alpha_N} \neq 0$ for any $0 \le i \le k$. In other words, m is the lowest order of partial derivation occuring in $\mathcal{R}$.

In Theorem 1, a first necessary condition is obtained in terms of N, q and m. In Theorem 2, if more information on the coefficients $a^i_{\alpha_1\cdots\alpha_N}$ is available, the result of Theorem 1 is improved by replacing the space dimension N by the number P of variables really appearing in the derivatives of γ in (2).

THEOREM 1. *Assume that* $N \le \frac{q}{q-1} m$. *Then the only solution* γ *in* $(L^q_{\rm loc}(\mathbb{R}^N))^k$ *of the inequation* $\mathcal{R}\gamma \le 0$ *is* $\gamma = 0$. (*a.e. with respect to the Lebesgue measure*).

REMARK. We assume above that $q > 1$. Indeed the result of Theorem 1 fails when $q = 1$ (consider $N = 1$, $k = 1$ and $R\gamma = \gamma' + |\gamma| = 0$).

PROOF. The proof is inspired from a technique developed by H. BRÉZIS in [4] (see also [5] where similar questions are investigated).

Let γ be in $L^q_{\rm loc}(\mathbb{R}^N)$ such that $R\gamma \le 0$. The result lies in proving that $\gamma = 0$ (a.e.).

Let φ be a positive $\mathcal{C}^\infty(\mathbb{R}_+)$-function such that $\varphi(r) = 1$ for $r \le 1$, $\varphi(r) \le 1$ for any r and supp$\varphi = [0, 2]$. Consider then the sequence of $\mathcal{C}^\infty(\mathbb{R}^N)$-functions φ_n from $\mathbb{R}^N$ into $[0, 1]$ defined through

$$\forall n \ge 1 \qquad \forall x \in \mathbb{R}^N \qquad \varphi_n(x) = \varphi\left(\frac{\|x\|}{n}\right) . \tag{5}$$

Remark that, for any n, the function φ_n has a compact support since supp$\varphi_n = B_{2n}$ where B_τ denotes the closed ball of radius τ in $\mathbb{R}^N$.

According to $\mathcal{R}\gamma \le 0$, we have, for every $n \in \mathbb{N}^*$ and every integer $\beta > 0$,

$$\int_{R^N} \|\gamma(x)\|^q \ \varphi_n^\beta(x)dx \le -\Big\langle \sum_{i=1}^k \sum_{\substack{(\alpha_1,\cdots,\alpha_N)\\ |\alpha|\le M}} a^i_{\alpha_1\cdots\alpha_N} \frac{\partial^{|\alpha|}\gamma_i}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}} \ , \varphi_n^\beta \Big\rangle \tag{6}$$

where $\langle .,. \rangle$ denotes the duality brackets between the space of distributions $\mathcal{D}'(\mathbb{R}^N)$ and the space $\mathcal{C}_0^\infty(\mathbb{R}^N)$ of $\mathcal{C}^\infty(\mathbb{R}^N)$-functions with compact support. From the very definition of the distribution derivatives of $(L^q_{\mathrm{loc}}(\mathbb{R}^N))^k$-functions, it follows that

$$(7)\quad \int_{\mathbb{R}^N} \|\gamma(x)\|^q \varphi_n^\beta(x)dx \leq - \sum_{\substack{(\alpha_1,\cdots,\alpha_N)\\ |\alpha|\leq M}} \Big\langle \sum_{i=1}^k a^i_{\alpha_1\cdots\alpha_N}(-1)^{|\alpha|}\gamma_i, \frac{\partial^{|\alpha|}\varphi_n^\beta}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}} \Big\rangle .$$

Since the distribution $\sum_{i=1}^k a^i_{\alpha_1\cdots\alpha_N}(-1)^{|\alpha|}\gamma_i$ lies in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$, inequality (7) yields

$$(8)\quad \begin{aligned} &\int_{\mathbb{R}^N} \|\gamma(x)\|^q \varphi_n^\beta(x)dx \leq \\ &\leq -\int_{\mathbb{R}^N} \sum_{i=1}^k \Big(\gamma_i(x) \sum_{\substack{(\alpha_1,\cdots,\alpha_N)\\ |\alpha|\leq M}} a^i_{\alpha_1\cdots\alpha_N}(-1)^{|\alpha|} \frac{\partial^{|\alpha|}\varphi_n^\beta(x)}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}}\Big)dx \leq \\ &\leq \int_{\mathbb{R}^N} \|\gamma(x)\| \cdot \Big\| \sum_{\substack{(\alpha_1,\cdots,\alpha_N)\\ |\alpha|\leq M}} a_{\alpha_1\cdots\alpha_N}(-1)^{|\alpha|} \frac{\partial^{|\alpha|}\varphi_n^\beta(x)}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}}\Big\| dx \end{aligned}$$

where $a_{\alpha_1\cdots\alpha_N}$ denotes the vector of $\mathbb{R}^k$ with components $a^i_{\alpha_1\cdots\alpha_N}$.

Now, the properties of the function φ and (5) ensure that, for $\beta \geq M$, there exists a constant $K > 0$, depending on φ and β such that

$$(9)\quad \Big| \frac{\partial^{|\alpha|}\varphi_n^\beta(x)}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}} \Big| \leq \frac{K}{n^m}\varphi_n^{\beta-M}(x)$$

for any multi-index $(\alpha_1,\cdots,\alpha_N)$ such that $m \leq |\alpha| \leq M$.

Indeed the inequality (9) has to be established only for $n \leq \|x\| \leq 2n$ because $\sup\varphi \subset [1,2]$. Next a recurrence argument together with Leibniz's formula and the estimate

$$\Big| \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\cdots\partial x_{x_N}^{\alpha_N}} \Big(\frac{\|x\|}{n}\Big) \Big| \leq \mathcal{C}_1 \frac{1}{n^{|\alpha|}}$$

for $n \leq \|x\| \leq 2n$ lead to

$$\Big| \frac{\partial^{|\alpha|}\varphi_n^\beta}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}} \Big(\frac{\|x\|}{n}\Big) \Big| \leq \frac{\mathcal{C}_2}{n^{|\alpha|}} \max(\varphi_n^{\beta-1}(x), \varphi_n^{\beta-2}(x)\cdots\varphi_n^{\beta-|\alpha|}(x))$$

where the constant $\mathcal{C}_1$ depends on N and the constant $\mathcal{C}_2$ depends on the maximum of the successive derivatives of φ and on β. Now inequality (9) is an easy consequence of the above estimate since $0 \leq \varphi_n \leq 1$ and $m \leq |\alpha| \leq M$.

Hence (8) and (9) give, for some constant $C > 0$,

$$\int_{\mathbb{R}^N} \|\gamma(x)\|^q \varphi_n^\beta(x)dx \leq \frac{C}{n^m} \int_{\mathbb{R}^N} \|\gamma(x)\| \varphi_n^{\beta-M}(x)dx \,. \tag{10}$$

Applying Hölder's inequality with $\frac{1}{r} + \frac{1}{q} = 1$, $\beta > Mr$ and $\varphi_n^{\beta-M} = \varphi_n^{\beta/r-M} \cdot \varphi_n^{\beta/q}$, inequality (10) gives

$$\begin{aligned} \int_{\mathbb{R}^N} \|\gamma(x)\|^q \varphi_n^\beta(x)dx &\leq \frac{C}{n^m}\Big(\int_{\mathbb{R}^N} \varphi_n^{\beta-rM}(x)dx\Big)^{1/r} \times \\ &\times \Big(\int_{\mathbb{R}^N} \|\gamma(x)\|^q \varphi_n^\beta(x)dx\Big)^{1/q}. \end{aligned} \tag{11}$$

It is easy to see that (11) is equivalent to

$$\int_{\mathbb{R}^N} \|\gamma(x)\|^q \varphi_n^\beta(x)dx \leq \frac{C^r}{n^{rm}} \int_{\mathbb{R}^N} \varphi_n^{\beta-rM}(x)dx \,. \tag{12}$$

Since $1\!\!1_{B_n} \leq \varphi_n \leq 1$ with $\mathrm{supp}\varphi_n \subset B_{2n}$ and restricting the integration of the first integral of (12) over B_n, it follows that

$$\int_{B_n} \|\gamma(x)\|^q dx \leq A n^{N-rm} \tag{13}$$

for some constant $A > 0$.

Letting n go to infinity, we deduce from (13) that, if $N < rm$, the function γ is equal to 0 almost everywhere and that, if $N = rm$, $\gamma \in (L^q(\mathbb{R}^N))^k$.

Thus the result of the theorem is proved for $N < rm = \frac{q}{q-1}m$. Consider the last case where $N = rm$. Notice that all the derivatives of the function φ vanishing out of the compact set $[1, 2]$ and φ being bounded by 1, inequality (9) can be refined in

$$\Big|\frac{\partial^{\alpha_1+\cdots+\alpha_N}\varphi_n^\beta(x)}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}}\Big| \leq \frac{K}{n^m} 1\!\!1_{C_n}(x) \tag{14}$$

where C_n is the annulus $\{x \in \mathbb{R}^N / n \leq \|x\| \leq 2n\}$ and where $\mathbb{1}_{C_n}$ holds for its characteristic function. Then we deduce from (8) and (14) that, for some constant $C > 0$,

$$\int_{B_n} \|\gamma(x)\|^q dx \leq \frac{C}{n^m} \int_{C_n} \|\gamma(x)\| dx \,. \tag{15}$$

By Hölder inequality it follows from (15) that

$$\int_{B_n} \|\gamma(x)\|^q dx \leq \frac{C}{n^m} \Big(\int_{C_n} dx \Big)^{1/r} \Big(\int_{C_n} \|\gamma(x)\|^q dx \Big)^{1/q} . \tag{16}$$

Using, as an upper bound, the measure of B_{2n} for the first integral of the right hand side of (16), the fact that $N = rm$ gives

$$\int_{B_n} \|\gamma(x)\|^q dx \leq A \Big(\int_{C_n} \|\gamma(x)\|^q dx \Big)^{1/q} \tag{17}$$

for some constant $A > 0$.

Now we have seen above that $\gamma \in (L^q(\mathbb{R}^N))^k$. Hence

$$\lim_{n \to \infty} \int_{C_n} \|\gamma(x)\|^q dx = 0 \,.$$

Thus (17) implies that

$$0 = \lim_{n \to \infty} \int_{B_n} \|\gamma(x)\|^q dx = \int_{\mathbb{R}^N} \|\gamma(x)\|^q dx$$

and, finally, that $\gamma = 0$ (a.e) which is the desired result. □

As already mentioned at the beginning of this section, Theorem 1 is now improved through taking into account the very structure of the operator $\mathcal{R}$. Loosely speaking, the dimension N in the statement of Theorem 1 may be replaced by the number P of variables x_j with respect to which partial derivatives really intervene in the operator $\mathcal{R}$. Through a reordering of the variables $x_1, \cdots, x_N$, the operator $\mathcal{R}$ may be rewritten as

$$\mathcal{R}(\gamma) = \sum_{i=1}^{k} \sum_{\substack{(\alpha_1, \cdots, \alpha_P) \\ |\alpha| \leq M}} a^i_{\alpha_1, \cdots, \alpha_P} \frac{\partial^{|\alpha|} \gamma_i}{\partial x_1^{\alpha_1} \cdots \partial x_P^{\alpha_P}} \,, \tag{18}$$

which means that no derivatives with respect to the $(N-P)$ variables $x_{P+1}, \cdots, x_N$ appear in the expression of $\mathcal{R}$.

THEOREM 2. *The result of Theorem* 1 *is still valid when* N *is replaced by* P.

PROOF. The proof of Theorem 2 is very similar to that of Theorem 1 and only the modifications to be made will be detailed.

Denote by $X=(x_1, \cdots, x_P) \in \mathbb{R}^P$ and $Y=(x_{P+1}, \cdots, x_N) \in \mathbb{R}^{N-P}$.

Consider the sequence of $\mathcal{C}_0^\infty(\mathbb{R}^P)$-functions φ_n from $\mathbb{R}^P$ into $[0,1]$ defined through

$$\forall\, n \geq 1 \qquad \forall\, X \in \mathbb{R}^P \qquad \varphi_n(X) = \varphi\Big(\frac{\|X\|}{n}\Big)$$

where φ is specified at the beginning of the proof of Theorem 1. Let ψ be a positive $\mathcal{C}_0^\infty(\mathbb{R}^{N-P})$-function such that $\psi \leq 1$.

Proceeding as when deriving (8), we obtain, for any $n \in \mathbb{N}^*$ and any integer $\beta > 0$,

$$\begin{aligned} &\int_{\mathbb{R}^P \times \mathbb{R}^{N-P}} \|\gamma(X,Y)\|^q \varphi_n^\beta(X) \psi(Y) dX\, dY \leq \\ &\leq \int_{\mathbb{R}^P \times \mathbb{R}^{N-P}} \psi(Y) \|\gamma(X,Y)\| \Big\| \sum_{\substack{(\alpha_1, \cdots, \alpha_P) \\ |\alpha| \leq M}} a_{\alpha_1 \cdots \alpha_N} (-1)^{|\alpha|} \times \\ &\times \frac{\partial^{|\alpha|} \varphi_n^\beta}{\partial x_1^{\alpha_1} \cdots \partial x_P^{\alpha_P}}(X) \Big\| dX\, dY\,. \end{aligned} \tag{19}$$

In view of (9), since ψ has a compact support, we deduce from (19) that

$$\begin{aligned} &\int_{\mathbb{R}^P \times \mathrm{supp}\psi} \|\gamma(X,Y)\|^q \varphi_n^\beta(X) \psi(Y) dX\, dY \leq \\ &\leq \frac{C}{n^m} \int_{\mathbb{R}^P \times \mathrm{supp}\psi} \|\gamma(X,Y)\| \varphi_n^{\beta - M}(X) \psi(Y) dX\, dY \end{aligned}$$

for some constant $C > 0$.

Hölder's inequality gives here

$$\begin{aligned}\int_{\mathbb{R}^P\times\mathrm{supp}\psi} &\|\gamma(X,Y)\|^q\varphi_n^\beta(X)\psi(Y)dX\,dY\le\\ &\le\frac{C}{n^m}\Big(\int_{\mathbb{R}^P\times\mathrm{supp}\psi}\|\gamma(X,Y)\|^q\varphi_n^\beta(X)\psi(Y)^q dX\,dY\Big)^{1/q}\times\\ &\quad\times\Big(\int_{\mathbb{R}^P\times\mathrm{supp}\psi}\varphi_n^{\beta-rM}(X)dX\,dY)\Big)^{1/r}.\end{aligned}$$

Since $\psi^q\le\psi$, it follows that

$$\int_{\mathbb{R}^P\times\mathrm{supp}\psi}\|\gamma(X,Y)\|^q\varphi_n^\beta(X)\psi(Y)dX\,dY\le An^{P-mr}\quad\mathrm{meas}(\mathrm{supp}\psi)\tag{20}$$

for some constant A>0.

If $P<rm$, letting n go to infinity leads to $\gamma\cdot\psi=0$ almost everywhere on $\mathbb{R}^N$, for any function ψ with compact support. It follows that $\gamma=0$ (a.e.).

In the case where $P=rm$, we first remark that inequality (20) does not imply that $\gamma\in(L^q(\mathbb{R}^N))^k$, but $\|\gamma\|^q\psi\in L^1(\mathbb{R}^N)$, for any function ψ defined such as above.

In view of (14) with X in place of x and the annulus C_n being now in $\mathbb{R}^P$, inequality (19) yields

$$\int_{B_n\times\mathrm{supp}\psi}\|\gamma(X,Y)\|^q\psi(Y)dX\,dY\le\frac{C}{n^m}\int_{C_n\times\mathrm{supp}\psi}\|\gamma(X,Y)\|\psi(Y)dX\,dY$$

(the ball B_n being now in $\mathbb{R}^P$).

Then Hölder's inequality gives, using $P=rm$ and $\psi^q\le\psi$,

$$\begin{aligned}&\int_{B_n\times\mathrm{supp}\psi}\|\gamma(X,Y)\|^q\psi(Y)dX\,dY\le\\ &\quad\le K[\mathrm{meas}(\mathrm{supp}\psi)]^{1/r}\Big(\int_{C_n\times\mathrm{supp}\psi}\|\gamma(X,Y)\|^q\psi(Y)dX\,dY\Big)^{1/q}\end{aligned}\tag{21}$$

where K is a positive constant. Since $\|\gamma\|^q\psi\in L^1(\mathbb{R}^N)$, the right hand side of (21) goes to 0 as n goes to infinity, from which we deduce that, as previously, $\gamma=0$ (a.e.). $\square$

3 – Application to dimension cut-off problems

As far as the dimension cut-off problem introduced in Section 1 is concerned, let us first consider the cases of the two inequalities in $\mathbb{R}^N$

$$A \;\mathrm{div}\gamma + \|\gamma\|^q \le 0 \tag{22}$$

and

$$B \;\Delta\gamma + |\gamma|^q \le 0\,. \tag{23}$$

In (22) and (23), A and B are two non zero real numbers. Indeed (22) (respectively (23)) recovers (3) (respectively (4)). Since the parameter m associated to (22) (respectively to (23)) is equal to 1 (respectively 2) *whatever the dimension* N, Theorem 1 shows that the critical value N_c of the dimension N is such that $N_c \ge [\frac{q}{q-1}]$ for (22) and $N_c \ge [\frac{2q}{q-1}]$ for (23) (where we denote by [y] the integer part of any real number y).

Now easy calculations show that the vector field $\gamma(x) = \frac{-cx}{\|x\|^{\frac{q}{q-1}}}$ satisfies (22) for $N > \frac{q}{q-1}$ and $0 < c \le \{A(N-\frac{q}{q-1})\}^{\frac{1}{q-1}}$ (respectively $-\{-A(N-\frac{q}{q-1})\}^{\frac{1}{q-1}} \le c < 0$) if $A > 0$ (respectively $A < 0$).

It is worth noting that the bound on the dimension $N > \frac{q}{q-1}$ coincides with the required condition so that γ belongs to $(L^q_{\mathrm{loc}}(\mathbb{R}^N))^N$.

Similar calculations yield that the real valued function $\gamma(x) = \frac{c}{\|x\|^{\frac{2}{q-1}}}$ lies in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ and satisfies (23) for $N > \frac{2q}{q-1}$ and $0 < c \le \{\frac{2B}{q-1}(N - \frac{2q}{q-1})\}^{\frac{1}{q-1}}$ (respectively $-\{\frac{-2B}{q-1}(N-\frac{2q}{q-1})\}^{\frac{1}{q-1}} \le c < 0$) if $B > 0$ (respectively $B < 0$).

It follows from the above considerations that $N_c = [\frac{q}{q-1}]$ for (22) and that $N_c = [\frac{2q}{q-1}]$ for (23). In the specific case $q = 2$, we obtain $N_c = 2$ for (3) and $N_c = 4$ for (4).

Remark that in both cases (22) and (23) the quantity $\frac{q}{q-1}$ is independent on the dimension N. Theorems 1 and 2 provide a lower bound for the initial value N_c of the dimension cut-off problem as soon as the sequence of operators $\mathcal{R}$ leads to such an N-independence.

In this setting and to illustrate how the use Theorem 2, consider the sequence $\mathcal{R}\gamma = 2\sum_{i=1}^{N-1} \frac{\partial^2\gamma}{\partial x_i^2} + |\gamma|^q$ for $\gamma \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$. Clearly $P = N-1$ and $m = 2$, so that Theorem 2 leads to $N_c \ge [\frac{2q}{q-1}]+1$ (remark that Theorem 1 would give $N_c \geq [\frac{2q}{q-1}]$). Actually for this example we have $N_c = [\frac{2q}{q-1}]+1$. Indeed, it is easy to check that the real valued function $\gamma(x) = \frac{c}{\|x\|^{\frac{2}{q-1}}}$ satisfies $\mathcal{R}\,\gamma \leq 0$ for $P > \frac{2q}{q-1}$ and $0 < c \leq \{\frac{4}{q-1}(P - \frac{2q}{q-1})\}^{\frac{1}{q-1}}$.

4 – Concluding remarks

The method presented in this paper quite fits to a general partial differential operator with constant coefficients, but it does not seem to work for space dependent (even smooth) coefficients unless they satisfy some very strong assumptions. As an example, case (26), considered in Section 5, cannot be investigated by the method used in Theorem 1 and Theorem 2.

5 – Appendix

The issue we are dealing with in this article has its origin in Statistics, more specifically in Statistical Decision Theory, the feature of this statistical problem being connected with differential inequalities. Our goal, in this section, is to briefly introduce the underlying statistical framework and to indicate how differential inequations are crucial in solving the statistical problem under consideration.

In Subsection 5.1, we first investigate the classical problem of estimating an unknown parameter. Subsection 5.2 sheds light on another statistical problem namely estimating a loss.

5.1 – Point estimation

Assume we observe a random quantity x in a space X coming from a probability distribution P_θ where θ is an unknown parameter which belongs to a set Θ, called the parameter space (X is called the sample space). We wish to estimate θ, that is, to give a value $\varphi(x)$ which assesses θ. Formally, any estimation of θ is given through a measurable application φ from X into Θ; φ is called an *estimator* of θ.

A criterion for the choice of an estimator φ lies in the consideration of a *loss function* L which is an application from $\Theta \times \Theta$ into $\mathbb{R}^+$; the quantity $L(\theta, \varphi(x))$ represents the loss incurred by the estimation $\varphi(x)$

when the “true value” of the parameter is θ. The frequentist approach to Statistics suggests that the performance of any estimator φ should be evaluated, not only on the basis of the current observation x, but on all the possible values of x, that is X. Precisely, the mean of the loss with respect to P_θ is considered, that is the *risk* of φ at θ:

$$R(\varphi, \theta) = \int_X L(\theta, \varphi(x)) P_\theta(dx)\,.$$

It will be convenient in the following to denote by $E_\theta[f]$ the integral of f with respect to P_θ; thus the risk of φ at θ will be written as $R(\varphi, \theta) = E_\theta[L(\theta, \varphi)]$.

When a particular estimator φ_0 is having good properties (with respect to its risk or other criterions) and is considered as “standard”, the question arises whether there exists another estimator φ which improves upon φ_0, that is, such that the risk difference

$$R(\varphi, \theta) - R(\varphi_0, \theta) = E_\theta[L(\theta, \varphi) - L(\theta, \varphi_0)] \tag{24}$$

is nonpositive for any $\theta \in \Theta$ and negative for some $\theta \in \Theta$. A difficulty with solving this question is that, in the integral of the right hand side of (24), the integrand term depends on θ. However, in many cases of interest (such as P_θ is Gaussian or belongs to the exponential family), the development of powerful tools of analysis (see C. Stein [11], H.M. Hudson [8]), leads to a representation of the risk difference as the mean value of an expression $\mathcal{R}\gamma$ involving the difference $\gamma = \varphi - \varphi_0$ and its derivatives but not the parameter θ. In other words, equation (24) can be written as

$$R(\varphi, \theta) - R(\varphi_0, \theta) = E_\theta[\mathcal{R}\gamma]\,. \tag{25}$$

Thus a sufficient condition for improving on φ_0 is to find a solution γ to the differential inequation $\mathcal{R}\gamma < 0$.

As a first basic example we will show below that, when P_θ is Gaussian and L is quadratic, the risk difference in (24) corresponds to the partial differential operator $\mathcal{R}_\gamma$ given by (3) in Section 1; that is, $\mathcal{R}\gamma = 2\text{div}\,\gamma + \|\gamma\|^2$.

An interesting phenomenon is that it can be proved that, for the corresponding statistical problem, there is no estimator φ such that, for any $\theta \in \Theta$, $R(\varphi,\theta)-R(\varphi_0,\theta) \leq 0$ (with strict inequality for some θ) when the dimension N is 1 or 2 (see C. STEIN [10] for the pionnered proof). Consequently, according to (25), we cannot have $\mathcal{R}\gamma(x) \leq 0$ with strict inequality on some x-set of positive P_θ-measure. However we have already seen in Section 1 that, when $N \geq 3$, $\gamma(x) = \frac{c}{\|x\|^2}$ leads to $\mathcal{R}\gamma(x) < 0$ for any $x \neq 0$ and for any constant c such that $0 < c < 2(N-2)$, and hence $R(\varphi,\theta) - R(\varphi_0,\theta) < 0$ for any θ.

We now develop the specific statistical context giving rise to a risk difference such as (25) with a differential operator of the form (3). Assume that P_θ is absolutely continuous with respect to the Lebesgue measure with density $x \to \frac{1}{(2\pi)^{N/2}} \exp\left(-\frac{1}{2}\|x-\theta\|^2\right)$ and that the loss function L is quadratic, that is $L(\varphi,\theta) = \|\varphi - \theta\|^2$.

The simplest candidate φ_0 to be a standard estimator is $\varphi_0(x) = x$ whose risk is easily derived to give

$$R(\varphi_0,\theta) = N\,.$$

One is then led to compute the quantity appearing in (24)

$$R(\varphi,\theta)-R(\varphi_0,\theta) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \left\{\|\varphi(x)-\theta\|^2 - \|x-\theta\|^2\right\} e^{-\|x-\theta\|^2/2} dx\,,$$

for any competing estimator φ. In the following, we show that the right hand side of the above equality can be derived through $\gamma = \varphi - \varphi_0$ and its divergence. Of course we only consider estimator φ with finite risk and a straightforward application of Schwarz's inequality shows that this is the case if and only if $E_\theta[\|\gamma\|^2] < +\infty$.

Then the risk difference between φ and φ_0 can be written as

$$R(\varphi,\theta) - R(\varphi_0,\theta) = E_\theta[\|\gamma\|^2] + 2E_\theta[(\varphi_0 - \theta)\cdot\gamma]\,.$$

Next

$$\begin{aligned} E_\theta[(\varphi_0-\theta)\cdot\gamma] &= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} (x-\theta)\cdot\gamma(x) e^{-\|x-\theta\|^2/2} dx = \\ &= \frac{1}{(2\pi)^{N/2}} \int_0^{+\infty} e^{-R^2/2} \int_{S_{R,\theta}} (x-\theta)\cdot\gamma(x) d\sigma_{R,\theta}(x) dR \end{aligned}$$

where $\sigma_{R,\theta}$ is the area measure on the sphere $S_{R,\theta}=\{x\in\mathbb{R}^N;\|x-\theta\|=R\}$.

Then Stokes' formula leads to

$$E_\theta[(\varphi_0-\theta)\cdot\gamma]=\frac{1}{(2\pi)^{N/2}}\int_0^{+\infty}Re^{-R^2/2}\int_{B_{R,\theta}}\operatorname{div}\gamma(x)dx\,dR$$

where $B_{R,\theta}=\{x\in\mathbb{R}^N;\|x-\theta\|\leq R\}$.

It easily follows that

$$\begin{aligned}E_\theta[(\varphi_0-\theta)\cdot\gamma]&=\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N}\operatorname{div}\gamma(x)\int_{\|x-\theta\|}^{+\infty}Re^{-R^2/2}dR\,dx=\\&=\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N}\operatorname{div}\gamma(x)e^{-\|x-\theta\|^2/2}dx=\\&=E_\theta[\operatorname{div}\gamma]\,.\end{aligned}$$

Finally, we obtain

$$\mathcal{R}(\varphi,\theta)-\mathcal{R}(\varphi_0,\theta)=E_\theta[\mathcal{R}\gamma]\,,$$

with $\mathcal{R}\gamma=2\operatorname{div}\gamma+\|\gamma\|^2$, which is the stated result.

The above example is not the only one where differential inequalities occur. J.O. Berger [2] yields different differential inequalities corresponding to a few distributions and estimated parameters under consideration. He mentions that solving these inequations seems difficult. However he shows that, in many cases, only a few terms of the differential expressions are important, in the sense that they determine the basic nature of the solution. These terms are of the form

$$\mathcal{R}\gamma(x)=\sum_{i=1}^N a_i x_i^{1-m}\frac{\partial\gamma_i(x)}{\partial x_i}+\sum_{i=1}^N b_i x_i^{-m}\gamma_i^2(x)\,. \tag{26}$$

where m denotes some integer in the range $[-2,1]$ and a_i and $b_i(1\leq i\leq N)$ are constant.

Then J.O. Berger [2] establishes a few sufficient conditions under which a function γ satisfies $\mathcal{R}\gamma(x)<0$. An interesting fact is that these conditions require that the dimension N is sufficiently large.

L.D. Brown [6] studies thoroughly a general class of differential inequalities and their relationship with the admissibility of corresponding estimators. These inequalities are of the form

$$2\sum_{i=1}^{N} a_i\gamma_i(x) + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\frac{\partial\gamma_i}{\partial x_j}(x) + \gamma^t(x)B\gamma(x) \leq 0 \tag{27}$$

where $\gamma^t(x)$ denotes the transpose of $\gamma(x)$, $B = (b_{ij})$ is a symmetric positive definitive matrix valued function, $A = (a_{ij})$ is a non-singular matrix valued function and a_i, a_{ij}, b_{ij} are everywhere continuously differentiable functions. Notice that, as far as the dimension cut-off is concerned, case (27) (even for constant coefficients) seems, when $a_i \neq 0$, to behave as the case $q = 1$ of (2) (see the remark after Theorem 1). Actually consider the simplest version of (27), that is $2a\cdot\gamma + 2\mathrm{div}\gamma + \|\gamma\|^2 \leq 0$, which by translation reduces to $2\,\mathrm{div}\,\Gamma + \|\Gamma\|^2 \leq \|a\|^2$. Indeed, for this last inequality no dimension cut-off may be expected (consider the case $N = 1$).

5.2 – Loss estimation

Another statistical context giving rise to differential inequalities is the estimation of a loss. Keeping the notations introduced at the beginning of this section, an estimator φ of the unknown parameter θ being choosen, one would like to calculate the incurred loss $L(\theta, \varphi(x))$. The reason of this concern is that the risk of φ at θ is a mean of the loss taken on all possible observations, when the only interesting thing at hand is the observation x itself. However $L(\theta, \varphi(x))$ is not available since it depends on the unknown parameter θ. Hence it is of interest to estimate the loss of $\varphi(x)$ with a measurable function λ from X into $\mathbb{R}^+$.

Of course there remains to assess the performance of λ itself. A simple way is to consider the quadratic error incurred by the loss estimation $\lambda(x)$ when the loss equals $L(\theta, \varphi(x))$, that is, $(\lambda(x) - L(\theta, \varphi(x)))^2$. Then the mean of this error is calculated with respect to the distribution P_θ ; this is a new notion of risk:

$$R(\lambda, \theta) = E_\theta\left[(\lambda - L(\theta, \varphi))^2\right] .$$

It is clear that the same notions developed for a point estimator have their analog for a loss estimator.

A framework in which loss estimation leads to the consideration of differential inequality is the following (cf. I. Johnstone [9]). Assume we are in the context of the example developed in Subsection 5.1, that is, P_θ is the normal distribution on $\mathbb{R}^N$ with density $x \to \frac{1}{(2\pi)^{N/2}} \exp - \frac{1}{2}\|x-\theta\|^2$ and unknown mean $\theta \in \mathbb{R}^N$. The considered estimator of θ is the identity function φ_0, that is $\varphi_0(x) = x$, and the loss used to evaluate φ_0 is the quadratic loss $L(\theta, \varphi_0(x)) = \|\varphi_0(x) - \theta\|^2$.

A possible criterion for the choice of a loss estimator λ is to require that λ is unbiased i.e.

$$\forall\, \theta \in \mathbb{R}^N \qquad E_\theta[\lambda] = E_\theta[\|\varphi - \theta\|^2]\,.$$

It is easy to check that the constant estimator $\lambda_0 = N$ is unbiased and that its risk is constant (with respect to θ) and equals $2N$.

Now the question of the existence of competing estimators arises. Note that every estimator λ can be written under the form $\lambda = \lambda_0 - \gamma = N - \gamma$ where γ is a measurable function from $\mathbb{R}^N$ into $\mathbb{R}$. Then I. Johnstone [9] proves that, under some analytic properties for γ (twice weak differentiability and integrability for γ^2 and for the different weak derivatives of γ), the risk difference between λ and λ_0 equals

$$R(\lambda, \theta) - R(\lambda_0, \theta) = E_\theta\left[2\Delta\gamma + \gamma^2\right] \tag{28}$$

for any θ in $\mathbb{R}$ (here $\Delta\gamma$ denotes the Laplacian of γ). Such a result expresses the risk difference in (28) through the partial differential operator $\mathcal{R}\gamma$ given by (4) in Section 1 (i.e. $\mathcal{R}\gamma = 2\Delta\gamma + \gamma^2$) and it can be obtained through a repeated application of Stokes' formula as in Subsection 5.1.

Thus there will exist an estimator which improves on λ_0 if there exists a function γ such that the differential inequality $2\Delta\gamma + \gamma^2 \leq 0$ holds true with strict inequality on a set of positive measure (with respect to the Lebesgue measure). Again it is interesting to note that such an improvement cannot occur when the dimension N is less than or equal to 4 (see I. Johnstone [9]). Nevertheless we know from Section 1 that, when $N \geq 5$, $\gamma(x) = \frac{d}{\|x\|^2}$ leads to $\mathcal{R}\gamma(x) < 0$ for any $x \neq 0$ and for any constant d such that $0 < d < 4(N-4)$, and hence $R(\lambda, \theta) - R(\lambda_0, \theta) < 0$ for any θ.

As a last example which also leads to partial differential inequalities involving the Laplacian, D. Fourdrinier and M.T. Wells [7] show that, for a more general class of distributions (i.e. a class of distributions invariant by orthogonal transformations translated by θ such that θ belongs to a proper linear subspace Θ of dimension $k < N$), the differential inequality

$$\frac{2}{(N-k+4)(N-k+6)}\Delta\gamma + \gamma^2 \leq 0 \tag{29}$$

is a sufficient condition for improving on an unbiased estimator of the loss of the least square estimator (i.e. the orthogonal projector from $\mathbb{R}^N$ onto Θ).

REFERENCES

[1] H. Berestycki – I. Capuzzo Dolcetta – L. Nirenberg: *Superlinear indefinite elliptic problems and nonlinear Liouville theorems*, Topological Methods in Nonlinear Analysis, **4** (1994), 59-78.

[2] J.O. Berger: *Improving on inadmissible estimators in continuous exponential families with applications to simultaneous estimation of gamma scale parameters*, The Annals of Statistics, **8** (1980), 545-571.

[3] I. Birindelli – I. Capuzzo Dolcetta – A. Cutri: *Liouville theorems for semilinear equations on the Heisenberg group*, Annales de l'Institut Henri Poincaré, Analyse Nonlinéaire, **14**, 3 (1997), 295-308.

[4] H. Brézis: *Postgraduate course*, Université Pierre et Marie Curie, Paris, (1988).

[5] H. Brézis: *Semilinear equations in* $\mathbb{R}^N$ *without condition at infinity*, Applied Mathematics and Optimization, **12** (1984), 271-282.

[6] L.D. Brown: *The differential inequality of a statistical estimation problem*, in: *Statistical Decision Theory and Related Topics*, **IV**, vol. **1**, (S.S. Gupta and J.O. Berger, eds.) (1988), 299-324.

[7] D. Fourdrinier – M.T. Wells: *Estimation of a loss function for spherically symmetric distributions in the general linear model*, The Annals of Statistics, **23**, 2 (1995), 571-592.

[8] H.M. Hudson: *A natural identity for exponential families with applications in multiparameter estimation*, The Annals of Statistics, **6** (1978), 473-484.

[9] I. Johnstone: *On inadmissibility of some unbiased estimates of loss*, in: *Statistical Decision Theory and Related Topics*, **IV**, vol. **1** (S.S. Gupta and J.O. Berger, eds.) (1988), 361-379.

[10] C. Stein: *Inadmissibility of the usual estimator for the mean of a multivariate normal distribution*, Proc. Third Berkeley Symp. Math. Statist. Probability, **1** (1956), 197-206.

[11] C. Stein: *Estimation of the mean of a multivariate distribution*, Proc. Prague Symp. Asymptotic Statist. (1973), 345-381.

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