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The 2-D Neumann problem in a domain with cuts

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Riassunto: *Si studia il problema di Neumann per l'equazione di Laplace in una regione piana connessa con tagli. Si dimostra l'esistenza della soluzione con la teoria classica del potenziale. Il problema è ricondotto ad una equazione di Fredholm di seconda specie, che è risolvibile univocamente.*

Abstract: *The Neumann problem for the Laplace equation in a connected plane region with cuts is studied. The existence of classical solution is proved by potential theory. The problem is reduced to a Fredholm equation of the second kind, which is uniquely solvable.*

1 – Introduction

The boundary value problems in domains containing cuts were not treated in the theory of 2-D PDEs before. Even in the case of Laplace and Helmholtz equations the problems in domains bounded by closed curves [2], [5]-[8], [12]-[14] and problems in the exterior of cuts [5], [9]-[11] were treated separately, because different methods were used in their analysis. Previously the Neumann problem in the exterior of a cut was reduced to the hypersingular integral equation [9]-[10] or to the infinite algebraic sys-

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tem of equations [11], while the Neumann problem in domains bounded by closed curves was reduced to the Fredholm equation of the second kind [6]-[8], [12]-[14]. The combination of these methods in case of domains containing cuts leads to the integral equation, which is algebraic or hypersingular on cuts, and it is an equation of the second kind with compact integral operators on the closed curves. The integral equation on the whole boundary is rather complicated to be effectively studied by standard methods. The approach suggested in the present paper enables to reduce the Neumann problem in domains with cuts to the Fredholm integral equation on the whole boundary with the help of the nonclassical angular potential. The Fredholm equation is uniquely solvable and can be computed by standard codes. Our approach is based on [3]-[4], where the problems in the exterior of cuts were reduced to the Fredholm integral equations using the angular potential. In [16]-[18] our approach has been applied to the Dirichlet and Neumann problems for the Helmholtz equation in domains with cuts. Some nonlinear problems of fluid dynamics were studied in [15]. From practical stand-point domains with cuts have great significance because cuts model cracks, screens or wings in physics, mechanics and engineering.

In the present paper we consider the Neumann problem for the Laplace equation in a plane domain with cuts. This problem is not uniquely solvable unlike [16]-[18] and therefore is more complicated. Nevertheless we reduce this problem to the uniquely solvable Fredholm equation, which can be computed by the direct numerical inversion of its integral operator.

2 – Formulation of the problem

By a simple open curve we mean a non-closed smooth arc of finite length without self-intersections [5].

In the plane $x = (x_1, x_2) \in R^2$ we consider the multiply connected domain bounded by simple open curves $\Gamma_1^1, ..., \Gamma_{N_1}^1 \in C^{2,\lambda}$ and simple closed curves $\Gamma_1^2, ..., \Gamma_{N_2}^2 \in C^{2,0}$, $\lambda \in (0, 1]$, so that the curves do not have points in common and the curve Γ_1^2 encloses all other. We put

$$\Gamma^1 = \bigcup_{n=1}^{N_1} \Gamma_n^1, \quad \Gamma^2 = \bigcup_{n=1}^{N_2} \Gamma_n^2, \quad \Gamma = \Gamma^1 \cup \Gamma^2.$$

The connected domain bounded by Γ^2 will be called $\mathcal{D}$. We assume that each curve Γ_n^k is parametricized by the arc length s : $\Gamma_n^k = \{x : x = x(s) = (x_1(s), x_2(s)),\ s \in [a_n^k, b_n^k]\}$, $n = 1, ..., N_k$, $k = 1, 2$, so that $a_1^1 < b_1^1 < ... < a_{N_1}^1 < b_{N_1}^1 < a_1^2 < b_1^2 < ... < a_{N_2}^2 < b_{N_2}^2$ and the domain $\mathcal{D}$ is to the right when the parameter s increases on Γ_n^2. Therefore points $x \in \Gamma$ and values of the parameter s are in one-to-one correspondence except a_n^2, b_n^2, which correspond to the same point x for $n = 1, ..., N_2$. Below the sets of the intervals on the Os axis

$$\bigcup_{n=1}^{N_1} [a_n^1, b_n^1], \quad \bigcup_{n=1}^{N_2} [a_n^2, b_n^2], \quad \bigcup_{k=1}^{2} \bigcup_{n=1}^{N_k} [a_n^k, b_n^k],$$

will be denoted by the same symbols as corresponding sets of curves, that is by Γ^1, Γ^2 and Γ respectively.

We put $C^0(\Gamma_n^2) = \{\mathcal{F}(s) :\ \mathcal{F}(s) \in C^0[a_n^2, b_n^2],\ \ \mathcal{F}(a_n^2) = \mathcal{F}(b_n^2)\ \}$, and

$$C^0(\Gamma^2) = \bigcap_{n=1}^{N_2} C^0(\Gamma_n^2).$$

By $\mathcal{D}_n$ we denote the internal domain bounded by the curve Γ_n^2, if $n = 2, ..., N_2$. The external domain bounded by Γ_1^2 will be called $\mathcal{D}_1$.

The tangent vector to Γ at the point $x(s)$ we denote by $\tau_x = (\cos\alpha(s), \sin\alpha(s))$, where $\cos\alpha(s) = x_1'(s)$, $\sin\alpha(s) = x_2'(s)$. Let $\mathbf{n}_x = (\sin\alpha(s), -\cos\alpha(s))$ be a normal vector to Γ at $x(s)$. The direction of $\mathbf{n}_x$ is chosen such that it will coincide with the direction of τ_x if $\mathbf{n}_x$ is rotated anticlockwise through an angle of $\pi/2$.

We consider the curves Γ^1 as a set of cuts. The side of Γ^1 which is on the left, when the parameter s increases, will be denoted by $(\Gamma^1)^+$ and the opposite side will be denoted by $(\Gamma^1)^-$.

We say, that the function $u(x)$ belongs to the smoothness class $\mathbf{K}$ if

1) $u \in C^0(\overline{\mathcal{D}\backslash\Gamma^1}) \cap C^2(\mathcal{D}\backslash\Gamma^1)$,
2) $\nabla u \in C^0(\overline{\mathcal{D}\backslash\Gamma^1}\backslash\Gamma^2\backslash X)$, where X is a point-set, consisting of the end-points of Γ^1 :

$$X = \bigcup_{n=1}^{N_1} \left(x(a_n^1) \cup x(b_n^1)\right),$$

3) in the neighbourhood of any point $x(d) \in X$ for some constants $\mathcal{C} > 0$, $\epsilon > -1$ the inequality holds

$$|\nabla u| \leq \mathcal{C}\, |x - x(d)|^{\epsilon}, \tag{1}$$

where $x \to x(d)$ and $d = a_n^1$ or $d = b_n^1$, $n = 1, ..., N_1$.
4) there exists a uniform for all $x(s) \in \Gamma^2$ limit of $(\mathbf{n}_x, \nabla_{\overline{x}} w(\overline{x}))$ as $\overline{x} \in \mathcal{D}\backslash\Gamma^1$ tends to $x \in \Gamma^2$ along the normal $\mathbf{n}_x$.

By the designation $\overline{x} \stackrel{\mathbf{n}}{\to} x(s) \in \Gamma^2$ we will stress that $\overline{x}$ tends to $x(s) \in \Gamma^2$ along the normal $\mathbf{n}_x$.

REMARK. By $C^0(\overline{\mathcal{D}\backslash\Gamma^1})$ we denote functions, which are continuously extended on cuts Γ^1 from the left and right, but their values on Γ^1 from the left and right can be different, so that the functions may have a jump on Γ^1.

Let us formulate the Neumann problem for the Laplace equation in the domain $\mathcal{D}\backslash\Gamma^1$.

Problem $\mathbf{U}$. To find a function $u(x)$ of the class $\mathbf{K}$, so that $w(x)$ satisfies the Laplace equation

$$\Delta u(x) = 0, \quad x \in \mathcal{D}\backslash\Gamma^1 ; \qquad \Delta = \partial_{x_1}^2 + \partial_{x_2}^2 , \tag{2a}$$

and the boundary condition

$$\left.\frac{\partial u(x)}{\partial \mathbf{n}_x}\right|_{x(s)\in(\Gamma^1)^+} = F^+(s), \quad \left.\frac{\partial u(x)}{\partial \mathbf{n}_x}\right|_{x(s)\in(\Gamma^1)^-} = F^-(s),$$
$$\left.\frac{\partial u(x)}{\partial \mathbf{n}_x}\right|_{x(s)\in\Gamma^2} = F(s). \tag{2b}$$

All conditions of the problem **U** must be satisfied in the classical sense. By $\partial u/\partial \mathbf{n}_x$ on Γ^2 we mean the limit ensured in the point 4) of the definition of the smoothness class **K**.

The edge condition (1) ensures the absence of point sources at the ends of Γ^1. If $N_1 = 0$ and cuts Γ^1 are absent, then the problem **U** transforms to the classical Neumann problem in a domain $\mathcal{D}$.

Using the energy equalities and the technique of equidistant curves [6] for Γ^2 we can easily prove the following assertion.

THEOREM 1. *Let* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$, $\lambda \in (0,1]$. *The necessary condition for the solvability of the problem* **U** *is the equality*

$$\int_{\Gamma^2} F(s)ds + \int_{\Gamma^1} (F^-(s) - F^+(s))ds = 0. \tag{3a}$$

If the solution of the problem **U** *exists, then it is defined up to an arbitrary additive constant.*

By $\int_{\Gamma^k} ... \, d\sigma$ we mean $\sum_{n=1}^{N_k} \int\limits_{a_n^k}^{b_n^k} \ldots \, d\sigma$.

PROOF. We envelope each cut Γ_n^1 $(n = 1, ..., N_1)$ by a closed contour and construct an equidistant contour for each closed curve Γ_n^2 $(n = 1, ..., N_2)$, so that all contours lie in $\mathcal{D}\backslash\Gamma^1$. Then we tend contours to Γ and use the smoothness of the solution of the problem **U**. The condition (3*a*) can be obtained in this way, if we take into account the following well-known property of the harmonic functions [6]. If the function $W(x)$ is harmonic in the domain Ω and enough smooth in $\overline{\Omega}$, then

$$\int_{\partial\Omega} \lim_{x \to x(s) \in \partial\Omega} \frac{\partial W}{\partial \mathbf{n}_x} ds = 0.$$

The limit is understood along the normal on those parts of the boundary $\partial\Omega$, where $\partial W/\partial \mathbf{n}_x$ exists in a sense of a uniform limit along the normal. Consequently, if $u(x)$ is a solution of the problem **U**, then in the domain $\mathcal{D}\backslash\Gamma^1$ we have

$$\int_{\Gamma^1} \left[\left(\frac{\partial u}{\partial \mathbf{n}_x} \right)^+ - \left(\frac{\partial u}{\partial \mathbf{n}_x} \right)^- \right] ds - \int_{\Gamma^2} \frac{\partial u}{\partial \mathbf{n}_x} ds = 0.$$

Here we consider Γ^1 as a set of cuts. The limit values of functions on Γ^+ and Γ^- are denoted by the superscripts "+" and "−" respectively. The equality (3*a*) follows from the boundary condition (2*b*).

If $u_0(x)$ is a solution of the homogeneous problem **U**, then we write the energy equalities for a domain, bounded by our auxiliary contours, tend them to Γ and use the smoothness of $u_0(x)$ ensured by the class **K**. In this way we obtain

$$\|\nabla u_0\|^2_{L_2(\mathcal{D}\setminus\Gamma^1)} = \int_{\Gamma^1} \left[u_0^+ \left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^+ - u_0^- \left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^- \right] ds - \int_{\Gamma^2} u_0 \frac{\partial u_0}{\partial \mathbf{n}_x} ds.$$

Taking into account the homogeneous boundary conditions (2*b*), we have

$$\|\nabla u_0\|^2_{L_2(\mathcal{D}\setminus\Gamma^1)} = 0.$$

Hence $u_0(x) \equiv \text{const}$ and the theorem is proved thanks to the linearity of the problem **U**.

3 – Integral equations at the boundary

Below we assume that

$$(3b) \qquad F^+(s),\, F^-(s) \in C^{0,\lambda}(\Gamma^1), \qquad F(s) \in C^0(\Gamma^2), \qquad \lambda \in (0,1],$$

and $F^\pm(s)$, $F(s)$ meet condition (3*a*).

If $\mathcal{B}_1(\Gamma^1)$, $\mathcal{B}_2(\Gamma^2)$ are Banach spaces of functions given on Γ^1 and Γ^2, then for functions given on Γ we introduce the Banach space $\mathcal{B}_1(\Gamma^1) \cap \mathcal{B}_2(\Gamma^2)$ with the norm $\|\cdot\|_{\mathcal{B}_1(\Gamma^1)\cap\mathcal{B}_2(\Gamma^2)} = \|\cdot\|_{\mathcal{B}_1(\Gamma^1)} + \|\cdot\|_{\mathcal{B}_2(\Gamma^2)}$. Examples of such Banach spaces are $C^{0,\lambda}(\Gamma) = C^{0,\lambda}(\Gamma^1) \cap C^{0,\lambda}(\Gamma^2)$, $C^0(\Gamma) = C^0(\Gamma^1) \cap C^0(\Gamma^2)$.

We consider an angular potential [1] for the equation (2*a*):

$$(4) \qquad w_1[\mu](x) = -\frac{1}{2\pi} \int_{\Gamma^1} \mu(\sigma) V(x,\sigma) d\sigma.$$

The kernel $V(x,\sigma)$ is defined (up to indeterminacy $2\pi m,\ m = \pm 1,\ \pm 2, ...$) by the formulae

$$\cos V(x,\sigma) = \frac{x_1 - y_1(\sigma)}{|x - y(\sigma)|}, \quad \sin V(x,\sigma) = \frac{x_2 - y_2(\sigma)}{|x - y(\sigma)|},$$

where

$$y = y(\sigma) = (y_1(\sigma), y_2(\sigma)) \in \Gamma^1,$$
$$|x - y(\sigma)| = \sqrt{(x_1 - y_1(\sigma))^2 + (x_2 - y_2(\sigma))^2}.$$

One can see, that $V(x,\sigma)$ is the angle between the vector $\overrightarrow{y(\sigma)x}$ and the direction of the Ox_1 axis. More precisely, $V(x,\sigma)$ is a many-valued harmonic function of x connected with $\ln|x - y(\sigma)|$ by the Cauchy-Riemann relations.

Below by $V(x,\sigma)$ we denote an arbitrary fixed branch of this function, which varies continuously with σ along each curve Γ^1_n $(n = 1, ..., N_1)$ for given fixed $x \notin \Gamma^1$.

Under this definition of $V(x,\sigma)$, the potential $w_1[\mu](x)$ is a many-valued function. In order that the potential $w_1[\mu](x)$ be single-valued, it is necessary to impose the following additional conditions

$$\int_{a^1_n}^{b^1_n} \mu(\sigma)\, d\sigma = 0, \quad n = 1, ..., N_1. \tag{5}$$

Below we suppose that the density $\mu(\sigma)$ belongs to the Banach space $C^\omega_q(\Gamma^1)$, $\omega \in (0,1]$, $q \in [0,1)$ and satisfies conditions (5).

We say, that $\mu(s) \in C^\omega_q(\Gamma^1)$ if

$$\mu(s) \prod_{n=1}^{N_1} |s - a^1_n|^q \, |s - b^1_n|^q \in C^{0,\omega}(\Gamma^1),$$

where $C^{0,\omega}(\Gamma^1)$ is a Hölder space with the index ω and

$$\|\mu(s)\|_{C^\omega_q(\Gamma^1)} = \left\| \mu(s) \prod_{n=1}^{N_1} |s - a^1_n|^q \, |s - b^1_n|^q \right\|_{C^{0,\omega}(\Gamma^1)}.$$

As shown in [1], [3], for such $\mu(\sigma)$ the angular potential $= w_1[\mu](x)$ belongs to the class **K**. In particular, the inequality (1) holds with $\epsilon = -q$, if $q \in (0,1)$. Moreover, integrating $w_1[\mu](x)$ by parts and using (5), we express the angular potential in terms of a double layer potential

$$w_1[\mu](x) = \frac{1}{2\pi} \int\limits_{\Gamma^1} \rho(\sigma) \frac{\partial}{\partial \mathbf{n}_y} \ln |x - y(\sigma)| \, d\sigma,$$

with the density $\rho(\sigma) = \int_{a_n^1}^{\sigma} \mu(\xi)\, d\xi$, $\sigma \in [a_n^1, b_n^1]$, $n = 1, ..., N_1$. Consequently, $w_1[\mu](x)$ satisfies equation $(2a)$ outside Γ^1.

Let us construct a solution of the problem **U** . This solution can be obtained with the help of potential theory for the equation $(2a)$. We seek a solution of the problem in the following form

$$u[\nu, \mu](x) = v_1[\nu](x) + w[\mu](x) + C \ , \tag{6}$$

where C is an arbitrary constant,

$$v_1[\nu](x) = -\frac{1}{2\pi} \int\limits_{\Gamma^1} \nu(\sigma) \ln |x - y(\sigma)| \, d\sigma,$$

$$w[\mu](x) = w_1[\mu](x) + w_2[\mu](x) \,,$$

$$w_2[\mu](x) = -\frac{1}{2\pi} \int\limits_{\Gamma^2} \mu(\sigma) \ln |x - y(\sigma)| \, d\sigma \,,$$

and $w_1[\mu](x)$ is an angular potential given by (4).

We will look for $\nu(s)$ in the space $C^{0,\lambda}(\Gamma^1)$.

We will seek $\mu(s)$ from the Banach space $C_q^{\omega}(\Gamma^1) \cap C^0(\Gamma^2)$, $\omega \in (0,1]$, $q \in [0,1)$ with the norm $\|\cdot\|_{C_q^{\omega}(\Gamma^1) \cap C^0(\Gamma^2)} = \|\cdot\|_{C_q^{\omega}(\Gamma^1)} + \|\cdot\|_{C^0(\Gamma^2)}$. Besides, $\mu(s)$ must satisfy conditions (5).

It follows from [1], [3], [6], that for such $\mu(s)$, $\nu(s)$ the function (6) belongs to the class **K** and satisfies all conditions of the problem **U** except the boundary condition $(2b)$.

To satisfy the boundary condition we put (6) in $(2b)$, use the limit formulas for the angular potential from [1], [3] and arrive at the integral equation for the densities $\mu(s),\ \nu(s)$:

$$\pm\frac{1}{2}\nu(s)+\frac{1}{2\pi}\int\limits_{\Gamma^1}\nu(\sigma)\frac{\cos\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}d\sigma+$$

$$(7a)\qquad -\frac{1}{2\pi}\int\limits_{\Gamma^1}\mu(\sigma)\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}d\sigma+$$

$$+\frac{1}{2\pi}\int\limits_{\Gamma^2}\mu(\sigma)\frac{\cos\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}d\sigma=F^{\pm}(s),\qquad s\in\Gamma^1,$$

$$\frac{1}{2\pi}\int\limits_{\Gamma^1}\nu(\sigma)\frac{\cos\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}d\sigma+$$

$$(7b)\qquad -\frac{1}{2\pi}\int\limits_{\Gamma^1}\mu(\sigma)\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}d\sigma+$$

$$-\frac{1}{2}\mu(s)+\frac{1}{2\pi}\int\limits_{\Gamma^2}\mu(\sigma)\frac{\cos\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}d\sigma=F(s),\qquad s\in\Gamma^2.$$

By $\varphi_0(x,y)$ we denote the angle between the vector $\overrightarrow{xy}$ and the direction of the normal $\mathbf{n}_x$. The angle $\varphi_0(x,y)$ is taken to be positive if it is measured anticlockwise from $\mathbf{n}_x$ and negative if it is measured clockwise from $\mathbf{n}_x$. Besides, $\varphi_0(x,y)$ is continuous in $x,y\in\Gamma$ if $x\neq y$. Note, that for $x(s),y(\sigma)\in\Gamma$ and $x\neq y$ we have the relationships

$$\frac{\partial}{\partial\mathbf{n}_x}\ln|x(s)-y(\sigma)|=\frac{\partial}{\partial\tau_x}V(x(s),\sigma)=\frac{\partial}{\partial s}V(x(s),\sigma)=$$
$$=-\frac{\cos\varphi_0\left(x(s),y(\sigma)\right)}{|x(s)-y(\sigma)|}=-\frac{\sin\left(V(x(s),\sigma)-\alpha(s)\right)}{|x(s)-y(\sigma)|},$$

$$\frac{\partial}{\partial\mathbf{n}_x}V(x(s),\sigma)=-\frac{\partial}{\partial\tau_x}\ln|x(s)-y(\sigma)|=-\frac{\partial}{\partial s}\ln|x(s)-y(\sigma)|=$$
$$=\frac{\sin\varphi_0\left(x(s),y(\sigma)\right)}{|x(s)-y(\sigma)|}=-\frac{\cos\left(V(x(s),\sigma)-\alpha(s)\right)}{|x(s)-y(\sigma)|},$$

where $\alpha(s)$ is the inclination of the tangent τ_x to the Ox_1 axis, and $V(x,\sigma)$ is the kernel of the angular potential from (4).

The third term in ($7a$) is a Cauchy singular integral. Equation ($7a$) is obtained as $x \to x(s) \in (\Gamma^1)^{\pm}$ and comprises two integral equations. The upper sign denotes the integral equation on $(\Gamma^1)^+$, the lower sign denotes the integral equation on $(\Gamma^1)^-$.

In addition to the integral equations written above we have the conditions (5).

Subtracting the integral equations ($7a$) we find

$$\nu(s) = \left(F^+(s) - F^-(s)\right) \in C^{0,\lambda}(\Gamma^1). \tag{8}$$

We note that $\nu(s)$ is found completely and satisfies all required conditions. Hence, the potential $v_1[\nu](x)$ is found completely as well.

We introduce the function $f(s)$ on Γ by the formula

$$f(s) = F(s) - \frac{1}{2\pi}\int\limits_{\Gamma^1} \left(F^+(\sigma) - F^-(\sigma)\right) \frac{\cos\varphi_0\left(x(s), y(\sigma)\right)}{|x(s) - y(\sigma)|} d\sigma, \quad s \in \Gamma, \tag{9}$$

where

$$F(s) = \frac{1}{2}\left(F^+(s) + F^-(s)\right), \quad s \in \Gamma^1,$$

and $F(s)$ on Γ^2 is specified in the boundary condition ($2b$). As shown in [4], if $s \in \Gamma^1$, then $f(s) \in C^{0,\lambda}(\Gamma^1)$. Hence, $f(s) \in C^{0,\lambda}(\Gamma^1) \cap C^0(\Gamma^2)$.

Adding the integral equations ($7a$) and taking into account ($7b$) we obtain the integral equation for $\mu(s)$ on Γ

$$\begin{gathered} -\frac{1}{2\pi}\int\limits_{\Gamma^1} \mu(\sigma)\frac{\sin\varphi_0(x(s), y(\sigma))}{|x(s) - y(\sigma)|} d\sigma + \\ -\frac{1}{2}\delta(s)\mu(s) + \frac{1}{2\pi}\int\limits_{\Gamma^2} \mu(\sigma)\frac{\cos\varphi_0\left(x(s), y(\sigma)\right)}{|x(s) - y(\sigma)|} d\sigma = f(s), \quad s \in \Gamma, \end{gathered} \tag{10}$$

where $f(s)$ is given by (9) and

$$\delta(s) = \begin{cases} 0, & \text{if } s \in \Gamma^1 \\ 1, & \text{if } s \in \Gamma^2 \end{cases}$$

Thus, if $\mu(s)$ is a solution of equations (5), (10) from the space $C_q^{\omega}(\Gamma^1) \cap C^0(\Gamma^2)$, $\omega \in (0,1]$, $q \in [0,1)$, then the potential (6) with $\nu(s)$ from (8) satisfies all conditions of the problem **U**.

The following theorem holds.

Theorem 2. *Let* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$ *and conditions* (3) *hold. If the system of equations* (10), (5) *has a solution* $\mu(s)$ *from the Banach space* $C_q^\omega(\Gamma^1) \cap C^0(\Gamma^2)$ *for some* $\omega \in (0,1]$ *and* $q \in [0,1)$, *then a solution of the problem* $\mathbf{U}$ *is given by* (6), *where* $\nu(s)$ *is defined in* (8).

If $s \in \Gamma^2$, then (10) is an equation of the second kind with a weak singularity in the kernel. If $s \in \Gamma^1$, then (10) is a Cauchy singular integral equation of the first kind [5].

Our further treatment will be aimed to the proof of the solvability of the system (5), (10) in the Banach space $C_q^\omega(\Gamma^1) \cap C^0(\Gamma^2)$. Moreover, we reduce the system (5), (10) to a Fredholm equation of the second kind, which can be easily computed by classical methods.

Equation (10) on Γ^2 we rewrite in the form

$$\mu(s) + \int_\Gamma \mu(\sigma) A_2(s,\sigma) d\sigma = -2f(s), \quad s \in \Gamma^2, \tag{11}$$

where

$$\begin{aligned} A_2(s,\sigma) = &-\Big\{-\frac{1}{\pi}\left(1-\delta(\sigma)\right)\frac{\partial}{\partial \mathbf{n}_x} V\left(x(s),\sigma\right) + \\ &-\frac{1}{\pi}\delta(\sigma)\frac{\partial}{\partial \mathbf{n}_x}\ln|x(s)-y(\sigma)|\Big\} = \\ &= -\Big\{-\frac{1}{\pi}(1-\delta(\sigma))\,\frac{\sin\varphi_0\left(x(s),y(\sigma)\right)}{|x(s)-y(\sigma)|} + \frac{1}{\pi}\delta(\sigma)\frac{\cos\varphi_0\left(x(s),y(\sigma)\right)}{|x(s)-y(\sigma)|}\Big\}, \end{aligned}$$

and $V(x,\sigma)$ is the kernel of the angular potential (4). Note, $A_2(s,\sigma) \in C^0(\Gamma^2 \times \Gamma)$, because $\Gamma^2 \in C^{2,0}$.

Remark. Evidently, $f(a_n^2) = f(b_n^2)$, and $A_2(a_n^2,\sigma) = A_2(b_n^2,\sigma)$ for $\sigma \in \Gamma$, $\sigma \neq a_n^2, b_n^2$ $(n = 1,\dots,N_2)$. Hence, if $\mu(s)$ is a solution of equation (11) from $C^0\left(\bigcup_{n=1}^{N_2}[a_n^2,b_n^2]\right)$, then, according to the equality (11), $\mu(s)$ automatically satisfies matching conditions $\mu(a_n^2) = \mu(b_n^2)$ for $n = 1,\dots,N_2$ and, therefore, belongs to $C^0(\Gamma^2)$. This observation is true for equation (10) also and can be helpful in finding numerical solutions, since we may drop matching conditions $\mu(a_n^2) = \mu(b_n^2)$, $(n = 1, ..., N_2)$, which are fulfilled automatically.

It can be easily proved that

$$\frac{\sin\varphi_0(x(s), y(\sigma))}{|x(s) - y(\sigma)|} - \frac{1}{\sigma - s} \in C^{0,\lambda}(\Gamma^1 \times \Gamma^1)$$

(see [3], [4] for details). Therefore we can rewrite (10) on Γ^1 in the form

$$\frac{1}{\pi}\int_{\Gamma^1} \mu(\sigma)\frac{d\sigma}{\sigma - s} + \int_{\Gamma} \mu(\sigma)Y(s,\sigma)d\sigma = -2f(s), \quad s \in \Gamma^1, \tag{12}$$

where

$$Y(s,\sigma) = \Big\{(1 - \delta(\sigma))\,\frac{1}{\pi}\left(\frac{\sin\varphi_0(x(s), y(\sigma))}{|x(s) - y(\sigma)|} - \frac{1}{\sigma - s}\right) + \\ + \frac{1}{\pi}\delta(\sigma)\frac{\partial}{\partial \mathbf{n}_x}\ln|x(s) - y(\sigma)|\Big\} \in C^{0,\lambda}(\Gamma^1 \times \Gamma).$$

In the next section we will study the solvability of equations (5), (10). In this section we will prove two lemmas to analyse properties of functions in equation (10).

By the subscript (0) we will denote the Banach spaces of functions $\mathcal{F}(s)$, which satisfy the condition

$$\int_{\Gamma^2} \mathcal{F}(s)ds = 0,$$

for example, $C^0_{(0)}(\Gamma^2)$, $C^0_{(0)}(\Gamma)$, and so on.

LEMMA 1. *If* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$, *and conditions* (3) *hold, then the function* $f(s)$ *from* (9) *belongs to* $C^{0,\lambda}(\Gamma^1) \cap C^0_{(0)}(\Gamma^2)$.

PROOF. As stated above, $f(s) \in C^{0,\lambda}(\Gamma^1) \cap C^0(\Gamma^2)$. So, to prove the lemma we must show, that

$$\int_{\Gamma^2} f(s)ds = 0.$$

We have

$$\int\limits_{\Gamma^2} f(s)ds = \int\limits_{\Gamma^2} F(s)ds - \int\limits_{\Gamma^2} \lim_{x\to x(s)\in\Gamma^2} \frac{\partial}{\partial \mathbf{n}_x} v_1[\nu](x)ds,$$

where $\nu(s)$ is given by (8). As noted above, the function $v_1[\nu](x)$ is harmonic in $\mathcal{D}\backslash\Gamma^1$ and belongs to the class $\mathbf{K}$. Moreover, $v_1[\nu](x) \in C^1(\overline{\mathcal{D}}\backslash\Gamma^1)$. With the help of the property of harmonic functions, which was used in the Theorem 1 to derive condition $(3a)$, we obtain

$$\int\limits_{\Gamma^2} \lim_{x\to x(s)\in\Gamma^2} \frac{\partial}{\partial \mathbf{n}_x} v_1[\nu](x)ds =$$

$$= \int\limits_{\Gamma^1} \left\{ \lim_{x\to x(s)\in(\Gamma^1)^+} \frac{\partial}{\partial \mathbf{n}_x} v_1[\nu](x) - \lim_{x\to x(s)\in(\Gamma^1)^-} \frac{\partial}{\partial \mathbf{n}_x} v_1[\nu](x) \right\} ds =$$

$$= \int\limits_{\Gamma^1} \nu(s)ds = \int\limits_{\Gamma^1} \left(F^+(s) - F^-(s)\right) ds\,.$$

Hence,

$$\int\limits_{\Gamma^2} f(s)ds = \int\limits_{\Gamma^2} F(s)ds - \int\limits_{\Gamma^1} \left(F^+(s) - F^-(s)\right) ds$$

and the lemma is proved since we assume that condition $(3a)$ holds. All limits in the proof exist thanks to smoothness properties of $v_1[\nu](x)$.

Now we prove

LEMMA 2. *Let* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$, $\lambda \in (0,1]$. *If* $\mu(s) \in C^{\omega}_q(\Gamma^1) \cap C^0(\Gamma^2)$, $\omega \in (0,1]$, $q \in [0,1)$, *and conditions* (5) *hold, then the identity holds*

$$\int\limits_{\Gamma^2} \left[\mu(s) + \int\limits_{\Gamma} \mu(\sigma) A_2(s,\sigma) d\sigma \right] ds = 0.$$

The proof is based on the same property of harmonic functions, which was used to derive condition (3a) in Theorem 1. Clearly,

$$\frac{1}{2}\int_{\Gamma^2}\left[\mu(s)+\int_{\Gamma}\mu(\sigma)A_2(s,\sigma)d\sigma\right]ds=$$
$$=\int_{\Gamma^2}\lim_{x\to x(s)\in\Gamma^2}\frac{\partial}{\partial\mathbf{n}_x}w_1[\mu](x)ds+\int_{\Gamma^2}\lim_{x\overset{\mathbf{n}}{\to}x(s)\in\Gamma^2}\frac{\partial}{\partial\mathbf{n}_x}w_2[\mu](x)ds.$$

The second term is equal to zero because $w_2[\mu](x)$ is a harmonic function in the domain $\mathcal{D}$ bounded by Γ^2. Let us show that the first term is also equal to zero. The function $w_1[\mu](x)$ is harmonic in $\mathcal{D}\backslash\Gamma^1$, and for the first term we obtain

$$\int_{\Gamma^2}\lim_{x\to x(s)\in\Gamma^2}\frac{\partial}{\partial\mathbf{n}_x}w_1[\mu](x)ds=$$
$$=\int_{\Gamma^1}\lim_{x\to x(s)\in(\Gamma^1)^+}\frac{\partial}{\partial\mathbf{n}_x}w_1[\mu](x)ds-\int_{\Gamma^1}\lim_{x\to x(s)\in(\Gamma^1)^-}\frac{\partial}{\partial\mathbf{n}_x}w_1[\mu](x)ds=0,$$

since there is no jump of the normal derivative of the angular potential $w_1[\mu](x)$ on Γ^1. The proof is complete. Note, that all limits in the proof exist because our potentials belong to the class $\mathbf{K}$, and, in addition, $w_1[\mu](x)\in C^1(\overline{\mathcal{D}}\backslash\Gamma^1)$.

4 – The Fredholm integral equation and the solution of the problem

Inverting the singular integral operator in (12), we arrive at the following integral equation of the second kind [5]:

$$\mu(s)+\frac{1}{Q_1(s)}\int_{\Gamma}\mu(\sigma)A_0(s,\sigma)d\sigma+\frac{1}{Q_1(s)}\sum_{n=0}^{N_1-1}G_ns^n=$$
$$=\frac{1}{Q_1(s)}\Phi_0(s),\quad s\in\Gamma^1, \tag{13}$$

where

$$A_0(s,\sigma) = -\frac{1}{\pi}\int\limits_{\Gamma^1} \frac{Y(\xi,\sigma)}{\xi - s} Q_1(\xi)d\xi,$$

$$Q_1(s) = \prod_{n=1}^{N_1} \left|\sqrt{s-a_n^1}\sqrt{b_n^1 - s}\right| \text{sign}(s-a_n^1)\ ,$$

$$\Phi_0(s) = \frac{1}{\pi}\int\limits_{\Gamma^1} \frac{2Q_1(\sigma)f(\sigma)}{\sigma - s} d\sigma,$$

and $G_0,...,G_{N_1-1}$ are arbitrary constants.

To derive equations for $G_0,...,G_{N_1-1}$, we substitute $\mu(s)$ from (13) in the conditions (5), then we obtain

$$\int\limits_{\Gamma} \mu(\sigma)l_n(\sigma)d\sigma + \sum_{m=0}^{N_1-1} B_{nm}G_m = H_n, \ \ n = 1,...,N_1\ \ , \tag{14}$$

where

$$l_n(\sigma) = -\int\limits_{\Gamma_n^1} Q_1^{-1}(s)A_0(s,\sigma)ds,$$

$$B_{nm} = -\int\limits_{\Gamma_n^1} Q_1^{-1}(s)s^m ds, \tag{15}$$

$$H_n = -\int\limits_{\Gamma_n^1} Q_1^{-1}(s)\Phi_0(s)ds.$$

By B we denote the $N_1 \times N_1$ matrix with the elements B_{nm} from (15). As shown in [4, Lemma 7], the matrix B is invertible. The elements of the inverse matrix will be called $(B^{-1})_{nm}$. Inverting the matrix B in (14), we express the constants $G_0,...,G_{N_1-1}$ in terms of $\mu(s)$

$$G_n = \sum_{m=1}^{N_1} (B^{-1})_{nm} \left[H_m - \int\limits_{\Gamma} \mu(\sigma)l_m(\sigma)d\sigma \right].$$

We substitute G_n in (13) and obtain the following integral equation for

$\mu(s)$ on Γ^1

$$\mu(s)+\frac{1}{Q_1(s)}\int\limits_{\Gamma}\mu(\sigma)A_1(s,\sigma)d\sigma=\frac{1}{Q_1(s)}\Phi_1(s),\quad s\in\Gamma^1, \tag{16}$$

where

$$A_1(s,\sigma)=A_0(s,\sigma)-\sum_{n=0}^{N_1-1}s^n\sum_{m=1}^{N_1}(B^{-1})_{nm}l_m(\sigma),$$

$$\Phi_1(s)=\Phi_0(s)-\sum_{n=0}^{N_1-1}s^n\sum_{m=1}^{N_1}(B^{-1})_{nm}H_m\ .$$

It can be shown using the properties of singular integrals [2], [5], that $\Phi_0(s)$, $A_0(s,\sigma)$ are Hölder functions if $s\in\Gamma^1$, $\sigma\in\Gamma$. Therefore, $\Phi_1(s)$, $A_1(s,\sigma)$ are also Hölder functions if $s\in\Gamma^1$, $\sigma\in\Gamma$. Consequently, any solution of (16) belongs to $C^{\omega}_{1/2}(\Gamma^1)$, and below we look for $\mu(s)$ on Γ^1 in this space.

We put

$$Q(s)=(1-\delta(s))\,Q_1(s)+\delta(s),\quad s\in\Gamma.$$

Instead of $\mu(s)\in C^{\omega}_{1/2}(\Gamma^1)\cap C^0(\Gamma^2)$ we introduce the new unknown function $\mu_*(s)=\mu(s)Q(s)\in C^{0,\omega}(\Gamma^1)\cap C^0(\Gamma^2)$ and rewrite (11), (16) in the form of one equation

$$\mu_*(s)+\int\limits_{\Gamma}\mu_*(\sigma)Q^{-1}(\sigma)A(s,\sigma)d\sigma=\Phi(s),\quad s\in\Gamma, \tag{17}$$

where

$$A(s,\sigma)=(1-\delta(s))\,A_1(s,\sigma)+\delta(s)A_2(s,\sigma),$$

$$\Phi(s)=(1-\delta(s))\,\Phi_1(s)-2\delta(s)f(s).$$

Thus, the system of equations (5), (10) for $\mu(s)$ has been reduced to the equation (17) for the function $\mu_*(s)$. It is clear from our consideration that any solution of (17) gives a solution of system (5), (10).

As noted above, $\Phi_1(s)$ and $A_1(s,\sigma)$ are Hölder functions if $s\in\Gamma^1$, $\sigma\in\Gamma$. More precisely (see [4], [5]), $\Phi_1(s)\in C^{0,p}(\Gamma^1)$, $p=\min\{1/2,\lambda\}$, and $A_1(s,\sigma)$ belongs to $C^{0,p}(\Gamma^1)$ in s uniformly with respect to $\sigma\in\Gamma$.

We arrive at the following assertion.

Lemma 3. *Let* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$, $\lambda \in (0,1]$, *and* $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0_{(0)}(\Gamma^2)$, *where* $p = \min\{\lambda, 1/2\}$. *If* $\mu_*(s)$ *from* $C^0(\Gamma)$ *satisfies the equation (17), then* $\mu_*(s)$ *belongs to* $C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$.

The condition $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0_{(0)}(\Gamma^2)$ holds if $f(s) \in C^{0,\lambda}(\Gamma^1) \cap C^0_{(0)}(\Gamma^2)$. According to the Lemma 1, $f(s) \in C^{0,\lambda}(\Gamma^1) \cap C^0_{(0)}(\Gamma^2)$ if conditions (3) hold.

Hence below we will seek $\mu_*(s)$ from $C^0(\Gamma)$.

Since $A(s,\sigma) \in C^0(\Gamma \times \Gamma)$, the integral operator from (17):

$$\mathbf{A}\mu_*(s) = \int_\Gamma \mu_*(\sigma) Q^{-1}(\sigma) A(s,\sigma)\, d\sigma \tag{18}$$

is a compact operator mapping $C^0(\Gamma)$ into itself. Therefore, (17) is a Fredholm equation of the second kind in the Banach space $C^0(\Gamma)$.

Let us show that if $\mu^0_*(s)$ is a solution of the homogeneous equation (17) from $C^0_{(0)}(\Gamma)$, then it is a trivial solution, that is $\mu^0_*(s) \equiv 0$. We will prove this by a contradiction. Let $\mu^0_*(s) \in C^0_{(0)}(\Gamma)$ be a non-trivial solution of the homogeneous equation (17). According to the Lemma 3, $\mu^0_*(s) \in C^{0,p}(\Gamma^1) \cap C^0_{(0)}(\Gamma^2)$, $p = \min\{\lambda, 1/2\}$. Therefore the function $\mu^0(s) = \mu^0_*(s) Q^{-1}(s) \in C^p_{1/2}(\Gamma^1) \cap C^0_{(0)}(\Gamma^2)$ converts the homogeneous equations (11), (16) into identities. Using the homogeneous identity (16), we check, that $\mu^0(s)$ satisfies conditions (5). Besides, acting on the homogeneous identity (16) with a singular operator with the kernel $(s-t)^{-1}$, we find that $\mu^0(s)$ satisfies the homogeneous equation (12). Consequently, $\mu^0(s)$ satisfies the homogeneous equation (10).

On the basis of Theorem 2, $u[0,\mu^0](x) = w[\mu^0](x)$ is a solution of the homogeneous problem $\mathbf{U}$. According to Theorem 1: $w[\mu^0](x) \equiv c_0 = const$, $x \in \mathcal{D} \backslash \Gamma^1$. Using the limit formulas for tangent derivatives of an angular potential [1,3], we obtain

$$\lim_{x \to x(s) \in (\Gamma^1)^+} \frac{\partial}{\partial \tau_x} w[\mu^0](x) - \lim_{x \to x(s) \in (\Gamma^1)^-} \frac{\partial}{\partial \tau_x} w[\mu^0](x) = \mu^0(s) \equiv 0, \quad s \in \Gamma^1.$$

Hence, $w[\mu^0](x) = w_2[\mu^0](x) \equiv c_0$, $x \in \mathcal{D}$. Clearly, $w_2[\mu^0](x) \in C^2(R^2 \backslash \Gamma^2) \cap C^0(R^2)$ and the potential $w_2[\mu^0](x)$ satisfies the following Dirichlet problem

$$\Delta w_2 = 0 \quad \text{in } \mathcal{D}_n \, ; \qquad w_2|_{\Gamma^2_n} = c_0 \, ,$$

for $n = 2, ..., N_2$. According to the uniqueness theorem for the Dirichlet problem, $w_2[\mu^0](x) \equiv c_0,\;\; x \in \mathcal{D}_n,\;\; n = 2, ..., N_2$. Using the jump of the normal derivative of the single layer potential $w_2[\mu^0](x)$ on Γ^2, we obtain $\mu^0(s) \equiv 0,\;\; s \in \Gamma^2_n,\;\; n = 2, ..., N_2$. Since in our assumptions the function $\mu^0(s)$ meets the identity

$$\int\limits_{\Gamma^2} \mu^0(s)ds = \int\limits_{\Gamma^2_1} \mu^0(s)ds = 0,$$

the potential $w_2[\mu^0](x)$ satisfies the following external Dirichlet problem in $\mathcal{D}_1$

$$\Delta w_2 = 0 \;\;\text{in}\;\; \mathcal{D}_1; \quad w_2|_{\Gamma^2_1} = c_0; \quad |w_2| < Const,$$

which has unique solution $w_2[\mu^0](x) \equiv c_0,\;\; x \in \mathcal{D}_1$. Using the jump of the normal derivative of $w_2[\mu^0](x)$ on Γ^2_1, we obtain $\mu^0(s) \equiv 0,\;\; s \in \Gamma^2_1$, and therefore $\mu^0(s) \equiv 0,\;\; s \in \Gamma$.

Consequently, if $s \in \Gamma$, then $\mu^0(s) \equiv 0$, $\mu^0_*(s) = \mu^0(s)Q^{-1}(s) \equiv 0$, and we arrive at the contradiction to the assumption that $\mu^0_*(s)$ is a non-trivial solution of the homogeneous equation (17). Thus, the homogeneous Fredholm equation (17) has only a trivial solution in $C^0_{(0)}(\Gamma)$.

We have proved the following assertion.

THEOREM 3. *If* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$, $\lambda \in (0,1]$, *then* (17) *is a Fredholm equation of the second kind in the space* $C^0(\Gamma)$. *Moreover, if* $\mu^0_*(s) \in C^0_{(0)}(\Gamma)$ *is a solution of the homogeneous equation* (17), *then it is a trivial solution, that is* $\mu^0_*(s) \equiv 0,\;\; s \in \Gamma$.

If Γ^1 is absent, then the homogeneous equation (17) has a non-trivial solution $r(s)$, which is known as a Roben density.

In our case, when the boundary contains open curves Γ^1 in $\mathcal{D}$, equation (17) has a non-trivial solution $r_0(s)$, which is equal to zero on Γ^1 and is equal to the Roben density on Γ^2, that is $r_0(s) = \delta(s)r(s)$.

Consequently, the operator of equation (17) is not invertible in $C^0(\Gamma)$. This is not convenient for practical purposes, when we need numerical solution. Instead of equation (17) we consider the following equation

$$\mu_*(s) + \hat{\mathbf{A}}\mu_*(s) = \Phi(s)\,, \quad s \in \Gamma\,, \tag{19}$$

where

$$\hat{\mathbf{A}}\mu_*(s) = \mathbf{A}\mu_*(s) + \delta(s)\int\limits_{\Gamma^2}\mu_*(\sigma)d\sigma,$$

and $\mathbf{A}$ is the integral operator from (18).

Obviously, (19) is a Fredholm equation of the second kind in $C^0(\Gamma)$. Now we prove the following statement.

LEMMA 4. *Let* $\Gamma^1 \in C^{2,\lambda},\ \Gamma^2 \in C^{2,0},\ \lambda \in (0,1]$.

1) *The Fredholm equation* (19) *is uniquely solvable in* $C^0(\Gamma)$ *for any* $\Phi(s) \in C^0(\Gamma)$.
2) *If* $\Phi(s) \in C^{0,p}(\Gamma^1)\cap C^0(\Gamma^2)$, *where* $p = \min\{\lambda, 1/2\}$, *then the solution of equation* (19) *in* $C^0(\Gamma)$ *belongs to* $C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$.
3) *If* $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0_{(0)}(\Gamma^2)$, *then the solution of equation* (19) *in* $C^0(\Gamma)$ *satisfies equation* (17).

PROOF.

1) Since (19) is a Fredholm equation, we must prove that the homogeneous equation (19) has only a trivial solution. We give a proof by a contradiction. Let $\mu^0_*(s) \in C^0(\Gamma)$ be a non-trivial solution of the homogeneous equation (19). Repeating the proof of the Lemma 3, we can show that $\mu^0_*(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{\lambda, 1/2\}$ and therefore $\mu^0(s) = \mu^0_*(s)Q^{-1}(s) \in C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$. Besides, $\mu^0(s)$ satisfies the homogeneous equation (16), because (19) transforms into (16) if $s \in \Gamma^1$. With the help of the homogeneous identity (16) we check, that $\mu^0(s)$ satisfies conditions (5). Integrating the homogeneous identity (19) on Γ^2 and using Lemma 2, we have

$$\int\limits_{\Gamma^2}\left[\mu^0_*(s) + \hat{\mathbf{A}}\mu^0_*(s)\right]ds =$$

$$= \int\limits_{\Gamma^2}\left[\mu^0(s) + \int\limits_{\Gamma}\mu^0(\sigma)A_2(s,\sigma)d\sigma + \int\limits_{\Gamma^2}\mu^0_*(\sigma)d\sigma\right]ds =$$

$$= \int\limits_{\Gamma^2} 1ds \int\limits_{\Gamma^2}\mu^0_*(s)ds = 0.$$

From the latter identity it follows that $\mu^0_*(s)$ belongs to $C^0_{(0)}(\Gamma)$ and satisfies the homogeneous equation (17), because $\hat{\mathbf{A}}\mu^0_*(s) = \mathbf{A}\mu^0_*(s)$.

It follows from the Theorem 3 that $\mu_*^0(s) \equiv 0$, and we arrive at the contradiction to the assumption that $\mu_*^0(s)$ is a nontrivial solution of equation (19). The 1-st point of the lemma is proved.

2) If $\mu_*(s)$ is an arbitrary function from $C^0(\Gamma)$, then repeating the proof of the Lemma 3, we show that the integral term in (19) belongs to $C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ in s. If $\mu_*(s) \in C^0(\Gamma)$ is a solution of equation (19) for $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, then it follows from the equality (19) that $\mu_*(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$. The 2-nd point of the lemma is proved.

3) Let $\mu_*(s) \in C^0(\Gamma)$ be a solution of equation (19) with $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0_{(0)}(\Gamma^2)$, $p = \min\{\lambda, 1/2\}$. According to point 2) of this lemma, $\mu_*(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{\lambda, 1/2\}$ and therefore $\mu(s) = \mu_*(s)Q^{-1}(s) \in C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$. If $s \in \Gamma^1$, then the identity (19) transforms to the identity (16) for $\mu(s)$. Based on (16) we check that $\mu(s)$ satisfies conditions (5). We integrate the identity (19) on Γ^2, and with the help of Lemma 2 we obtain

$$\int_{\Gamma^2} \left[\mu_*(s) + \hat{\mathbf{A}}\mu_*(s)\right] ds =$$

$$= \int_{\Gamma^2} \left[\mu(s) + \int_{\Gamma} \mu(\sigma) A_2(s,\sigma) d\sigma + \int_{\Gamma^2} \mu_*(\sigma) d\sigma\right] ds =$$

$$= \int_{\Gamma^2} 1 ds \int_{\Gamma^2} \mu_*(s) ds = 0.$$

Therefore $\hat{\mathbf{A}}\mu_*(s) = \mathbf{A}\mu_*(s)$, and $\mu_*(s)$ is a solution of (17). The lemma is proved.

As a consequence of the Lemma 4 we obtain the corollary.

Corollary. *Let* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$, $\lambda \in (0,1]$. *If* $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0_{(0)}(\Gamma^2)$, *where* $p = \min\{\lambda, 1/2\}$, *then equation* (17) *has a solution* $\mu_*(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, *which is a unique solution of the Fredholm equation* (19) *in* $C^0(\Gamma)$.

We recall that $\Phi(s)$ belongs to the class of smoothness required in the corollary if $f(s) \in C^{0,\lambda}(\Gamma^1) \cap C^0_{(0)}(\Gamma^2)$.

As mentioned above, if $\mu_*(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ is a solution of (17), then $\mu(s) = \mu_*(s)Q^{-1}(s) \in C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ is a solution of system (5), (10). We obtain the following statement.

THEOREM 4. *If* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$, $f(s) \in C^{0,\lambda}(\Gamma^1) \cap C^0_{(0)}(\Gamma^2)$, $\lambda \in (0,1]$, *then the system of equations* (5), (10) *has a solution* $\mu(s) \in C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{1/2, \lambda\}$, *which is expressed by the formula* $\mu(s) = \mu_*(s)Q^{-1}(s)$, *where* $\mu_*(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ *is the unique solution of the Fredholm equation* (19) *in* $C^0(\Gamma)$.

We note, that the function $\mu_*(s)$ mentioned in the Theorem 4 satisfies equation (17), though this solution is not unique for (17). According to the Lemma 1, if conditions (3) hold, then $f(s) \in C^{0,\lambda}(\Gamma^1) \cap C^0_{(0)}(\Gamma^2)$, and so the Theorem 4 is true.

On the basis of the Theorems 2, 4 we arrive at the final result.

THEOREM 5. *If* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$, *and conditions* (3) *hold, then the solution of the problem* **U** *exists and is given by* (6), *where* $\nu(s)$ *is defined in* (8) *and* $\mu(s)$ *is a solution of equations* (5), (10) *from* $C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{1/2, \lambda\}$ *ensured by the Theorem 4.*

According to the Theorem 1, the solution of the problem **U** is defined up to an arbitrary additive constant.

It can be checked directly that the solution of the problem **U** satisfies condition (1) with $\epsilon = -1/2$. Explicit expressions for singularities of the solution gradient at the end-points of the open curves can be easily obtained with the help of formulas presented in [4].

Theorem 5 ensures existence of a classical solution of the problem **U** when $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$, and conditions (3) hold. On the basis of our consideration we suggest the following scheme for solving the problem **U**. First, we find the unique solution $\mu_*(s)$ of the Fredholm equation (19) from $C^0(\Gamma)$. This solution satisfies Fredholm equation (17) and automatically belongs to $C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{\lambda, 1/2\}$. Second, we construct the solution of equations (5), (10) from $C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ by the formula $\mu(s) = \mu_*(s)Q^{-1}(s)$. Finally, substituting $\nu(s)$ from (8) and $\mu(s)$ in (6), we obtain the solution of the problem **U**.

5 – Additional remarks

Consider the integral equation

$$
\begin{aligned}
&-\frac{1}{2\pi}\int\limits_{\Gamma^1}\mu(\sigma)\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}d\sigma-\frac{1}{2}\delta(s)\mu(s)+\\
&+\frac{1}{2\pi}\int\limits_{\Gamma^2}\mu(\sigma)\frac{\cos\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}d\sigma-\frac{1}{2}\delta(s)\int\limits_{\Gamma^2}\mu(\sigma)d\sigma=\\
&=f(s),\quad s\in\Gamma,
\end{aligned}
\tag{20}
$$

which differs from (10) only by the term

$$
-\frac{1}{2}\delta(s)\int\limits_{\Gamma^2}\mu(\sigma)d\sigma.
$$

Let us show, that the system of equations (5), (20) is equivalent to the equation (19). Indeed, if $s\in\Gamma^2$, then $\mu(s)=\mu_*(s)$ and equation (19) coincides with (20). If $s\in\Gamma^1$, then equation (20) coincides with equation (10), which takes the form of equation (12). As stated in the section 4, if $s\in\Gamma^1$, then the system (12), (5) is equivalent to the equation (16), which, in turn, is equivalent to equation (19) on Γ^1.

So, any solution $\mu(s)$ of the system (5), (20) yields a solution $\mu_*(s)=\mu(s)Q(s)$ of equation (19), and, conversely, any solution of (19) gives a solution of (5), (20).

It follows from points 1), 2) of the Lemma 4, that if $f(s)\in C^{0,\lambda}(\Gamma^1)\cap C^0(\Gamma^2)$, then the unique solution $\mu_*(s)$ of equation (19) in $C^0(\Gamma)$ belongs to $C^{0,p}(\Gamma^1)\cap C^0(\Gamma^2)$, $p=\min\{1/2,\lambda\}$. Since the solution $\mu(s)=\mu_*(s)Q^{-1}(s)$ of system (20), (5) is defined in weighted Hölder spaces on Γ^1, we arrive at

Theorem 6. *Let* $\Gamma^1\in C^{2,\lambda}$, $\Gamma^2\in C^{2,0}$, $f(s)\in C^{0,\lambda}(\Gamma^1)\cap C^0(\Gamma^2)$, $\lambda\in(0,1]$. *There exists the unique solution* $\mu(s)$ *of the system* (5), (20) *in* $C^{p_o}_{1/2}(\Gamma^1)\cap C^0(\Gamma^2)$ *for any* $p_o\in(0,p]$, *where* $p=\min\{1/2,\lambda\}$. *This solution automatically belongs to* $C^{p}_{1/2}(\Gamma^1)\cap C^0(\Gamma^2)$ *and the function* $\mu_*(s)=\mu(s)Q(s)$ *is a unique solution of equation* (19) *in* $C^0(\Gamma)$.

It is evident from the Lemma 1 that if conditions (3) hold, then $f(s) \in C^{0,\lambda}(\Gamma^1) \cap C^0_{(0)}(\Gamma^2)$, and so the Theorem 6 is true. According to the Theorem 4, if $f(s) \in C^{0,\lambda}(\Gamma^1) \cap C^0_{(0)}(\Gamma^2)$, then the function $\mu(s)$ mentioned in the Theorem 6 is a solution of the system (5), (10), though this solution is not unique for (5), (10), because the system (5), (10) is equivalent to the equation (17) and a solution of (17) in $C^0(\Gamma)$ is not unique. On the basis of the Theorem 2, we arrive at

Theorem 7. *If* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$, *and conditions* (3) *hold, then the solution of the problem* **U** *is given by* (6), *where* $\nu(s)$ *is defined in* (8) *and* $\mu(s)$ *is a unique solution of the system* (5), (20) *in* $C^{p_o}_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$, $p_o \in (0, p]$, $p = \min\{1/2, \lambda\}$, *ensured by the Theorem 6.*

Theorems 6, 7 propose another way for solving problem **U**, than Theorems 4, 5. The way, suggested in Theorems 6, 7, can be helpful for finding numerical solution of the problem **U**. Indeed, it follows from the Theorem 6, that the numerical solution of the system (20), (5) can be obtained by the direct numerical inversion of the integral operator of this system. In doing so, Hölder functions can be approximated by continuous piecewise linear functions, since they also obey Hölder inequality. Numerical analysis of Cauchy singular integral equations is developed, for example, in [7]. The simplification for numerical solving the system (20), (5) is suggested in the remark to the equation (11).

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