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Some properties for eigenvalues and eigenfunctions of nonlinear weighted problems

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RIASSUNTO: *Si studiano alcune proprietà del primo autovalore e delle corrispondenti autofunzioni per una classe di problemi non lineari pesati. Utilizzando tecniche di simmetrizzazione si ottiene una disuguaglianza di Faber-Krahn per il primo autovalore e una disuguaglianza di Payne-Rayner per le corrispondenti autofunzioni.*

ABSTRACT: *We study some properties of the first eigenvalue and of the corresponding eigenfunctions for a class of non linear weighted problems. Using symmetrization techniques, we give a Faber-Krahn inequality for the first eigenvalue and a Payne-Rayner inequality for the corresponding eigenfunctions.*

1 – Introduction

In this paper, we study the properties of the first eigenvalue of the Dirichlet problem:

$$(1.1) \qquad \begin{cases} -\operatorname{div}(|Du|^{p-2}Du) = \lambda m(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open, bounded and connected subset of $\mathbb{R}^n$, $1 < p < n$ and m is a positive function in $L^r(\Omega)$, $r > n/p$.

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It is well known, by general theory on non linear eigenvalue problems (see, for example, [14], [5]), that the problem (1.1) has, at least, a divergent sequence of positive eigenvalues. These eigenvalues can be evaluated by Lyusternik-Schnirelman method, as critical values of the functional

$$\mathcal{R}(u) = \frac{\| Du \|_p^p}{\displaystyle\int_\Omega m(x)|u|^p(x)dx}$$

on a family of subsets of $W_0^{1,p}(\Omega)$ with special topological properties. Moreover, the first eigenvalue λ_p is the minimum of $\mathcal{R}$ on $W_0^{1,p}(\Omega)$. In particular, if $m \equiv 1$, λ_p is the best constant C in the Sobolev-Poincarè inequality

$$\| Du \|_p^p \geq C \| u \|_p^p \, .$$

In the case $p = 2$ and $m \equiv 1$, it is well known that λ_p is positive and simple, that is all the corresponding eigenfunctions are multiple of each other. Moreover, various inequalities have been given for λ_p and the corresponding eigenfunctions. For example, in [17] (see also [21], [22], [1]) it has been shown that, if u is an eigenfunction corresponding to λ_p, then, for any $0 < q < r \leq +\infty$, there exists a constant $K = K(r, q, n, \lambda_p)$ such that

$$\| u \|_r \leq K \| u \|_q \tag{1.2}$$

and the equality holds if, and only if, Ω is a ball.

More in general, we consider the problem

$$\begin{cases} -\operatorname{div}((ADu, Du)^{(p-2)/2} ADu) = \lambda m(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is an open, bounded and connected subset of $\mathbb{R}^n$, $1 < p < n$, $A = \{a_{ij}\}_{ij}$ is a matrix such that

$$\begin{cases} a_{ij} \in L^\infty(\Omega),\ a_{ij}(x) = a_{ji}(x) & \text{for a.e. } x \in \Omega,\ \forall i, j = 1, \cdots, n \\ a_{ij}(x)\xi_i\xi_j \geq |\xi|^2 & \forall \xi \in \mathbb{R}^n,\ \text{for a.e. } x \in \Omega, \end{cases} \tag{1.4}$$

and

$$m > 0, \quad m \in L^r(\Omega), \quad r > n/p. \tag{1.5}$$

The aim of this paper is to prove that some of the properties, described above, of the first eigenvalue for the problem (1.1) in the case $m = 1$, still hold for the first eigenvalue of (1.3) under the assumptions (1.4) and (1.5). The first eigenvalue of the problem (1.3) is positive and simple; this result is essentially contained in [7]. Similar results for problems of the type (1.1) have already been obtained by many authors also in the case m changes its sign (see, for example, [29], [24], [6], [7], [19], [20], [18], [1]). Secondly, by using symmetrization techniques, we prove a Faber-Krahn inequality for the first eigenvalue λ_p of (1.3) (see [11] for the case $p = 2$, $m \equiv 1$ and [1] for the nonlinear case and $m \equiv 1$). Such a result can be summarized as follows. Let us consider the class of problems of the type (1.3) where Ω is an open, bounded and connected subset of $\mathbb{R}^n$, with prescribed measure, the matrix $A = \{a_{ij}\}_{ij}$ satisfies (1.4) and m is a function satisfying (1.5), with prescribed rearrangement. Then the first eigenvalue of any problem in the class described above attains the lowest value $\lambda_p^\#$ when Ω is a ball, $a_{ij}x_j = x_i$ and $m = m^\#$ a.e. in Ω modulo translations.

In the last section, using a comparison result and properties of rearrangements, we obtain a Payne-Rayner inequality for the eigenfunctions corresponding to λ_p, that is an inequality of the type (1.2). More precisely, we compare u by a suitable eigenfunction v of the problem (1.1) with Ω replaced by a ball B, centered at the origin, such that the corresponding first eigenvalue is equal to λ_p, and m replaced by its spherically symmetric decreasing rearrangement $m^\#$, whose definition is given in Section 2. We remark that all the inequalities we will prove are *isoperimetric*, in the sense that they are equalities only in the spherical situation, that is, if and only if Ω is a ball, $a_{ij}x_j = x_i$ and $m = m^\#$ a.e. in Ω modulo translations.

2 – Notations and preliminary results

Let E be a bounded and measurable subset of $\mathbb{R}^n$, $n \geq 1$, let $a \in L^1(E)$ be a non negative function and let $1 \leq p \leq +\infty$. We put

$$L^p(E,a) = \left\{ u : E \to \mathbb{R} \ \textit{measurable} : \|u\|^p_{L^p(E,a)} = \int_E a(x)|u(x)|^p dx < +\infty \right\},$$

if $1 \leq p < +\infty$, and

$$L^\infty(E,a) \equiv L^\infty(E). \tag{2.1}$$

We denote by $W_0^{1,p}(E,a)$ the completion of the space $C_0^\infty(E)$ with respect to the norm

$$\|u\|_{W^{1,p}(E,a)} = \|u\|_{L^p(E,a)} + \|Du\|_{L^p(E,a)}.$$

Now, let Ω be an open, bounded subset of $\mathbb{R}^n$, $n \geq 2$. If $|\Omega|$ is the n-dimensional Lebesgue measure of Ω, let us denote by Ω^* the open interval $(0,|\Omega|)$ and by $\Omega^\#$ the $n-$dimensional ball, centered at the origin and with the same measure of Ω. If $f : \Omega \to \mathbb{R}$ is a measurable function, *the decreasing rearrangement of* f is defined by

$$f^*(s) = \sup\{t > 0 : \mu_f(t) > s\}, \qquad s \in \Omega^*,$$

where $\mu_f(t) = |\{x \in \Omega : |f(x)| > t\}|$ is the distribution function of f and *the spherically symmetric decreasing rearrangement of* f is defined by

$$f^\#(x) = f^*(C_n|x|^n), \qquad x \in \Omega^\#,$$

where C_n denotes the measure of the unit ball of $\mathbb{R}^n$. Among all the properties of rearrangements , we recall the well known *Hardy-Littlewood inequality*, that is

$$\int_\Omega |f(x)g(x)|dx \leq \int_{\Omega^\#} f^\#(x)g^\#(x)dx. \tag{2.2}$$

Moreover the following result holds (see, for example, [12], [2]):

PROPOSITION 2.1. *Let $f \in L^1_+(\Omega)$, $g \in L^1_+(\Omega)$ and suppose that μ_f is a continuous function on $[0,\sup f[$. If*

$$\int_\Omega f(x)g(x)dx = \int_{\Omega^\#} f^\#(x)g^\#(x)dx$$

then for all $\tau \geq 0$ there exists $t \geq 0$ such that

$$\{x \in \Omega : |g(x)| > \tau\} = \{x \in \Omega : |f(x)| > t\} \tag{2.3}$$

up to zero measure set.

Let $f, g \in L^q(0,a)$, $1 \le q \le +\infty$. We say that g is *dominated by f* if and only if

$$\begin{cases} \displaystyle\int_0^s g^*(t)dt \le \int_0^s f^*(t)dt \qquad \forall s \in (0,a) \\ \displaystyle\int_0^a g^*(s)ds = \int_0^a f^*(s)ds. \end{cases}$$

and we write

$$f \prec g.$$

The following result is well known (see, for example, [12], [3]):

Proposition 2.2. *Let $f, g \in L^q_+(0,a)$, $q \ge 1$ and let q' be the conjugate exponent of q. If $\psi \in L^{q'}_+(0,a)$ and $g \prec f$, then*

$$\int_0^a \psi^*(t)g^*(t)\,dt \le \int_0^a \psi^*(t)f^*(t)\,dt. \tag{2.4}$$

Furthermore an immediate consequence of Proposition 2.2 is the following:

Proposition 2.3. *Let f, g, h be positive and decreasing functions on $(0,a)$ and let hf, $hg \in L^1_+(0,a)$. Let F be a convex, nonnegative, Lipschitz function such that $F(0) = 0$. If $hg \prec hf$ then*

$$\int_0^a h^*(t)F(g^*(t))\,dt \le \int_0^a h^*(t)F(f^*(t))\,dt. \tag{2.5}$$

Moreover, if F is strictly convex, we have equality in (2.5) if and only if $f^* \equiv g^*$ a.e. in $(0,a)$.

Another useful property of rearrangements is (see [23], [10]):

Theorem 2.1. *Let Ω be an open, bounded subset of $\mathbb{R}^n$, $n \ge 2$. Let u be a positive function in $W^{1,p}_0(\Omega)$, $1 < p < +\infty$. Then*

$$\int_\Omega |Du|^p(x)dx \ge \int_{\Omega^\#} |Du^\#|^p(x)dx. \tag{2.6}$$

If $|\{x \in \Omega : |Du|(x) = 0\} \bigcap {u^\#}^{-1}(0, \operatorname{ess\,sup} u)| = 0$, and equality holds in (2.6), then $\Omega = \Omega^\#$, $u = u^\#$ a.e. in Ω, modulo translations.

The inequality (2.6) is known as the *Pólya-Szegö inequality.* For an exhaustive treatment of rearrangements, see, for example, [12], [26], [8], [15], [28].

Let us now consider the nonlinear eigenvalue problem (1.3) under the assumptions (1.4) and (1.5). Let $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$. Then u is an *eigenfuction* of the problem (1.3) if

$$\int_\Omega (ADu, Du)^{\frac{(p-2)}{2}}(ADu, D\varphi)dx = \lambda \int_\Omega mu|u|^{p-2}\varphi \, dx, \tag{2.7}$$

for all $\varphi \in W_0^{1,p}(\Omega)$ and the corresponding real number λ is called an *eigenvalue* of the problem (1.3).

It is well known, by general theory of nonlinear eigenvalue problems (see for example [14], [5], [18]) that there exists a sequence $\{\lambda_p^k\}_{k\in\mathbb{N}}$ of eigenvalues such that $\lim_k \lambda_p^k = +\infty$. Moreover the first eigenvalue λ_p is the minimum of the Rayleigh quotient

$$\lambda_p = \min_{\substack{u\in W_0^{1,p}(\Omega)\\ u\not\equiv 0}} \frac{\displaystyle\int_\Omega (ADu, Du)^{\frac{p}{2}}dx}{\displaystyle\int_\Omega m|u|^p dx}, \tag{2.8}$$

and all the functions that realize the minimum of this quotient are eigenfunctions of (1.3).

In the case $p = 2$ and $m \in L^\infty(\Omega)$, it is well known (see, for example, [13], [7]) that the first eigenvalue λ_p is positive and simple. Moreover all the eigenfunctions associated to λ_p do not change sign. Similar results have been proven in [1] in the case $p \neq 2$ and $m \equiv 1$.

More in general, the following result, which is essentially contained in [7] holds (see also [18], [1]):

THEOREM 2.2. *Let Ω be an open, bounded and connected subset of $\mathbb{R}^n$, $1 < p < n$. Let λ_p be the first eigenvalue of the problem* (1.3) *under the assumptions* (1.4) *and* (1.5). *Then λ_p is simple, that is if u and v are eigenfunctions associated to λ_p, then there exists $\alpha \in \mathbb{R}$ such that $u = \alpha v$.*

3 – Faber-Krahn inequality

It is well known that the membrane with lowest principal frequency is the circular one. This is the so called Faber-Krahn inequality. Using Pólya-Szegö inequality and rearrangements properties, it is possible to prove a generalization of the Faber-Krahn inequality. The following result holds:

THEOREM 3.1. *Let Ω be an open, bounded and connected subset of* $\mathbb{R}^n$, $1 < p < n$. *Let λ_p be the first eigenvalue of* (1.3) *under the assumptions* (1.4) *and* (1.5). *If $\lambda_p^\#$ is the first eigenvalue of the problem*

$$\left\{ \begin{array}{ll} -\mathrm{div}\,(|Dv|^{p-2}Dv) = \lambda m^\#(x) v|v|^{p-2} & \text{in } \Omega^\# \\ v = 0 & \text{on } \partial\Omega^\#, \end{array} \right. \tag{3.1}$$

then

$$\lambda_p \geq \lambda_p^\# \tag{3.2}$$

and equality holds if and only if $\Omega = \Omega^\#$, $a_{ij}(x)x_j = x_i$ and $m(x) = m^\#(x)$ a.e. in Ω, modulo translations.

PROOF. The proof of (3.2) is similar the one given in [29] in the case $a_{ij}(x) = \delta_{ij}$. We reproduce it for completeness. By definition, we have that

$$\lambda_p^\# = \min_{\substack{v \in W_0^{1,p}(\Omega^\#) \\ v \neq 0}} \frac{\displaystyle\int_{\Omega^\#} |Dv|^p(x)dx}{\displaystyle\int_{\Omega^\#} m^\#(x)|v|^p(x)dx}.$$

If $u > 0$ is an eigenfunction associated to the first eigenvalue of (1.3), using ellipticity condition, Pòlya-Szegö principle and Hardy-Littlewood inequality, we have:

$$\lambda_p = \frac{\displaystyle\int_\Omega (ADu(x), Du(x))^{p/2}dx}{\displaystyle\int_\Omega m(x)|u|^p(x)dx} \geq \frac{\displaystyle\int_\Omega |Du|^p(x)dx}{\displaystyle\int_\Omega m(x)|u|^p(x)dx} \geq \tag{3.3}$$

$$\geq \frac{\displaystyle\int_{\Omega^\#} |Du^\#|^p(x)dx}{\displaystyle\int_{\Omega^\#} m^\#(x)|u^\#|^p(x)dx} \geq \lambda_p^\#.$$

This completes the proof of the first part of the theorem.

About the equality, let us suppose that $\lambda_p = \lambda_p^{\#}$. So, by (3.3), Hardy-Littlewood inequality and Pólya-Szegö inequality, we have:

$$1 \leq \frac{\displaystyle\int_{\Omega} |Du|^p(x)dx}{\displaystyle\int_{\Omega^{\#}} |Du^{\#}|^p(x)dx} = \frac{\displaystyle\int_{\Omega} m(x)|u|^p(x)dx}{\displaystyle\int_{\Omega^{\#}} m^{\#}(x)|u^{\#}|^p(x)dx} \leq 1,$$

so that

$$\int_{\Omega} m(x)|u|^p(x)dx = \int_{\Omega^{\#}} m^{\#}(x)|u^{\#}|^p(x)dx \tag{3.4}$$

and

$$\int_{\Omega} |Du|^p(x)dx = \int_{\Omega^{\#}} |Du^{\#}|^p(x)dx. \tag{3.5}$$

By (3.3), $u^{\#}$ is an eigenfunction of the problem (3.1) corresponding to the first eigenvalue $\lambda_p^{\#}$. On the other hand $u^{\#}(x) = u^*(C_n|x|^n)$ so that the equation of problem (3.1) can be reduced to the one dimensional equation

$$-\frac{d}{d|x|}\left[\left(\frac{du^{\#}}{d|x|}\right)^{p-1}|x|^{n-1}\right] = \lambda_p^{\#} m^{\#}(x)(u^{\#}(x))^{p-1}|x|^{n-1}. \tag{3.6}$$

Integrating both sides of (3.6) between 0 and $|x|$ we have

$$-\frac{du^{\#}}{d|x|} = \left[\frac{\lambda_p^{\#}}{nC_n|x|^{n-1}}\int_{|y|<|x|} m^{\#}(y)(u^{\#}(y))^{p-1}\,dy\right]^{\frac{1}{p-1}}. \tag{3.7}$$

On the other hand, by Harnack inequality we have that $u^{\#} > 0$ and then, by (3.7), it is strictly decreasing along the radious. In particular

$$|\{x \in \Omega : 0 < u^{\#}(x) < \operatorname{ess\,sup}\, u, |Du^{\#}|(x) = 0\}| = 0. \tag{3.8}$$

By (3.5), (3.8) and Theorem 2.1 we can conclude that

$$\Omega = \Omega^{\#}, \quad u(x) = u^{\#}(x) \quad \text{a.e.} \quad x \in \Omega, \quad \text{modulo translations.} \tag{3.9}$$

Finally, by (3.4), (3.7), (3.8), using Proposition 2.1 and (3.9), it follows that

$$m(x) = m^{\#}(x) \quad \text{a.e.in } \Omega.$$

By standard arguments (see [2], [16] and [1]) we get the thesis. ◻

4 – Payne-Rayner type inequalities

In this section we establish an inverse Hölder inequality for the eigenfunctions of the problem (1.3) corresponding to the first eigenvalue λ_p. The main tool we use is a comparison result between u and an eigenfunction v_q of a suitable problem. More precisely, let B be the ball centered at the origin, such that λ_p is the first eigenvalue of the following problem:

$$\begin{cases} -\operatorname{div}(|Dv|^{p-2}Dv) = \lambda m^{\#}(x) v|v|^{p-2} & \text{in } B \\ v = 0 & \text{on } \partial B, \end{cases} \tag{4.1}$$

Let v_q be a positive eigenfunction of the problem (4.1), corresponding to λ_p, such that

$$\|v_q^*\|_{L^q(B^*,m^*)} = \|u^*\|_{L^q(\Omega^*,m^*)}, \tag{4.2}$$

for all $0 < q \leq +\infty$. In particular, if $q = +\infty$, by (2.1), we have

$$v_\infty^*(0) = u^*(0). \tag{4.3}$$

A straightforward calculation shows that

$$B = \left\{ x \in \mathbb{R}^n : |x| < (K_p/\lambda_p)^{1/p} \right\},$$

where K_p denotes the first eigenvalue of the problem (4.1) in the unit ball. By Theorem 3.1 and the characterization of the first eigenvalue, it follows that $|B| \leq |\Omega|$.

The following comparison result between u and v_q holds:

Theorem 4.1. *Let Ω be an open, bounded and connected subset of $\mathbb{R}^n$, $1 < p < n$. Let u be a positive eigenfunction of the problem* (1.3) *corresponding to the first eigenvalue* λ_p *under the assumptions* (1.4) *and* (1.5) *and let* v_q *be a positive eigenfunction of the problem* (4.1) *corresponding to* λ_p *such that* (4.2) *holds. Then:*

i) *if* $1 \le q < +\infty$,

$$\int_0^s m^*(t)(u^*(t))^q dt \le \int_0^s m^*(t)(v_q^*(t))^q dt, \quad s \in [0, |B|]; \tag{4.4}$$

ii) *if* $q = +\infty$,

$$u^*(s) \ge v_\infty^*(s), \quad s \in [0, |B|].$$

If any of the above inequalities hold as an equality, then $\Omega = B$, $u(x) = u^{\#}(x) = v_q(x)$, $a_{ij}(x)x_j = x_i$ *and* $m(x) = m^{\#}(x)$ *a.e. in* Ω, *modulo translations.*

The first step to get Theorem 4.1 is to obtain a differential inequality involving the decreasing rearrangement of u. More precisely, the following proposition holds.

Proposition 4.1. *Let* Ω *be an open, bounded and connected subset of* $\mathbb{R}^n$, $1 < p < n$. *Let* u *be a positive eigenfunction of the problem* (1.3) *corresponding to the first eigenvalue* λ_p *under the assumptions* (1.4) *and* (1.5) *and let* v_q *be a positive eigenfunction of the problem* (4.1) *corresponding to* λ_p *such that* (4.2) *holds. Let*

$$U(s) = \int_0^s m^*(t)(u^*(t))^{p-1} dt, \quad s \in [0, |\Omega|], \tag{4.5}$$

and

$$\mu_p = \left(\frac{\lambda_p^{1/p}}{n C_n^{1/n}} \right)^{p'} . \tag{4.6}$$

Then

$$\begin{cases} -\left(\left(\dfrac{U'(s)}{m^*(s)} \right)^{\frac{1}{p-1}} \right)' \le \mu_p (s^{1-1/n})^{-p'} (U(s))^{\frac{1}{p-1}} \qquad a.e. \ in \ \Omega^* \\ U(0) = U'(|\Omega|) = 0. \end{cases} \tag{4.7}$$

Proof. By well known comparison results (see, for example [26], [27], [4], [9]), we have

$$-(u^*(s))' \leq \mu_p (s^{1-1/n})^{-p'} \left(\int_0^s m^*(t)(u^*(t))^{p-1} dt \right)^{\frac{1}{p-1}}, \tag{4.8}$$

a.e. in Ω^*. Writing (4.8) in terms of U, we get the thesis. ◻

Remark 4.1. We observe that any eigenfunction associated to the first eigenvalue of the problem (4.1) is spherically symmetric because of the symmetry of the problem (4.1) and the simplicity of the first eigenvalue. In particular, $v = v_q$ is spherically symmetric, that is $v_q(x) = v_q(|x|)$ and then the equation of problem (4.1) can be reduced to the one dimensional equation

$$-\frac{d}{d|x|}\left[\left(\frac{dv_q}{d|x|} \right)^{p-1} |x|^{n-1} \right] = \lambda_p m^{\#}(x)(v_q(x))^{p-1}|x|^{n-1}. \tag{4.9}$$

Integrating both sides of (4.9) between 0 and $|x|$ we have

$$-\frac{dv_q}{d|x|} = \left[\frac{\lambda_p}{nC_n|x|^{n-1}} \int_{|y|<|x|} m^{\#}(y)(v_q(y))^{p-1} dy \right]^{\frac{1}{p-1}}, \tag{4.10}$$

then since v_q is positive by (4.10) it follows that v_q is decreasing along the radious, that is $v_q(x) = v_q^{\#}(x) = v_q^*(C_n|x|^n)$. By (4.10) we get

$$-\frac{dv_q^*}{ds} = \left[\frac{\lambda_p}{(nC_n^{1/n} s^{1-1/n})^p} \int_0^s m^*(t)(v_q^*(t))^{p-1} dt \right]^{\frac{1}{p-1}},$$

where $s = C_n|x|^n$.

Then, if we put

$$V(s) = \int_0^s m^*(t)(v_q^*(t))^{p-1} dt, \quad s \in [0, |B|], \tag{4.11}$$

we have

$$\begin{cases} -\left(\left(\dfrac{V'(s)}{m^*(s)} \right)^{\frac{1}{p-1}} \right)' = \mu_p (s^{1-1/n})^{-p'} \left(V(s) \right)^{\frac{1}{p-1}} & \text{in } (0, |B|) \\ V(0) = V'(|B|) = 0 \end{cases} \tag{4.12}$$

where we put

$$\left(\frac{V'(s)}{m^*(s)}\right)^{1/(p-1)} = v_q^*(s) \quad s \in (0, |B|).$$

Let us consider the following problem

$$\text{(4.13)} \qquad \begin{cases} -\left(\left(\dfrac{w'(s)}{m^*(s)}\right)^{\frac{1}{p-1}}\right)' = \mu_p s^{-p'(1-\frac{1}{n})} \left(w(s)\right)^{\frac{1}{p-1}} & \text{in } (0, |B|) \\ w(0) = w'(|B|) = 0. \end{cases}$$

We will say that a function w is a solution of (4.13) if w is an absolutely continuous function such that $\left(\dfrac{w'(s)}{m^*(s)}\right)^{1/(p-1)}$ is an absolutely continuous function and it satifies problem (4.13) for all $s \in (0, |B|)$. By definition it follows that a function w is a solution of (4.13) if and only if there exists a summable function h such that

$$w(s) = \int_0^s m^*(t)(h(t))^{p-1} dt.$$

Following similar arguments to those used in [1], we have that λ_p is the first eigenvalue of the problem (4.1), if and only if μ_p is the first eigenvalue of the problem (4.13).

The second step is to compare the functions U and V defined in (4.5) and (4.11) respectively.

LEMMA 4.1. *Let Ω be an open, bounded and connected subset of $\mathbb{R}^n$, $1 < p < n$. Let u be a positive eigenfunction of the problem* (1.3) *corresponding to the first eigenvalue λ_p under the assumptions* (1.4) *and* (1.5) *and let v_q be a positive eigenfunction of the problem* (4.1) *corresponding to λ_p such that* (4.2) *holds. Let U, V and μ_p be as in* (4.5), (4.11) *and* (4.6) *respectively. Then*

$$U(s) \leq V(s), \quad \forall s \in [0, |B|].$$

Proof. First of all, by (4.2), we observe that $U(|B|) < U(|\Omega|) = V(|B|)$. Let us suppose, ab absurdo, that there is a positive maximum of $W(s) = U(s) - V(s)$ in $(0, |B|)$. Then there exists $s_1 \in (0, |B|)$ such that $W(s_1) > 0$ and $u^*(s_1) = v^*(s_1)$. Let us define

$$Z(s) = \begin{cases} U(s) & s \in (0, s_1) \\ V(s) + W(s_1) & s \in [s_1, |B|]. \end{cases} \tag{4.14}$$

The function $\left(\dfrac{Z'(s)}{m^*(s)}\right)^{\frac{1}{p-1}}$ is absolutely continuous in $[\epsilon, |B|]$, for all $\epsilon > 0$ and

$$\left(\frac{Z'(s)}{m^*(s)}\right)^{\frac{1}{p-1}} = \left(\frac{U'(s)}{m^*(s)}\right)^{\frac{1}{p-1}}, \qquad Z(s) = U(s), \quad s \in (0, s_1),$$

$$\left(\frac{Z'(s)}{m^*(s)}\right)^{\frac{1}{p-1}} = \left(\frac{V'(s)}{m^*(s)}\right)^{\frac{1}{p-1}}, \qquad Z(s) > U(s), \quad s \in [s_1, |B|).$$

This means that Z satisfies the problem:

$$\begin{cases} -\left(\left(\dfrac{Z'(s)}{m^*(s)}\right)^{\frac{1}{p-1}}\right)' \leq \mu_p s^{-p'(1-\frac{1}{n})} \left(Z(s)\right)^{\frac{1}{p-1}} \quad \text{a.e. in } (0, |B|) \\ Z(0) = Z'(|B|) = 0. \end{cases} \tag{4.15}$$

Multiplying both sides of the inequality in (4.15) by $Z(s)$ and integrating by parts, we obtain

$$\int_0^{|B|} \left(\frac{Z'(s)}{m^*(s)}\right)^{\frac{1}{p-1}} Z'(s) ds \leq \mu_p \int_0^{|B|} \frac{Z(s)^{p'}}{s^{(1-1/n)p'}} ds, \tag{4.16}$$

since, as we will show at the end of the proof,

$$\lim_{\varepsilon \to 0^+} \left(\frac{Z'(\varepsilon)}{m^*(\varepsilon)}\right)^{\frac{1}{p-1}} Z(\varepsilon) = 0. \tag{4.17}$$

By (4.16) we have:

$$\frac{\displaystyle\int_0^{|B|}\left(\frac{(Z'(s))^{p'}}{(m^*(s))^{\frac{1}{p-1}}}\right)ds}{\displaystyle\int_0^{|B|}\left(\frac{(Z(s))^{p}}{s^{(1-1/n)p'}}\right)ds}\leq\mu_p.$$

This implies that Z is an eigenfunction associated to the first eigenvalue μ_p of problem (4.12) and then it must be proportional to V. But, by its definition, it follows that Z has to coincide with V. So we get a contradiction, since we have supposed that $W(s_1)>0$.

In order to conclude the proof, we have to prove (4.17). First of all, we observe that the hypotheses on m imply that the embedding

$$W^{1,p'}(\Omega^*,(m^*)^{-\frac{1}{p-1}})\hookrightarrow L^{p'}(\Omega^*,s^{-(1-\frac{1}{n})p'}) \tag{4.18}$$

is compact. By absolute continuity of $\left(\dfrac{Z'(s)}{m^*(s)}\right)^{\frac{1}{p-1}}$ and by inequality (4.15), we have:

$$\left(\frac{Z'(\varepsilon)}{m^*(\varepsilon)}\right)^{\frac{1}{p-1}}Z(\varepsilon)=-\left[\int_\varepsilon^{|B|}\left(\left(\frac{Z'(s)}{m^*(s)}\right)^{\frac{1}{p-1}}\right)'ds\right]\left[\int_0^\varepsilon Z'(s)ds\right]\leq$$

$$\leq\frac{\lambda_p}{\left(nC_n^{1/n}\right)^p}\left[\int_\varepsilon^{|B|}\frac{(Z(s))^{\frac{1}{p-1}}}{s^{(1-\frac{1}{n})p'}}ds\right]\left[\int_0^\varepsilon m^*(s)\left(\int_s^{|B|}\frac{(Z(t))^{\frac{1}{p-1}}}{t^{(1-\frac{1}{n})p'}}dt\right)^{p-1}ds\right].$$

Using Hölder inequality and the embedding (4.18), we obtain

$$\left(\frac{Z'(\varepsilon)}{m^*(\varepsilon)}\right)^{\frac{1}{p-1}}Z(\varepsilon)\leq C\left\|Z\right\|^{1+\frac{1}{p-1}}_{L^{p'}(B^*,s^{-(1-1/n)p'})}\left(\int_\varepsilon^{|B|}\frac{1}{s^{(1-1/n)p'}}ds\right)^{\frac{1}{p'}}\times$$

$$\times\left(\int_0^\varepsilon m^*(s)\left[\int_s^{|B|}\frac{1}{t^{(1-\frac{1}{n})p'}}dt\right]^{\frac{p-1}{p'}}ds\right).$$

Again by Hölder inequality, (1.5) and using the fact that $1 < p < n$, we get

$$\left(\frac{Z'(\varepsilon)}{m^*(\varepsilon)}\right)^{\frac{1}{p-1}} Z(\varepsilon) \leq C\varepsilon^{\frac{1}{n}-\frac{1}{p}} \parallel m \parallel_r \left(\int_0^\varepsilon s^{(\frac{1}{n}-\frac{1}{p})(p-1)r'} ds\right)^{\frac{1}{r'}},$$

that is

$$\left(\frac{Z'(\varepsilon)}{m^*(\varepsilon)}\right)^{\frac{1}{p-1}} Z(\varepsilon) \leq C \parallel m \parallel_r \varepsilon^{\frac{p}{n}-\frac{1}{r}}. \tag{4.19}$$

Since $r > p/n$ passing to the limit as $\varepsilon \to 0^+$ in (4.19), we get the thesis. □

We are now able to prove Theorem 4.1.

Proof of Theorem 4.1. Let us consider the case $0 < q < +\infty$. We point out that, if $q = p - 1$, then the conclusion follows directly by Lemma 4.1. In the general case, let us define the following functions

$$U_q(s) = \int_0^s m^*(t)(u^*(t))^q dt, \quad s \in [0, |\Omega|]$$

and

$$V_q(s) = \int_0^s m^*(t)(v_q^*(t))^q dt, \quad s \in [0, |B|].$$

By definitions, we get that

$$U_q(0) = V_q(0) = 0,$$

and by Lemma 4.1,

$$U_q(|B|) \leq V_q(|B|).$$

If, ab absurdo, (4.4) does not hold, then there exists $s_1 \in (0, |B|)$ such that

$$U_q(s_1) > V_q(s_1) \tag{4.20}$$

and

$$u^*(s_1) \leq v_q^*(s_1). \tag{4.21}$$

On the other hand, by Lemma 4.1, it follows that, if U and V are the functions defined in (4.5) and (4.11), then

$$U(s) \leq V(s), \quad s \in [0, |B|] \tag{4.22}$$

and then, by (4.7), (4.12) and (4.21), we get

$$-\left(\left(\frac{U'(s)}{m^*(s)}\right)^{\frac{1}{p-1}}\right)' \leq -\left(\left(\frac{V'(s)}{m^*(s)}\right)^{\frac{1}{p-1}}\right)' \quad \text{a.e. } s \in [0, |B|],$$

that is

$$-(u^*(s))' \leq -(v_q^*(s))' \quad \text{a.e. } s \in [0, |B|]. \tag{4.23}$$

If $s \in]0, s_1[$, integrating (4.23) between s and s_1, and using (4.21), we get

$$u^*(s) \leq v_q^*(s), \quad s \in (0, s_1],$$

which contraddicts (4.20).

If $q = +\infty$, we get the thesis by similar arguments. Since $|B| < |\Omega|$ and $u^*(|B|) > v_\infty^*(|B|) = 0$, then we consider

$$s_1 = \inf\{s \in B^* : u^*(t) \geq v_\infty^*(t) \quad \forall t \in (s, |B|]\}.$$

If $s_1 = 0$, then Theorem 4.1 is proven. So let us suppose, ab absurdo, that $s_1 > 0$. Then

$$u^*(s_1) = v_\infty^*(s_1).$$

Moreover, as we have pointed out in Section 2, we have

$$u^*(0) = v_\infty^*(0).$$

If $s \in (0, s_1)$, integrating (4.23) between 0 and s we get

$$u^*(s) \geq v_\infty^*(s) \quad \forall s \in (0, s_1)$$

which contraddicts the assumption $s_1 > 0$.

Finally, the case of equality in i) and ii) follows immediately from Theorem 3.1. It is enough to observe that, if, for example, equality holds in part i), then $u^* = v_q^*$ and $|\Omega| = |B|$. This means that we can apply Theorem 3.1. $\square$

THEOREM 4.2. *Let Ω be an open, bounded and connected subset of $\mathbb{R}^n$, $1 < p < n$. Let u be a positive eigenfunction of the problem* (1.3) *corresponding to the first eigenvalue λ_p under the assumptions* (1.4) *and* (1.5). *Then, for $0 < q < r \leq +\infty$, we have:*

$$\| u^* \|_{L^r(\Omega^*,m^*)} \leq \beta(n,q,r,p,\lambda_p) \| u^* \|_{L^q(\Omega^*,m^*)}, \tag{4.24}$$

where $\beta(n,q,r,p,\lambda_p) = \| v^* \|_{L^r(\Omega^*,m^*)} \ / \ \| v^* \|_{L^q(\Omega^*,m^*)}$ and v is a non trivial eigenfunction of the problem (4.1). *Furthermore, equality in* (4.24) *holds if, and only if, $B = \Omega = \Omega^\#$, $u(x) = u^\#(x)$, $m(x) = m^\#(x)$ and $a_{ij}(x)x_j = x_i$ a.e. in Ω, modulo translations.*

PROOF. By Theorem 4.1, we get that $m^* u^{*q} \prec m^* v_q^{*q}$. Then by Corollary 2.1, we get

$$\| u^* \|_{L^r(\Omega^*,m^*)} \leq \| v_q^* \|_{L^r(\Omega^*,m^*)} = \frac{\| v_q^* \|_{L^r(\Omega^*,m^*)}}{\| v_q^* \|_{L^q(\Omega^*,m^*)}} \| u^* \|_{L^q(\Omega^*,m^*)}$$

and equality holds if, and only if, $u^*(s) = v_q^*(s)$, for all $s \in \Omega^*$. Then, by Theorem 4.1, we get the thesis.

Let us, now, consider the case $r = +\infty$. We get the thesis by the fact

$$\| u^* \|_{L^\infty(\Omega^*)} = \lim_{r\to+\infty} \| u^* \|_{L^r(\Omega^*,m^*)} \leq \lim_{r\to+\infty} \| v_q^* \|_{L^r(\Omega^*,m^*)} = \| v_q^* \|_{L^\infty(\Omega^*)} \ .$$

In particular if $\| u^* \|_{L^\infty(\Omega^*)} = \| v_q^* \|_{L^\infty(\Omega^*)}$, then $v_q = v_\infty$. Taking into account Theorem 4.1 we immediately have $u^* = v_\infty^*$. This completes the proof. $\square$

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