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A regularity theorem for ω-minimizers of integral functionals

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RIASSUNTO: *Si prova la locale Hölderianità degli ω-minimi del funzionale integrale $\int_\Omega f(x,u,Du)$ dove la funzione di Carathèodory f soddisfa la seguente condizione di crescita*

$$| Du |^p - b(x) | u |^\gamma - a(x) \le f(x,u,Du) \le L\left(| Du |^p + b(x) | u |^\gamma + a(x)\right),$$

con $L \ge 1$, $1 < p \le \gamma < p^*$ e dove $a(x)$, $b(x)$ sono funzioni non negative aventi opportune sommabilità.

ABSTRACT: *We prove local Hölder continuity of the ω-minimizers of the integral functional $\int_\Omega f(x,u,Du)$, where the Carathèodory function f, satisfies the following growth condition*

$$| Du |^p - b(x) | u |^\gamma - a(x) \le f(x,u,Du) \le L\left(| Du |^p + b(x) | u |^\gamma + a(x)\right),$$

where $L \ge 1$, $1 < p \le \gamma < p^*$ and $a(x)$, $b(x)$ are two non negative functions that lie in suitable L^p spaces.

– Introduction

Let us consider an integral functional of the type

$$F(u,\Omega) = \int_\Omega f(x,u,Du)\, dx, \tag{1}$$

where $u \in W^{1,p}(\Omega)$, $1 < p < n$, and f is Carathéodory integrand satisfy-

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ing the growth assumption

$$|\xi|^p - b(x)\,|u|^\gamma - a(x) \le f(x,u,\xi) \le L(|\xi|^p + b(x)\,|u|^\gamma + a(x)), \tag{2}$$

where $a(x), b(x)$ are two nonnegative functions, $a(x) \in L^s(\Omega)$, $\frac{1}{s} = \frac{p}{n} - \epsilon$, $b(x) \in L^\sigma(\Omega)$, $\frac{1}{\sigma} = 1 - \frac{\gamma}{p^*} - \epsilon$, $\epsilon > 0$ and $p \le \gamma < p^* = \frac{np}{n-p}$. We recall that u is an ω-minimizer of functional (1) if for any ball $B_R(x_0) \subset\subset \Omega$, and for any C^1 function φ with compact support in $B_R(x_0)$

$$F(u, B_R(x_0)) \le [1 + \omega(R)]\, F(u + \varphi, B_R(x_0))$$

where $\omega(R)$ is a continuous nondecreasing function such that $\omega(0) = 0$.

The notion of ω-minimizer was considered for the first time few years ago in a paper by G. Anzellotti (see [2]) and its introduction was motivated by the fact that several typical problems in calculus of variations lead to an inequality of the type above. For instance, if $u \in W_0^{1,2}(\Omega)$ is the solution of the obstacle problem

$$\mathrm{Min}\left\{\int_\Omega |\, Dv\,|^2 \;\; dx : \; v \in W_0^{1,2}(\Omega),\; v \ge \psi \;\; a.e. \text{ in } \Omega\right\}$$

where $\psi \in W_0^{1,2}(\Omega)$, then u is an ω-minimizer of the functional

$$\int_\Omega (|\, Du\,|^2 + \lambda^2)\; dx$$

for a suitable λ. Another nontrivial example of ω-minimizer is given by the solution of the volume-constrained problem (see [2])

$$\mathrm{Min}\left\{\int_\Omega |\, Dv\,|^2 \;\; dx \; : \; v \in W_0^{1,2}(\Omega)\,;\; \int_\Omega v \; dx = \mathrm{const}\right\}.$$

Anzellotti proved that ω-minimizers of the functional

$$\int_\Omega (|Du|^2 + \lambda^2) \;\; dx$$

are $C^{1,\alpha}$ in Ω. More generally (see Chapter 8, [9]) for a general functional of the type (1), under the usual regularity assumptions on the integrand

f, one can prove that ω-minimizers *are* $C^{1,\alpha}$ *provided they are locally Hölder continuous.* This result naturally leads to the question whether ω-minimizers are Hölder continuous or not. A first result in this direction has been given in a recent paper (see [3]), where f is assumed to satisfy the stronger growth assumption

$$\mid \xi \mid^p \leq f(x,u,\xi) \leq L(\mid \xi \mid^p + \mid u \mid^p + 1)$$

In this paper we prove Hölder continuity of ω-minimizers, for an integrand satisfying the fairly general conditions (2). The notion of ω-minimizer is clearly related to the notion of Quasi-minimizer introduced by Giaquinta and Giusti in [7] . However the problem of showing that ω-minimizers are also Quasi-minimizers is still open. If it where so Hölder continuity of ω-minimizers would follow from the results proved in [7] . It is also interesting to remark that a notion similar to the one of ω-minimizer has been previously given by Almgren in the context of geometric measure theory (see [1]).

To prove that ω-minimizers are Hölder continuous we follow a technique used in [3] and [5]. As in the papers quoted above we use a well known variational principle due to Ekeland (see Theorem 5). As far as we know the use of Ekeland principle to get not only existence but also some information about the regularity of minimizers goes back to a paper by Marcellini and Sbordone (see [12]). Here the idea is to compare the ω-minimizer u with the minimizer v in the Dirichlet class $u+W_0^{1,p}(B_{R(x_0)})$ of the functional

$$w \longrightarrow \; F(w, B_R(x_0)) + C_R \int_{B_R(x_0)} |Dw - Du| \; dx,$$

where C_R is a suitable constant depending on u and R.

The minimizer v turns out to be a Q-minimizer of the functional

$$w \longrightarrow \; \int_{B_R(x_0)} |Dw|^p + b(x)\,|w|^\gamma + a(x) + C'_R,$$

where C'_R depend on C_R , hence it is Hölder continuous and we show that

$$\int_{B_\rho(x_0)} \left(|Dv|^p + \rho^{-\mu}\,|v|^\gamma\right) dx$$

decays as $\rho^{n-p+\alpha p}$ for some $\alpha > 0$, $\mu > 0$ not depending on R. From this estimate we are able to prove a similar, but more complicated, decay estimate for the ω-minimizer u (see (20) below). And from this, using a nonstandard version of the usual iteration argument, we finally conclude that u is locally Hölder continuous.

– Preliminary results

From now on Ω will denote any open subset of R^n while $B_R(x_0), x_0 \in \Omega$ and $R <$dist$(x_0, \partial\Omega)$, will denote the open ball with radius R and center at x_0, $\{x \in \Omega : |x - x_0| < R\}$. If u is integrable in $B_R(x_0)$ we set:

$$(u)_{x_0,R} = \frac{1}{\omega_n R^n} \int_{B_R(x_0)} u \, dx = \fint_{B_R(x_0)} u \, dx.$$

Where no confusion may arise we shall simply write u_R in place of $(u)_{x_0,R}$ and B_R in place of $B_R(x_0)$ while c will denote a (possibly) varying constant and only the meaningful dependeces will be specified. We shall deal with variational integrals of the type:

$$F(u,\Omega) = \int_\Omega f(x,u,Du) \, dx, \tag{3}$$

where $u \in W^{1,p}(\Omega), 1 < p < n$,

$$f : \Omega \times R \times R^n \longrightarrow R$$

is a Carathéodory integrand satisfying the following growth conditions:

$$|\xi|^p - b(x)\,|u|^\gamma - a(x) \leq f(x,u,\xi) \leq L(|\xi|^p + b(x)\,|u|^\gamma + a(x)), \tag{4}$$

$$1 < p \leq \gamma < p^* = \frac{np}{n-p}$$

and $a(x), b(x)$ are two nonnegative functions such that:

$$a \in L^s(\Omega), \quad b \in L^\sigma(\Omega),$$

where for some $\epsilon > 0$

$$\frac{1}{s} = \frac{p}{n} - \epsilon, \quad \frac{1}{\sigma} = 1 - \frac{\gamma}{p^*} - \epsilon. \tag{5}$$

and $L > 1$.

We now recall the definitions of Q-minimizer and ω-minimizer:

Definition 1. A function $u \in W^{1,p}(\Omega)$ is a Q-minimizer of functional F, with $Q \geq 1$ if and only if

$$F(u, \operatorname{supp}\varphi) \leq QF(u + \varphi, \operatorname{supp}\varphi)$$

for any $\varphi \in W_0^{1,p}(\Omega)$.

Definition 2. A function $u \in W^{1,p}(\Omega)$ is an ω-minimizer of functional F if and only if

$$F(u, B_R) \leq [1 + \omega(R)]\, F(u + \varphi, B_R)$$

for any $B_R \subset\subset \Omega$ and $\varphi \in W_0^{1,p}(B_R)$, where $\omega : [0, +\infty[\to [0, +\infty[$ is a continuous non decreasing function such that $\omega(0) = 0$.

Definition 3. A function $u \in W^{1,p}(\Omega)$ is a spherical Q-minimizer of functional F if and only if:

$$F(u, B_R) \leq QF(u + \varphi, B_R)$$

for any $B_R \subset\subset \Omega$ and $\varphi \in W_0^{1,p}(B_R)$ with $Q \geq 1$.

The definitions above show that the concepts of Q-minimizer, ω-minimizer and spherical Q-minimizer are closely related and generalize, in different directions, the notion of classical minimizer. Under growth assumptions stated in (4) Q-minimizers of functional F turn out to be locally Hölder continuous (see [9], chapter 7, or [7], for a proof). So, the same question (partially answered in [3], under less general growth assumptions) naturally arises for ω-minimizers, the spherical Q-minimizers being known to be, in general, unbounded, even in the case of the Dirichlet functional (see [7]). The proof of the local Hölder continuity of Q-minimizers, due to Giaquinta-Giusti, is achieved through a fairly natural application of the classical De Giorgi's iteration technique (usually emploied in the theory of elliptic and parabolic PDE's) directly to the functional and rests on the use of the growth conditions and on a proper choice of the test function φ, in order to get suitable energy estimates. This is made possible by the fact that in the definition of Q-minimizer the support of the test function φ is directly involved. In the case of ω-minimzers, instead, this is not true anymore, so such a direct proof cannot be given and a local approximation argument through the Ekeland's variational principle, as considered here and in [3], must be used.

The following lemma, whose proof can be found, for example, in [7], Lemma 6.1, is a technical result that will be used in the proof of Theorem 1.

LEMMA 1. *Let $Z(t):\ [\rho,R]\longrightarrow[0,+\infty]$ be a bounded function and suppose that with $\rho\le t<s\le R$ such that:*

$$Z(t)\le\theta Z(s)+\frac{A}{(s-t)^{\alpha}}+B$$

where A,B,α are positive constants and $0\le\theta<1$, then it follows:

$$Z(t)\le c\frac{A}{(R-\rho)^{\alpha}}+B$$

where c is a positive constant depending on α and θ

Next result is a convenient version of the theorem concerning the higher integrability for the gradients of spherical Q-minimizers (see also [7], Theorem 3.1):

THEOREM 1. *Let $u\in W^{1,p}(\Omega)$ be a spherical Q-minimizer of functional* (3)*. Then there exist $r>1$, $R_0>0$ and c depending on $n,\gamma,p,Q,L,\|b\|_{\sigma},\|u\|_{p^*},\|Du\|_{p}$, such that, for any $R\le R_0$,*

$$\fint_{B_{\frac{R}{2}}}|Du|^{pr}\,dx\le c\left[\left(\fint_{B_R}|Du|^{p}\,dx\right)^{r}+\fint_{B_R}(a(x)+b(x)\,|u|^{\gamma})^{r}\,dx\right]\tag{6}$$

Proof. Let us consider $B_R \subset\subset \Omega$, $\frac{R}{2} < t < s < R$ and choose $\eta \in C_0^\infty(B_s)$ with $0 \le \eta \le 1$, $\eta = 1$ on B_t and $\mid D\eta \mid < \frac{2}{s-t}$; then we define

$$\varphi = \eta(u_s - u), \quad v = u + \varphi.$$

Now from the Q-minimality of u we get

$$F(u, B_s) \le QF(v, B_s)$$

and then, using the growth conditions (4), we have

$$\begin{aligned}\int_{Bs} \mid Du \mid^p \, dx \le c \left[\int_{Bs} \mid Dv \mid^p \, dx + \int_{Bs} a(x)\, dx \right] + \\ + \int_{B_s} b(x)(\mid u \mid^\gamma + \mid v \mid^\gamma)\, dx,\end{aligned}$$

Now we estimate

$$\begin{gathered}\mid v \mid \le \mid u \mid + \mid u - u_s \mid \\ \mid Dv \mid^p \le c \left[(1-\eta) \mid Du \mid^p + \frac{1}{(s-t)^p} \mid u - u_s \mid^p \right]\end{gathered}$$

and so we get

$$\begin{aligned}&\int_{B_s} \mid Du \mid^p \, dx \le \\ (7) \qquad &\le c \left[\int_{B_s - B_t} \mid Du \mid^p dx + \frac{1}{(s-t)^p} \int_{B_s} \mid u - u_s \mid^p dx \right] + \\ &+ c \left[\int_{B_s} a(x) dx + \int_{B_s} b(x) \mid u \mid^\gamma dx + \int_{B_s} b(x) \mid u - u_s \mid^\gamma \, dx \right].\end{aligned}$$

Finally we estimate, using Sobolev and Hölder inequalities, the term

$$\begin{aligned}&\int_{B_s} b(x) \mid u - u_s \mid^\gamma \, dx \le \\ &\le \left(\int_{B_s} \mid u - u_s \mid^{p^*} \, dx \right)^{\frac{p}{p^*}} \left(\int_{B_s} (b(x) \mid u - u_s \mid^{\gamma - p})^{\frac{n}{p}} \, dx \right)^{\frac{p}{n}} \le \\ &\le c\xi(R) \int_{B_s} \mid Du \mid^p \, dx\end{aligned}$$

where

$$\xi(R) = \left(\int_{B_R} (b(x) \mid u - u_s \mid^{\gamma - p})^{\frac{n}{p}} \; dx\right)^{\frac{p}{n}} \leq$$

$$\leq c\left(\int_{B_R} \mid u \mid^{p^*} \; dx\right)^{\frac{\gamma-p}{p^*}} \left(\int_{B_R} b(x)^{\frac{p^*}{p^*-\gamma}} \; dx\right)^{1-\frac{\gamma}{p^*}} \leq$$

$$\leq c \parallel u \parallel_{p^*}^{\gamma-p} \parallel b \parallel_{\sigma} \mid B_R \mid^{\epsilon} .$$

If we choose $R \leq R_0$ small enough, in such way that $c\xi(R_0) < \frac{1}{2}$, we can strike out this term on the left hand side in (7). Then we finally have:

$$\int_{B_t} \mid Du \mid^p \; dx \leq c\left[\int_{B_s - B_t} \mid Du \mid^p \; dx + \frac{1}{(s-t)^p} \int_{B_s} \mid u - u_s \mid^p \; dx\right] +$$
$$+ c \int_{B_s} (a(x) + b(x) \mid u \mid^{\gamma}) \; dx$$

and so adding to both sides the quantity

$$c \int_{B_t} \mid Du \mid^p \; dx$$

we get for $\theta = \frac{c}{c+1}$

$$\int_{B_t} \mid Du \mid^p \; dx \leq \theta \int_{B_s} \mid Du \mid^p \; dx + \frac{c}{(s-t)^p} \int_{B_R} \mid u - u_R \mid^p \; dx$$

$$+ c \int_{B_R} (a(x) + b(x) \mid u \mid^{\gamma}) \; dx$$

From this inequality, using Lemma 1 with:

$$Z(t) = \int_{B_t} \mid Du \mid^p \; dx; \qquad A = c \int_{B_R} \mid u - u_R \mid^p \; dx$$

$$B = c \int_{B_R} (a(x) + b(x) \mid u \mid^{\gamma}) \; dx$$

it follows

$$\int_{B_{\frac{R}{2}}} \mid Du \mid^p \; dx \leq \frac{c}{R^p} \int_{B_R} \mid u - u_R \mid^p \; dx + c \int_{B_R} (a(x) + b(x) \mid u \mid^{\gamma}) \; dx$$

Now the thesis easily follows by a standard applications of the Sobolev embedding theorem and Gehring's lemma in the version of GIAQUINTA and MODICA (see [8]) noticing that $b(x) \mid u \mid^{\gamma} \in L^{\tau}$ for some $\tau > 1$. ∎

REMARK 1. If estimate (6) holds for some $r > 1$ then it still holds for every $1 < r^{'} < r$ with the constant c being bounded from above.

From now on troughout the paper $r > 1$ will be chosen such that

$$\epsilon^{'} = \epsilon - (r-1) > 0, \;\; \gamma r < p^{*}.$$

We shall set also

$$\lambda = (1 - \frac{\gamma}{p^{*}})n, \quad \mu = \lambda - n\epsilon', \qquad (\mu < p).$$

We now recall some results from the theory of Q-minimizers that are crucial for our work.

The following Caccioppoli type inequality on level sets is stated in [9], Theorem 7.1 and contains all the information about Hölder continuity of Q-minimizers. We shall refer to this inequality for the proof of Theorem 3 and Theorem 4 (see [9] chapter 7).

THEOREM 2. *Let $u \in W^{1,p}(\Omega)$ be a Q-minimizer of functional* (3) *then, there exist R_0, and c depending on $n, p, \gamma, Q, \|u\|_{p^{*}}, \| b \|_{\sigma}$ such that for any $x_0 \in \Omega$ and $0 \le \rho < R \le \min\{R_{0,} dist(x_0, \partial\Omega)\}$ and $k \in R$:*

$$\begin{aligned} \int_{A(k,\rho)} |Du|^{p}\, dx \;\le\; & \frac{c}{(R-\rho)^{p}} \int_{A(k,R)} (k-u)^{p} dx + \\ & + c(\|a\|_{L^{s}(B_R)} + |k|^{p} R^{-n\epsilon})\, |A(k,R)|^{1-\frac{p}{n}+\epsilon}, \end{aligned} \tag{8}$$

$$\begin{aligned} \int_{B(k,\rho)} |Du|^{p}\, dx \;\le\; & \frac{c}{(R-\rho)^{p}} \int_{B(k,R)} (k-u)^{p} dx + \\ & + c(\|a\|_{L^{s}(B_R)} + |k|^{p} R^{-n\epsilon})\, |B(k,R)|^{1-\frac{p}{n}+\epsilon}, \end{aligned} \tag{9}$$

where

$$A(k,R) \equiv \{x \in B_R(x_0) \;\; : u(x) > k\}$$
$$B(k,R) \equiv \{x \in B_R(x_0) \;\; : u(x) < k\}$$

THEOREM 3. *Let $u \in W^{1,p}(\Omega)$ be a Q-minimizer of functional (3), then for any $q > 0$ there exist $c \equiv c(q)$ and $R_0 > 0$ as in Theorem 2, such that for any $x_0 \in \Omega$ and $0 \le \rho < R < \min\{R_0, dist(x_0, \partial\Omega)\}$:*

$$\sup_{B_\rho} |u| \le c(q) \left\{ \left[\frac{1}{(R-\rho)^n} \int_{B_R} |u|^q \ dx \right]^{\frac{1}{q}} + \|a\|_{L^s(B_R)}^{\frac{1}{p}} R^{\frac{n\epsilon}{p}} \right\}. \tag{10}$$

See [9] Chapter 7 for the proof.

In the following we shall also deal with Q-minimizers of the functional

$$\mathcal{G}(v, B_R) = \int_{B_R} (\mid Dv \mid^p + b(x) \mid v \mid^\gamma + a(x) + M^{\frac{p}{p-1}}) \ dx, \tag{11}$$

where $M \ge 0$, $B_R \subset\subset \Omega$.

The following technical lemma will be useful in the sequel.

LEMMA 2. *Let $u \in W^{1,p}(B_R)$ such that*

$$\begin{aligned} \int_{\mathrm{supp}(u-v)} \mid Du \mid^p dx \le \\ \le c \int_{\mathrm{supp}(u-v)} \Big(\mid Dv \mid^p + b(x) \mid v \mid^\gamma + a(x) + M^{\frac{p}{p-1}} \Big) dx + \\ + \int_{\mathrm{supp}(u-v)} b(x) \mid u \mid^\gamma dx \end{aligned}$$

for every $v \in u + W_0^{1,p}(B_R)$. Then there exists $R_0 > 0$ and $Q > 1$ depending on $\| b \|_\sigma, \| Du \|_p, \| u \|_{p^*}, c$, such that if $R \le R_0$, u is a Q-minimizer of the functional (11) in B_R.

PROOF. Let us fix $v \in u + W_0^{1,p}(B_R)$. From the assumption, adding the quantity:

$$\int_{\mathrm{supp}(u-v)} \Big(b(x) \mid u \mid^\gamma + a(x) + M^{\frac{p}{p-1}} \Big) dx$$

we get:

$$
\begin{aligned}
&\int_{\operatorname{supp}(u-v)} \left(| \, Du \,|^p + b(x) \,|\, u \,|^\gamma + a(x) + M^{\frac{p}{p-1}}\right) dx \le \\
(12) \qquad &\le c \int_{\operatorname{supp}(u-v)} \left(| \, Dv \,|^p + b(x) \,|\, v \,|^\gamma + a(x) + M^{\frac{p}{p-1}}\right) dx + \\
&\quad + c \int_{\operatorname{supp}(u-v)} b(x) \,|\, u \,|^\gamma \, dx.
\end{aligned}
$$

It is not restrictive to assume that :

$$
\int_{\operatorname{supp}(u-v)} (| \, Dv \,|^p + b(x) \,|\, v \,|^\gamma) \, dx \le \int_{\operatorname{supp}(u-v)} (| \, Du \,|^p + b(x) \,|\, u \,|^\gamma) \, dx
$$

otherwise the thesis would trivially follow. Therefore we may estimate:

$$
\begin{aligned}
\| \, u - v \, \|_{p^*}^{\gamma - p} &\le c \left(\int_{B_R} (| \, Du \,|^p + | \, Dv \,|^p) \; dx \right)^{\frac{\gamma}{p} - 1} \le \\
&\le c \left(\int_{B_R} (| \, Du \,|^p + b(x) \,|\, u \,|^\gamma) \; dx \right)^{\frac{\gamma}{p} - 1} = H(R).
\end{aligned}
$$

Applying Hölder inequality we have

$$
\begin{aligned}
&\int_{\operatorname{supp}(u-v)} b(x) \,|\, u \,|^\gamma \, dx \le \\
&\le c \int_{\operatorname{supp}(u-v)} b(x) \,|\, v \,|^\gamma \, dx + c \int_{\operatorname{supp}(u-v)} b(x) \,|\, u - v \,|^\gamma \, dx \le \\
&\le \int_{\operatorname{supp}(u-v)} b(x) \,|\, v \,|^\gamma \, dx + c \, \| \, b \, \|_\sigma \| \, u - v \, \|_{p^*}^{\gamma - p} \int_{\operatorname{supp} \varphi} | \, D(u - v) \,|^p \, dx \le \\
&\le c \left[\int_{\operatorname{supp}(u-v)} b(x) \,|\, v \,|^\gamma \, dx + \int_{\operatorname{supp}(u-v)} | \, Dv \,|^p \, dx \right] + \\
&\quad + c \, \| \, b \, \|_\sigma \, H(R) \int_{\operatorname{supp}(u-v)} | \, Du \,|^p \, dx.
\end{aligned}
$$

So choosing R small enough in such a way that $c \, \| \, b \, \|_\sigma \, H(R) < \frac{1}{2}$ and inserting the last inequality in (12) we obtain the assertion. $\square$

The aim of the next result is to give a suitable decay estimate for the quantity:

$$
\int_{B_\rho} (|Du|^p + \rho^{-\mu} \, |u|^\gamma) dx
$$

where u is a Q-minimizer of functional (11)

THEOREM 4. *Let $u \in W^{1,p}(\Omega)$ be a Q-minimizer of functional* (11), *then there exist two constants, $\alpha > 0$, $c > 0$ and $R_0 > 0$ depending on $n, p, \gamma, Q, \| a \|_s, \| b \|_\sigma, \|u\|_{p^*}, \| Du \|_p$ but not on M such that for any $0 \leq \rho \leq \frac{R}{2} \leq \frac{R_0}{2}$, if $B_R \subset \Omega$:*

$$
\begin{aligned}
\int_{B_\rho} (|Du|^p + \rho^{-\mu} |u|^\gamma)\, dx &\leq c \left(\frac{\rho}{R}\right)^{n-p+\alpha p} \int_{B_R} (|Du|^p + R^{-\mu} |u|^\gamma) dx \\
&+ c \left(\frac{\rho}{R}\right)^{n-p+\alpha p} (R^{n-p+\alpha p} + M^{\frac{p}{p-1}} R^n + M^{\frac{\gamma}{p-1}} R^{\frac{\gamma n}{p}})
\end{aligned}
\tag{13}
$$

PROOF. Let $u \in W^{1,p}(\Omega)$ be a Q-minimizer of functional (11). By formula (7.41) in [9] we see that there exist $R_0 > 0$ and an integer ν depending on $n, p, \gamma, Q, \| b \|_\sigma, \| a \|_s, \| u \|_{p^*}$ but not on M such that if:

$$
\operatorname{osc}\left(u, \frac{R}{4}\right) \leq \left(1 - \frac{1}{2^{\nu+2}}\right) \operatorname{osc}(u, R) + c\chi(R) 2^\nu R^\beta, \tag{14}
$$

where $\beta = \frac{n\epsilon}{p}$ and

$$
\chi^p(R) \leq c \left[\| a + M^{\frac{p}{p-1}} \|_{L^s(B_R)} + R^{-n\epsilon} \sup_{B_R} | u |^p \right] \tag{15}
$$

(see (7.29) and (7.30) in [9]), with

$$
\operatorname{osc}(u, s) = \sup_{B_s} u - \inf_{B_s} u
$$

Notice that by replacing the quantity ϵ appearing in (5) with something smaller, $\| b \|_\sigma$ will remain bounded, therefore the integer ν appearing in (14) would remain bounded too. So it is not restrictive to assume that $0 < \epsilon < \log_{\frac{1}{4}}(1 - \frac{1}{2^{\nu+2}})$. With such choice of ϵ, from (14), using Lemma 7.3 in [9], we get that if $0 < \rho < R < R_0$, and $B_R \subset \Omega$

$$
\operatorname{osc}(u, \rho) \leq c \left[\left(\frac{\rho}{R}\right)^{\frac{n\epsilon}{p}} \operatorname{osc}(u, R) + \chi(R) \rho^{\frac{n\epsilon}{p}} \right],
$$

where $\chi(R)$ is estimated in (15). From this inequality one can then easily prove that if $0 < \rho < \frac{R}{2}$, $R \leq R_0$, $B_R \subset \Omega$

$$(16)\qquad \int_{B_\rho} | \, Du \, |^p \leq c \left(\frac{\rho}{R}\right)^{n-p+n\epsilon} \left[\int_{B_{R_R}} | \, Du \, |^p + \chi^p \left(\frac{R}{2}\right) R^{n-p+n\epsilon}\right].$$

Now we use Theorem 3 to estimate

$$\begin{aligned}\sup_{B_{\frac{R}{2}}} |u|^p &\leq c \left[\frac{1}{R^n} \int_{B_R} |u|^p \, dx + \left\| a + M^{\frac{p}{p-1}} \right\|_{L^s(B_R)} R^{n\epsilon}\right] \leq \\ &\leq c \left[\frac{1}{R^n} \int_{B_R} |u|^p \, dx + \ \|a\|_{L^s(B_R)} R^{n\epsilon} + M^{\frac{p}{p-1}} R^p\right].\end{aligned}$$

If we put this estimate in (15) we get that if $0 < \rho < \frac{R}{2}$, $R \leq R_0$:

$$(17)\qquad \begin{aligned}\int_{B_\rho} |Du|^p \, dx \leq & c \left(\frac{\rho}{R}\right)^{n-p+n\epsilon} \left[\int_{B_R} |Du|^p \, dx + \frac{1}{R^p} \int_{B_R} |u|^p \, dx\right] \\ & + \left(\frac{\rho}{R}\right)^{n-p+n\epsilon} \left(\|a\|_{L^s(B_R)} R^{n-p+n\epsilon} + M^{\frac{p}{p-1}} R^n\right).\end{aligned}$$

Now, using Lemma 1 it is easy to check that if u is a Q-minimizer of functional (11) then $u - u_R$ is a $\tilde{Q}$-minimizer of the functional

$$w \to \int_{B_R} \left(|Dw|^p + b(x) \, |w|^\gamma + a(x) + b(x) \, |u_R|^\gamma + M^{\frac{p}{p-1}}\right) dx$$

for some $\tilde{Q}$ depending on $Q, \| \, b \, \|_\sigma, \| \, Du \, \|_p, \| \, u \, \|_{p^*}$. Therefore applying estimate (17) to $u - u_R$ we have:

$$\begin{aligned}(18) \int_{B_\rho} |Du|^p \ \, dx \leq \, & c \left(\frac{\rho}{R}\right)^{n-p+n\epsilon} \left[\int_{B_R} |Du|^p \, dx + \frac{1}{R^p} \int_{B_R} |u - u_R|^p \, dx\right] + \\ + c \left(\frac{\rho}{R}\right)^{n-p+n\epsilon} & \left[\|a\|_{L^s(B_R)} R^{n-p+n\epsilon} + M^{\frac{p}{p-1}} R^n + \| \, b \, \|_s \, |u_R|^\gamma R^{n-p+n\epsilon}\right] \leq \\ \leq c \left(\frac{\rho}{R}\right)^{n-p+n\epsilon} & \left[\int_{B_R} |Du|^p \, dx + \int_{B_R} R^{-\mu} \, |u|^\gamma \, dx\right] + \\ + c \left(\frac{\rho}{R}\right)^{n-p+n\epsilon} & \left[\|a\|_{L^s(B_R)} R^{n-p+n\epsilon} + M^{\frac{p}{p-1}} R^n + R^n\right].\end{aligned}$$

where we used Poincaré inequality and we estimated u_R using Hölder inequality:

$$\| b \|_s R^{n-p+n\epsilon} \mid u_R \mid^\gamma \leq c \| b \|_\sigma R^{n(\frac{1}{s}-\frac{1}{\sigma})-p+n\epsilon} \int_{B_R} \mid u \mid^\gamma \leq$$
$$\leq cR^{-\mu} \int_{B_R} \mid u \mid^\gamma \ dx.$$

Now, using Theorem 3 again, we estimate the term

$$\rho^{-\mu} \int_{B_\rho} \mid u \mid^\gamma \ dx.$$

Then we have

$$\begin{aligned} &\int_{B_\rho} |u|^\gamma dx \leq \omega_n \rho^n \int_{B_\rho} |u|^\gamma dx \leq \omega_n \rho^n \sup_{B_\rho} |u|^\gamma \\ &\leq c \left(\frac{\rho}{R}\right)^n \left[\int_{B_R} |u|^\gamma dx + \|a\|_{L^s(B_R)}^{\frac{\gamma}{p}} R^{\frac{n\epsilon\gamma}{p}+n} + \|M^{\frac{p}{p-1}}\|_{L^s(B_R)}^{\frac{\gamma}{p}} R^{n+\frac{n\epsilon\gamma}{p}}\right] \\ &\leq c \left(\frac{\rho}{R}\right)^n \left[\int_{B_R} |u|^\gamma dx + \|a\|^{\frac{\gamma}{p}} R^{\frac{\epsilon\gamma n}{p}+n} + M^{\frac{\gamma}{p-1}} R^{\gamma+n}\right]. \end{aligned}$$

Finally, by the previous inequality and the fact that $\mu < p$ we get:

$$\begin{aligned} &\rho^{-\mu} \int_{B_\rho} |u|^\gamma dx \leq \\ &\leq c \left(\frac{\rho}{R}\right)^{n-p+p\alpha} \left[R^{-\mu} \int_{B_R} |u|^\gamma dx + \| a \|_{L^s(B_R)}^{\frac{\gamma}{p}} R^{n-p+p\alpha} + M^{\frac{\gamma}{p-1}} R^{\frac{n\gamma}{p}}\right] \end{aligned} \tag{19}$$

for a suitable $\alpha > 0$. Now adding the last estimate to (18) we get the result. □

Remark 2. The estimate of the previous theorem is a generalization of the one found in [3], Theorem 6 (see also [5]) but the proof given here is different. In fact in that case the growth assumption were:

$$\mid \xi \mid^p \leq f(x, u, \xi) \leq L(\mid \xi \mid^p + \mid u \mid^p +1)$$

and the homogenity of the function on the right hand side allowed a rescaling argument. In our case it is not so and a careful use of the integral estimates available for Q-minimzers has been emploied.

Now we state a very well known variational principle due to Ekeland that will be the main tool in the proof of Theorem 6 (see [4], or [9], chapter 5, for the proof).

THEOREM 5 (Ekeland's variational principle). *Let (X, d) be a complete metric space and $F : X \longrightarrow]-\infty + \infty]$ a lower semicontinous functional such that*

$$-\infty < \inf_X F < +\infty.$$

Let $\sigma > 0$ and $x \in X$ such that

$$F(x) \le \inf_X \ F + \sigma.$$

Then there exixts $y \in X$ such that

$$\begin{aligned} d(x, y) &\le 1 \\ F(y) &\le F(x) \\ F(y) &\le F(z) + \sigma d(y, z) \quad \forall z \in X. \end{aligned}$$

Finally we state a technical lemma that will be useful in the proof of Theorem 6.

LEMMA 3. *Let $\varphi :]0, +\infty] \longrightarrow]0, +\infty[$ be a positive function and let $0 < \tau < 1$ and $R_0 > 0$ such that for every $R < R_0$:*

$$\begin{aligned} \varphi(\tau R) &\le \tau^\delta \varphi(R) + BR^\beta \\ \varphi(h) &\le c\varphi(\tau^k R) \end{aligned}$$

if $\tau^{k+1}R < h < \tau^k R$, for positive constants c, B and $0 < \beta < \delta$. Then if $\rho < R < R_0$ we also have:

$$\varphi(\rho) \le H \left[\left(\frac{\rho}{R} \right)^\beta \varphi(R) \ + \ B\rho^\beta \right],$$

where H depends on τ, δ, β, c.

PROOF. Just follow the proof of Lemma 7.3 of [9] using the inequality $\varphi(h) \le c\varphi(\tau^k R)$ instead of the fact that φ is nondecreasing. $\square$

– Proof of the main result

In this section we prove the following

THEOREM 6. *Let $u \in W^{1,p}(\Omega)$ be an ω-minimizer of functional (3), then u is locally Hölder continuous.*

In the following we refer to the quantity:

$$\phi\left(\rho, x_0\right) = \int_{B_\rho(x_0)} \left(\mid Du \mid^p + \rho^{-\mu} \mid u \mid^\gamma\right) dx$$

and for the sake of simplicity we will write:

$$\phi\left(\rho, x_0\right) \equiv \phi(\rho)$$

where no confusion will occour.

The proof of the theorem will be obtained once we get the following technical result:

PROPOSITION 1. *There exist R_0, c, α dependig on $n, p, \gamma, \omega, \parallel a \parallel_s$, $\parallel b \parallel_\sigma$,$u$ such that:*

$$\phi(\rho) \le c\left[\left(\frac{\rho}{R}\right)^{n-p+p\alpha} + (\omega(R))^d + \left(\frac{R}{\rho}\right)^\lambda R^{n\epsilon'}\right]\phi(R) + cR^{n-p+p\alpha} \tag{20}$$

for any $0 < \rho < R < R_0$, where $\lambda > 0$.

PROOF. We divide the proof in three steps.

STEP 1. Using Ekeland's variational principle. Let us fix $B_R(x_0) \subset\subset \Omega$ and consider the complete metric space (X, d) where X is the subset of the Dirichlet class $u + W_0^{1,1}(B_R(x_0))$ defined by the condition

$$w \in X \iff w \in u + W_0^{1,p}(B_R(x_0))$$

and

$$\int_{B_R} \left(\mid Dw \mid^p + b(x) \mid w \mid^\gamma\right) dx \le \int_{B_R} \left(\mid Du \mid^p + b(x) \mid u \mid^\gamma\right) dx$$

with

$$d(u_1, u_2) = C_R \int_{B_R} |Du_1 - Du_2|\, dx$$

$$C_R = \frac{1}{R^n \omega(R)} \left[\fint_{B_R} (|Du|^p + b(x)\,|u|^\gamma + a(x))\, dx \right]^{-\frac{1}{p}}.$$

Let us recall that the Dirichlet class $u + W_0^{1,1}(B_R(x_0))$ is a complete metric space when equipped with the metric d. We remark now, that X , endowed with the metric d, is a complete metric subspace of the Dirichlet class $u + W_0^{1,1}(B_R(x_0))$. Indeed let $w_n \subset X$ such that:

$$\int_{B_R} \mid Dw_n - Dw \mid \; dx \to 0$$

with $w \in u + W_0^{1,1}(B_R(x_0))$ then, eventually passing to a (not relabelled) subsequence, we may assume that:

$$w_n - w \to 0, \quad Dw_n - Dw \to 0 \; a.e. \; in \; B_R(x_0)$$

Applying Fotou's lemma we get:

$$\int_{B_R} (\mid Dw \mid^p + b(x) \mid w \mid^\gamma)\; dx \leq \liminf_n \int_{B_R} (\mid Dw_n \mid^p + b(x) \mid w_n \mid^\gamma)\; dx \leq$$
$$\leq \int_{B_R} (\mid Du \mid^p + b(x) \mid u \mid^\gamma)\; dx$$

so that $w \in X$. In this way (X, d) is a closed metric subspace of the complete metric space $(u + W_0^{1,1}(B_R(x_0)), d)$, and hence it is complete.

Let $\delta > 0$ and $v_\delta \in X$ such that

$$F(v_\delta, B_R) \leq \inf_X \; F + \delta.$$

Recalling that u is an ω-minimizer we have:

$$F(u, B_R) \leq [1 + \omega(R)]\, F(v_\delta, B_R) \leq$$
$$\leq \inf_X \; F + \delta + \omega(R)\, [F(u, B_R) + \delta]\,.$$

Letting δ go to zero we get

$$\begin{aligned} F(u, B_R) &\leq \inf_X \ F + \omega(R) F(u, B_R) \leq \\ &\leq \inf_X \ F + c\omega(R) \int_{B_R} (|Du|^p + b(x)\,|u|^\gamma + a(x))\,dx \leq \\ &\leq \inf_X \ F + cR^n \omega(R) \fint_{B_R} (|Du|^p + b(x)\,|u|^\gamma + a(x))\,dx. \end{aligned}$$

The functional $u \to F(u, B_R)$ turns out to be lower semicontinuous on (X, d) so we can use Ekeland's variational principle to find a function $v \in X$ (depending on R) such that:

$$\int_{B_R} |Du - Dv|\,dx \leq C_R^{-1}$$

$$\int_{B_R} (|\, Dv\,|^p + b(x)\,|\,v\,|^\gamma)\ dx \leq \int_{B_R} (|\, Du\,|^p + b(x)\,|\,u\,|^\gamma)\ dx \tag{21}$$

$$F(v, B_R) \leq F(w, B_R) + M \int_{B_R} |Dw - Dv|\,dx$$

for any $w \in X$ where

$$M = cR^n \omega(R) C_R \fint_{B_R} (|Du|^p + b(x)\,|u|^\gamma + a(x))dx =$$

$$= c \left[\fint_{B_R} (|Du|^p + b(x)\,|u|^\gamma + a(x))dx \right]^{1-\frac{1}{p}}.$$

In the proof of the theorem it will be clear that in order to apply Theorems 1, 2, 4 and Lemma 2, we will need that the quantity:

$$S(R) = \|\, Dv\,\|_{L^p(B_R)} + \|\, v\,\|_{L^{p^*}(B_R)} + \|\, b\,\|_{L^\sigma(B_R)},$$

(v is the comparison function given by Ekeland's theorem) remains bounded for R sufficiently small.

So in view of applying these results to v we check that

$$\sup_R \ S(R) < \infty\,. \tag{22}$$

In order to prove (22) we note that by $(21)_2$ it trivially follows:

$$\int_{B_R} \mid Dv \mid^p \; dx \leq \int_{B_R} (\mid Du \mid^p + b(x) \mid u \mid^\gamma) \, dx \tag{23}$$

while applying Sobolev-Poincaré inequalty and (23) we have:

$$\begin{aligned}\int_{B_R} \mid v \mid^{p^*} \; dx &\leq c \int_{B_R} \mid u - v \mid^{p^*} \; dx + \int_{B_R} \mid u \mid^{p^*} \; dx \leq \\ &\leq \left(\int_{B_R} (\mid Du \mid^p + \mid Dv \mid^p) \, dx \right)^{\frac{p^*}{p}} + \int_{B_R} \mid u \mid^{p^*} \; dx < \infty\end{aligned}$$

Now we prove the crucial fact that the minimizing property of the function v implies the fact that v is a Q-minimizer of a comparison functional determined by the growth conditions.

Let $w \in u + W_0^{1,p}(B_R)$ and $\varphi = w - u$. If $w \in X$ then, using growth conditions, Young and Sobolev-Poincaré inequalities with $(21)_3$ we get:

$$\begin{aligned}&\int_{\operatorname{supp}\varphi} |Dv|^p \, dx \leq \\ &\leq c \int_{\operatorname{supp}\varphi} (|Dv + D\varphi|^p + b(x) \, |v + \varphi|^\gamma + |v|^p + a(x)) \, dx + \\ &+ cM \int_{\operatorname{supp}\varphi} |D\varphi| \, dx \leq \\ &\leq c \int_{\operatorname{supp}\varphi} (|Dv + D\varphi|^p + b(x) \, |v + \varphi|^\gamma + |v|^\gamma + a(x)) \, dx + \\ &+ \int_{\operatorname{supp}\varphi} \left(C_\sigma M^{\frac{p}{p-1}} + \sigma \, |D\varphi|^p \right) dx \leq \\ &\leq c \int_{\operatorname{supp}\varphi} \left(|Dv + D\varphi|^p + b(x) \, |v + \varphi|^\gamma + b(x) \, |v|^\gamma + a(x) + M^{\frac{p}{p-1}} \right) dx + \\ &+ \sigma \int_{\operatorname{supp}\varphi} |Dv|^p \, dx.\end{aligned}$$

So that, if we choose $0 < \sigma < 1$, we can shift the last term to the first member, thus obtaining:

$$\begin{aligned}&\int_{\operatorname{supp}\varphi} |Dv|^p \, dx \leq \\ &\leq c \int_{\operatorname{supp}\varphi} \left(|Dv + D\varphi|^p + b(x) \, |v + \varphi|^\gamma + b(x) \, |v|^\gamma + a(x) + M^{\frac{p}{p-1}} \right) dx.\end{aligned}$$

On the other hand if $w \notin X$ then, by definition of X it follows:

$$\int_{B_R} (\mid Du \mid^p + b(x) \mid u \mid^\gamma)\ dx \le \int_{B_R} (\mid Dw \mid^p + b(x) \mid w \mid^\gamma)\ dx$$

and, a fortiori:

$$\int_{B_R} (\mid Dv \mid^p + b(x) \mid v \mid^\gamma)\ dx \le \int_{B_R} (\mid Dw \mid^p + b(x) \mid w \mid^\gamma)\ dx$$

hence, summing up the missing terms and passing to the support of φ we have:

$$\begin{aligned} &\int_{\operatorname{supp}\varphi} (\mid Dv \mid^p + b(x) \mid v \mid^\gamma + a(x) + M^{\frac{p}{p-1}})\ dx \le \\ &\le \int_{\operatorname{supp}\varphi} (\mid Dw \mid^p + b(x) \mid w \mid^\gamma + a(x) + M^{\frac{p}{p-1}})\ dx \end{aligned}$$

So that, in any case we have:

$$\begin{aligned} &\int_{\operatorname{supp}\varphi} \mid Dv \mid^p\ dx \le \\ &\le \int_{\operatorname{supp}\varphi} \Big(\mid Dv + D\varphi \mid^p + b(x) \mid v + \varphi \mid^\gamma + b(x) \mid v \mid^\gamma + a(x) + M^{\frac{p}{p-1}}\Big)\ dx \end{aligned}$$

for every $\varphi \in W_0^{1,p}(B_R)$. Now, by Lemma 1, it follows that v is a Q-minimizer of functional (11) with $Q \equiv Q(n, \gamma, p, \|v\|_{p^*}, \|Dv\|_p, \|\ b\ \|_\sigma\ )$, and by (22) the constants Q are uniformly bounded.

Step 2. Comparing the Q-minimizers with the ω-minimizer.

We will derive some estimates that will allow us to compare, in the last step, the ω-minimizer u and the function v, over the ball $B_R(x_0)$. We stress that these estimates are based on the higher integrability of the gradients of u and v.

From Definitions 1, 2, 3 it follows easily that both Q-minimizers and ω-minimizers are spherical Q-minimizers so that higher integrability result (Theorem 1) applies to both v and u and for $R < R_0$ we have, writing $q = pr$:

$$(24)\quad \left(\fint_{B_{\frac{R}{2}}} |Dv|^q \; dx\right)^{\frac{1}{q}} \le c\left(\fint_{B_R} |Dv|^p \; dx\right)^{\frac{1}{p}} + c\left(\fint_{B_R} (b(x)\,|v|^\gamma)^{\frac{q}{p}}\, dx\right)^{\frac{1}{q}} + + c\left(\fint_{B_R} a^{\frac{q}{p}} dx\right)^{\frac{1}{q}} + cM^{\frac{1}{p-1}}.$$

$$(25)\quad \left(\fint_{B_{\frac{R}{2}}} |Du|^q \; dx\right)^{\frac{1}{q}} \le c\left(\fint_{B_R} |Du|^p \; dx\right)^{\frac{1}{p}} + + c\left(\fint_{B_R} (b(x)\,|u|^\gamma)^{\frac{q}{p}}\, dx\right)^{\frac{1}{q}} + c\left(\fint_{B_R} a^{\frac{q}{p}} dx\right)^{\frac{1}{q}}.$$

We can now interpolate between 1 and q, with:

$$0 < \theta < 1\ , \quad \frac{\theta}{q} + 1 - \theta = \frac{1}{p}$$

obtaining , by (21_1) and (24), (25):

$$(26)\quad \begin{aligned} &\left(\fint_{B_{\frac{R}{2}}} |Du - Dv|^p\, dx\right)^{\frac{1}{p}} \le \left(\fint_{B_{\frac{R}{2}}} |Du - Dv|^q\, dx\right)^{\frac{\theta}{q}} \left(\fint_{B_{\frac{R}{2}}} |Du - Dv|\, dx\right)^{1-\theta} \le \\ &\le \Bigg[\left(\fint_{B_R} |Du|^p\, dx\right)^{\frac{1}{p}} + \left(\fint_{B_R} |\; Dv\; |^p \; dx\right)^{\frac{1}{p}} + \left(\fint_{B_R} (b(x) \mid u \mid^\gamma)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} + \\ &+ \left(\fint_{B_R} (b(x) \mid v \mid^\gamma)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} + \left(\fint_{B_R} a^{\frac{q}{p}} dx\right)^{\frac{1}{q}} + M^{\frac{1}{p-1}}\Bigg]^{\theta} \times \\ &\times \left[\omega(R)\left(\fint_{B_R} (|Du|^p + b(x)\,|u|^\gamma + a(x)) \; dx\right)^{\frac{1}{p}}\right]^{1-\theta}. \end{aligned}$$

Now we observe that by Hölder inequality we have:

$$M^{\frac{1}{p-1}} \leq c\left[\left(\fint_{B_R} \mid Du \mid^p + b(x) \mid u \mid^\gamma dx\right)^{\frac{1}{p}} + \left(\fint_{B_R} a(x)^{\frac{q}{p}} dx\right)^{\frac{1}{q}}\right]$$

therefore, raising both members of (26) to the power p, getting rid of the averages, we finally obtain

$$\begin{aligned}
&\int_{B_{\frac{R}{2}}} |Du - Dv|^p\, dx \leq c\,(\omega(R))^d \left[\int_{B_R} |Du|^p\, dx + \int_{B_R} b(x)\,|u|^\gamma\, dx\right] + \\
(27)\quad &+ (c\omega(R))^d \left[\left(\int_{B_R} (b(x) \mid u \mid^\gamma)^{\frac{q}{p}} dx\right)^{\frac{p}{q}} + \left(\int_{B_R} (b(x) \mid v \mid^\gamma)^{\frac{q}{p}} dx\right)^{\frac{p}{q}}\right] + \\
&+ cR^{n-p+n\epsilon}
\end{aligned}$$

for some $d > 0$. Now we put

$$A = \int_{B_R} b(x) \mid u \mid^\gamma dx, \qquad B = \left(\int_{B_R} (b(x) \mid u \mid^\gamma)^{\frac{q}{p}} dx\right)^{\frac{p}{q}},$$

$$C = \left(\int_{B_R} (b(x) \mid v \mid^\gamma)^{\frac{q}{p}} dx\right)^{\frac{p}{q}}$$

and estimate these three quantities. We have:

$$A \leq c \int_{B_R} b(x) \mid u - u_R \mid^\gamma dx + c \int_{B_R} b(x) \mid u_R \mid^\gamma dx$$

and so using Hölder, Sobolev and Poincaré inequalities we have

$$\begin{aligned}
&\int_{B_R} b(x) \mid u - u_R \mid^\gamma \; dx \leq \\
&\leq \left(\int_{B_R} (b(x) \mid u - u_R \mid^{\gamma - p})^{\frac{n}{p}} dx\right)^{\frac{p}{n}} \left(\int_{B_R} \mid u - u_R \mid^{p^*} dx\right)^{\frac{p}{p^*}} \leq \\
&\leq c \int_{B_R} \mid Du \mid^p dx,
\end{aligned}$$

while by Hölder inequality we obtain:

$$\int_{B_R} b(x) \mid u_R \mid^\gamma dx = \parallel b \parallel_1 \mid u_R \mid^\gamma \leq cR^{-\mu} \int_{B_R} \mid u \mid^\gamma dx.$$

Collecting the last two inequalities we have that

$$A \leq c \int_{B_R} (\mid Du \mid^p + R^{-\mu} \mid u \mid^\gamma) dx.$$

Now we estimate in the same way of A:

$$B \leq \left(\int_{B_R} (b(x) \mid u - u_R \mid^\gamma)^r dx \right)^{\frac{1}{r}} + \left(\int_{B_R} (b(x) \mid u_R \mid^\gamma)^r dx \right)^{\frac{1}{r}}.$$

As before we have

$$\int_{B_R} (b(x) \mid u - u_R \mid^\gamma)^r dx \leq \zeta(R) \int_{B_R} \mid Du \mid^p dx,$$

where

$$\zeta(R) \leq \left(\int_{B_R} \mid u - u_R \mid^{p^*} dx \right)^{\frac{\gamma r - p}{p^*}} \left(\int_{B_R} b^{\frac{p^*}{p^* - \gamma r}} dx \right)^{1 - \frac{\gamma r}{p^*}} \leq$$
$$\leq c \left(\int_{B_R} \mid Du \mid^p dx \right)^{\frac{\gamma r - p}{p}}.$$

So we have

$$\left(\int_{B_R} (b(x) \mid u - u_R \mid^\gamma)^r dx \right)^{\frac{1}{r}} \leq c \left(\int_{B_R} \mid Du \mid^p dx \right)^{\frac{\gamma}{p} - 1} \int_{B_R} \mid Du \mid^p dx \leq$$
$$\leq c \int_{B_R} \mid Du \mid^p dx.$$

Finally

$$\left(\int_{B_R} (b(x) \mid u_R \mid^\gamma)^r dx \right)^{\frac{1}{r}} \leq c \parallel b \parallel_r \mid u_R \mid^\gamma \leq cR^{-\mu} \int_{B_R} \mid u \mid^\gamma dx$$

and also in this case,

$$B \leq c \int_{B_R} (\mid Du \mid^p + R^{-\mu} \mid u \mid^\gamma) dx.$$

We now estimate the last term

$$C \leq \left(\int_{B_R} (b(x) \mid u - v \mid^\gamma)^r \; dx \right)^{\frac{1}{r}} + B.$$

As for B we have, using also the estimates for A and B:

$$\begin{aligned} \left(\int_{B_R} (b(x) \mid u - v \mid^\gamma)^r dx \right)^{\frac{1}{r}} &\leq c \parallel u - v \parallel_{p^*}^{\frac{\gamma r - p}{r}} \left(\int_{B_R} \mid Du - Dv \mid^p \, dx \right)^{\frac{1}{r}} \leq \\ &\leq c \int_{B_R} \mid Du - Dv \mid^p \, dx \leq \\ &\leq c \int_{B_R} (\mid Du \mid^p + b(x) \mid u \mid^\gamma) \; dx \leq \\ &\leq \int_{B_R} (\mid Du \mid^p + R^{-\mu} \mid u \mid^\gamma) \; dx \, . \end{aligned}$$

So we finally have

$$A + B + C \leq c \int_{B_R} (\mid Du \mid^p + R^{-\mu} \mid u \mid^\gamma) \; dx \, .$$

If we put this estimate in (27) we have

$$\begin{aligned} \int_{B_{\frac{R}{2}}} \mid Du - Dv \mid^p \, dx \leq c \, (\omega(R))^d \int_{B_R} (\mid Du \mid^p + R^{-\mu} \mid u \mid^\gamma) \; dx + \\ + cR^{n-p+n\epsilon}. \end{aligned} \tag{28}$$

Step 3. Getting the decay estimate for the ω-minimizer

Finally, by means of the comparison estimates of the previous step, we prove that the term:

$$\int_{B_\rho} \left(|Du|^p + \rho^{-\mu} \, |u|^\gamma \right) dx$$

inherits the nice decay estimate of the term:

$$\int_{B_\rho} \left(|Dv|^p + \rho^{-\mu}\,|v|^\gamma\right) dx$$

proved in Theorem 4. So, using also the estimate of the term A in the previous step and Hölder inequality we have:

$$\text{(29)}\quad M^{\frac{p}{p-1}} R^n + M^{\frac{\gamma}{p-1}} R^{\frac{\gamma n}{p}} \le c \int_{B_R} \left(|\,Du\,|^p + R^{-\mu}\,|u|^\gamma\right)\ dx + cR^{n-p+n\epsilon}.$$

By (23) and again the estimate of the term A we have:

$$\text{(30)}\qquad \left(\int_{B_R} |\,Du - Dv\,|^p\ dx\right)^{\frac{\gamma}{p}} \le c \int_{B_R} (|\,Du\,|^p + R^{-\mu}\,|\,u\,|^\gamma) dx\,.$$

Now from (29), (30) and Theorem 4 we get that for $0 < \rho < \frac{R}{2}$, $R < R_0$:

$$\begin{aligned}
&\int_{B_\rho} \left(|Du|^p + \rho^{-\mu}\,|u|^\gamma\right) dx \le c \int_{B_\rho} \left(|Dv|^p + \rho^{-\mu}\,|v|^\gamma\right) dx + \\
&\quad + c \int_{B_\rho} |Dv - Du|^p\, dx \ + \ \frac{c}{\rho^\mu} \int_{B_\rho} |u - v|^\gamma\, dx \le \\
&\le c \left(\frac{\rho}{R}\right)^{n-p+\alpha p} \left[\int_{B_{\frac{R}{2}}} (|Dv|^p + R^{-\mu}\,|v|^\gamma) dx \ + \ M^{\frac{p}{p-1}} R^n + M^{\frac{\gamma}{p-1}} R^{\frac{\gamma n}{p}}\right] + \\
&\quad + c \int_{B_{\frac{R}{2}}} |Dv - Du|^p\, dx \ + \frac{c}{\rho^\mu} \int_{B_{\frac{R}{2}}} |u - v|^\gamma\, dx \ + \ cR^{n-p+n\epsilon} \le \\
&\le c \left(\frac{\rho}{R}\right)^{n-p+\alpha p} \left[\int_{B_R} \left(|Du|^p + R^{-\mu}\,|u|^\gamma\right) dx + \right. \\
&\quad \left. + \int_{B_{\frac{R}{2}}} \left(|\,Du - Dv\,|^p + R^{-\mu}\,|\,u - v\,|^\gamma\right) dx\right] + \\
&\quad + c\,(\omega(R))^d \int_{B_R} \left(|Du|^p + R^{-\mu}\,|\,u\,|^\gamma\right)\ dx +
\end{aligned}$$

$$+ c\rho^{n\epsilon'}\left(\frac{R}{\rho}\right)^{\lambda}\left(\int_{B_R} | Du - Dv |^p \ dx\right)^{\frac{\gamma}{p}} + cR^{n-p+n\epsilon} \leq$$
$$\leq c\left[\left(\frac{\rho}{R}\right)^{n-p+p\alpha} + (\omega(R))^d + \left(\frac{R}{\rho}\right)^{\lambda} R^{n\epsilon'}\right]\int_{B_R} (| Du |^p + R^{-\mu} | u |^{\gamma})\, dx +$$
$$+ cR^{n-p+p\alpha}$$

thus proving Proposition 1. □

REMARK 3. The decay estimate found in the last proposition is an essential tool in order to prove our regularity result (see proof of Theorem 6, below) and is a more complicated version of an analogous decay estimate found in [3]. In our case a number of technical complications , due to the general growth assumptions (4), had to be faced. The most worth mentioning are the following. In the proof of boundedness of the quantity $S(R)$ a space of "bounded energy" functions X has been introduced to overcome the lackness of coercivity of the functional F so we are able to get a uniform estimate for the norms of the gradients.

Another technical point is the choice of the right quantity to make decay. In [3], with the particular growth assumptions reported in Remark 2, the "right" quantity was:

$$\phi(\rho) = \int_{B_\rho} (| Du |^p + | u |^p) \ dx$$

while in the case

$$|\xi|^p - |u|^{\gamma} - a(x) \leq f(x, u, \xi) \leq L(|\xi|^p + |u|^{\gamma} + a(x)),$$

some computation suggest to use:

$$\phi(\rho) = \int_{B_\rho} (| Du |^p + | u |^{\gamma}) \ dx\,.$$

In our general case, we multiply the u term in the definition of ϕ by a suitable (negative) power of the radius R chosen in order to keep into account the summabilty properties of $b(x)$.

Now we can prove the local Hölder continuity of the ω-minimizer u. The proof of the theorem is actually based on a more or less standard iteration argument starting from the decay estimate provided by the previous proposition.

Proof of the Theorem 6. In the Proposition 1 choose $0 < \tau < 1$ and $0 < \alpha' < \alpha'' < \alpha$ such that

$$c\tau^{p\left(\alpha-\alpha''\right)} \;<\; \frac{1}{2}$$

and then take $R < R_0$ small enough in such a way that:

$$c\left[\left(\omega(R)\right)^{d} + R^{n\epsilon'}\tau^{-\lambda}\right] < \frac{\tau^{n-p+p\alpha''}}{2}.$$

If we write estimate (19) for $\rho = \tau R$ we obtain:

$$\phi\left(\tau R\right) \leq \tau^{n-p+p\alpha''}\phi(R) + cR^{n-p+p\alpha'}\,.$$

Finally we observe that if $\tau^{k+1}R < h < \tau^{k}R$ we have:

$$\phi(h) \leq \frac{1}{\tau}\phi(\tau^{k}R).$$

So it is possible to apply Lemma 3 to our ϕ in order to get:

$$\phi(\rho) \leq c\left(\frac{\rho}{R}\right)^{n-p+p\alpha'}\phi(R) + c\rho^{n-p+p\alpha'}$$

recalling the definition of ϕ, we get:

$$\int_{B_\rho} \mid Du\mid^{p}\;dx \leq \left(\frac{\rho}{R}\right)^{n-p+p\alpha}\phi(R) + c\rho^{n-p+p\alpha} \leq c(R)\rho^{n-p+p\alpha}$$

and by Poincaré inequality we have:

$$\fint_{B_\rho} \mid u - u_\rho \mid^{p} dx \leq c(R)\rho^{p\alpha'}.$$

So, applying Campanato's characterization of Hölder continuity and a standard covering argument, we get the result. $\square$

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