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Continuous Sobolev inner products on the unit circle: canonical models

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RIASSUNTO: *Si studiano le proprietà algebriche ed asintotiche delle successioni di polinomi ortogonali secondo due forme di prodotto interno di tipo Lebesgue-Sobolev. Si tratta delle forme* (1.2) *e* (1.3) *che sono, in certo senso, canoniche.*

ABSTRACT: *We study the algebraic and asymptotic properties for the orthogonal polynomials with respect to Lebesgue-Sobolev inner products of two different types, which, in a certain sense, can be considered canonical.*

1 – Introduction

The theory of orthogonal polynomials on the real line with respect to an inner product of Sobolev type (discrete case) has been widely developped, while the study of the orthogonality with respect to a Sobolev inner product (continuous case) has been done for particular measures, generally classical measures ([12], [9], [14]). Moreover, an important part of the study has been done from the point of view of the coherence of measures (see [8]). The full description of the coherent pairs appears in [13], where the author proves that, at least, one of the measures that constitutes the coherent pair must be classical (Laguerre or Jacobi).

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With regard to the unit circle $\mathbb{T}$, the theory of orthogonal polynomials with respect to Sobolev products has been developped in the discrete case, that is, for inner products of the following type:

$$\langle f, g\rangle = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\mu(\theta) + f(Z)Ag(Z)^H,$$

where $f(Z) = (f(z_1), \ldots, f^{(l_1)}(z_1), \ldots, f(z_m), \ldots, f^{(l_m)}(z_m))$, $|z_i| \geq 1$ and A is a hermitian and positive definite matrix. The algebraic study has been extensively developped and recently asymptotic properties were found: relative asymptotics in [5] and strong asymptotics in [6].

The study of the coherent pairs on the unit circle is introduced in [4]. There it appears the characterization of the normalized Lebesgue measure like the unique Bernstein-Szegö measure which has a coherent companion. On the other hand, the first model in the Sobolev continuous case on the unit circle appears in [10]. It corresponds to the Szegö polynomials z^n which are orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\frac{d\theta}{2\pi} + \int_0^{2\pi} f'(e^{i\theta})\overline{g'(e^{i\theta})}\frac{d\theta}{2\pi},$$

because the sequence of derivatives is also orthogonal with respect to the Lebesgue measure.

As a previous step for the development of the general theory, it is interesting to know explicit examples of orthogonal polynomials for appropriate choices of the measures. Following this idea we have given in [1] a complete characterization of the orthogonal polynomials with respect to the inner product

$$\langle f, g\rangle_s = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\mu(\theta) + \frac{1}{\lambda}\int_0^{2\pi} f'(e^{i\theta})\overline{g'(e^{i\theta})}\frac{d\theta}{2\pi}, \tag{1.1}$$

with $d\mu(\theta) = \frac{d\theta}{|e^{i\theta}-\alpha|^2}$, $|\alpha| < 1$ and $\lambda > 0$.

Taking into account that the better known examples of orthogonal polynomial sequences on the unit circle, in the standard theory, are those corresponding to rational or polynomial modifications of the Lebesgue measure or the addition of a Dirac delta to the Lebesgue measure (see

[7], [15]); in this paper we discuss the properties of the orthogonal polynomials with respect to inner products like (1.1) for these other two different choices of the measure μ, which, in this sense, can be considered canonical.
In inner products like (1.1), the second measure has a very important role. Therefore we begin this study taking for the derivatives the normalized Lebesgue measure in both cases. In this situation generalizations for higher derivatives could be obtained in a natural way.

Case I.

$$\langle f, g\rangle_{s_1} = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\mu_1(\theta) + \frac{1}{\lambda}\int_0^{2\pi} f'(e^{i\theta})\overline{g'(e^{i\theta})}\frac{d\theta}{2\pi} \tag{1.2}$$

with $d\mu_1(\theta) = \frac{|e^{i\theta}-\alpha|^2}{1+|\alpha|^2}\frac{d\theta}{2\pi}$, $\alpha \neq 0$ and $\lambda > 0$.

Case II.

$$\langle f, g\rangle_{s_2} = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\mu_2(\theta) + \frac{1}{\lambda}\int_0^{2\pi} f'(e^{i\theta})\overline{g'(e^{i\theta})}\frac{d\theta}{2\pi} \tag{1.3}$$

with $d\mu_2(\theta) = \frac{d\theta}{2\pi} + d\delta_a(\theta)$, $|a| = 1$ and $\lambda > 0$.

2 – Orthogonal Polynomials related to $\langle\,,\rangle_{s_1}$

Theorem 1. *Let $\{\widetilde{\Phi}_n\}$ be the monic orthogonal polynomial sequence* (MOPS) *with respect to $\langle\,,\rangle_{s_1}$. Then for $n \geq 1$:*

i) $$\widetilde{\Phi}_n(z) = z^n + \alpha_n \widetilde{\Phi}_{n-1}(z)$$

ii) $$\| \widetilde{\Phi}_n \|_{s_1}^2 = 1 + \frac{n^2}{\lambda} - \frac{|c_1|^2}{\| \widetilde{\Phi}_{n-1} \|_{s_1}^2}$$

iii) $$\widetilde{\Phi}_{n+1}(z) = (z + \alpha_{n+1})\widetilde{\Phi}_n(z) - \alpha_n z \widetilde{\Phi}_{n-1}(z)$$

iv) $$\widetilde{\Phi}_{n-1}(z) = \Big((z + \alpha_{n+1})\widetilde{\Phi}_n(z) - \widetilde{\Phi}_{n+1}(z)\Big)\,\frac{1}{\alpha_n z}.$$

with $\| \widetilde{\Phi}_0 \|_{s_1}^2 = 1$, $c_1 = \langle z, 1 \rangle_{\mu_1}$ *and*

$$\alpha_n = -\frac{c_1}{\| \widetilde{\Phi}_{n-1} \|_{s_1}^2}. \tag{2.1}$$

Proof. i) The matrix of moments for $\langle\, , \rangle_{s_1}$ is tridiagonal, symmetric and the elements on the diagonal are $d_{i,i} = 1 + \frac{i^2}{\lambda}$ for $i \geq 0$. Thus $\langle z^n, z^k \rangle_{s_1} = 0$ for $k = 0, \ldots, n-2$ and therefore for $n \geq 1$ there exists α_n such that $\widetilde{\Phi}_n(z) = z^n + \alpha_n \widetilde{\Phi}_{n-1}(z)$. Besides we can compute α_n as follows

$$\begin{aligned} \alpha_n &= -\frac{\langle z^n, \widetilde{\Phi}_{n-1}(z) \rangle_{s_1}}{\| \widetilde{\Phi}_{n-1} \|_{s_1}^2} = -\frac{\langle z^n, \widetilde{\Phi}_{n-1}(z) \rangle_{\mu_1}}{\| \widetilde{\Phi}_{n-1} \|_{s_1}^2} = \\ &= -\frac{\langle z^n, z^{n-1} \rangle_{\mu_1}}{\| \widetilde{\Phi}_{n-1} \|_{s_1}^2} = -\frac{c_1}{\| \widetilde{\Phi}_{n-1} \|_{s_1}^2} \end{aligned}$$

ii) Taking norms in (i, Theorem 1) we get $\| \widetilde{\Phi}_n \|_{s_1}^2 = \| z^n \|_{s_1}^2 - |\alpha_n|^2 \| \widetilde{\Phi}_{n-1} \|_{s_1}^2$. Thus if we use (2.1) we conclude (ii, Theorem 1).
iii) Let us consider (i, Theorem 1) for n and $n+1$. Then if we eliminate z^{n+1} we obtain (iii, Theorem 1).
iv) It is immediate that $\alpha_n = 0$ if and only if $c_1 = 0$, which is equivalent to $\alpha = 0$. Then, since $\alpha \neq 0$, we can deduce the three-term backward descending relation. □

Corollary 1. *It holds that*

$$\widetilde{\Phi}_n(z) = \sum_{k=0}^{n} \left(\prod_{j=k+1}^{n} \alpha_j \right) z^k.$$

Proof. It is straightforward from (i, Theorem 1) using induction. □

Theorem 2. *Let* $\{\widetilde{\Phi}_n\}$ *be the MOPS with respect to* $\langle\, , \rangle_{s_1}$. *Then:*

i) $$\lim_{n \to \infty} \| \widetilde{\Phi}_n \|_{s_1}^2 = \infty.$$

ii) $$\lim_{n \to \infty} \frac{\| \widetilde{\Phi}_n \|_{s_1}^2}{n^2} = \frac{1}{\lambda}.$$

iii) *There exists* $N \in \mathbb{N}$ *such that* $\{\| \widetilde{\Phi}_n \|_{s_1}^2\}_{n=N}^{\infty}$ *is an increasing sequence.*

iv)
$$\lim_{n\to\infty} \Big(\| \widetilde{\Phi}_n \|_{s_1}^2 - \frac{n^2}{\lambda} \Big) = 1.$$

Proof. i) and ii) We apply the extremal property of the MOPS with respect to an inner product. Thus if we denote $c_0 = \langle z^n, z^n \rangle_{\mu_1}$ we have:

$$c_0 + \frac{n^2}{\lambda} = \| z^n \|_{s_1}^2 \geq \| \widetilde{\Phi}_n \|_{s_1}^2 \geq \min_{P(z)=z^n+\dots} \| P \|_{\mu_1}^2 + \frac{1}{\lambda} \min_{P(z)=z^n+\dots} \| P' \|_{\theta}^2 \geq \frac{n^2}{\lambda},$$

and dividing by n^2, we obtain $\frac{c_0}{n^2} + \frac{1}{\lambda} \geq \frac{\|\widetilde{\Phi}_n\|_{s_1}^2}{n^2} \geq \frac{1}{\lambda}$. Therefore we get i) and ii).

iii) It is easy to check that $\frac{n^2}{\lambda} \geq c_0 + \frac{(n-1)^2}{\lambda}$ if and only if $n \geq \frac{\lambda c_0 + 1}{2}$. Thus if we take N the greatest integer smaller than $\frac{\lambda c_0+1}{2}$ we have for $n \geq N$ that

$$\| \widetilde{\Phi}_n \|_{s_1}^2 \geq \frac{n^2}{\lambda} \geq c_0 + \frac{(n-1)^2}{\lambda} \geq \| \widetilde{\Phi}_{n-1} \|_{s_1}^2,$$

which proves iii).

Assertions i), ii) and iii) are valid for Sobolev products like (1.1) where μ is a Borel positive measure with infinite support on $[0, 2\pi]$.

iv) It is a consequence of i) and (ii, Theorem 1). □

Corollary 2. *Let* $\{\alpha_n\}$ *be the sequence in* $(i, Theorem 1)$. *Then*

i) *There exists* N *such that* $\{|\alpha_n|\}_{n\geq N}$ *is decreasing and* $\lim_{n\to\infty} \alpha_n = 0$.

ii) *The sequence* $\{\alpha_n\}_{n\geq N}$ *is on the straight line segment from* 0 *to* $-c_1$.

iii) $\widetilde{\Phi}_n(0) \neq 0$ *for* $n \geq 0$ *and* $\alpha_n = \frac{\widetilde{\Phi}_n(0)}{\widetilde{\Phi}_{n-1}(0)}$ *for* $n \geq 1$.

iv)
$$\alpha_n = \frac{-c_1}{1 + \alpha_{n-1}\overline{c_1} + \frac{(n-1)^2}{\lambda}} \quad \textit{for} \quad n \geq 1 \quad \textit{with} \quad \alpha_0 = 0.$$

Proof. i) and ii) Taking into account that $\alpha_n = -\frac{c_1}{\|\widetilde{\Phi}_{n-1}\|^2_{s_1}}$ for $n \geq 1$ and Theorem 2 we get the assertions.
iii) From (i, Theorem 1) $\widetilde{\Phi}_n(0) = \alpha_n \widetilde{\Phi}_{n-1}(0)$, with $\alpha_1 = -c_1 \neq 0$ and $\widetilde{\Phi}_0(0) = 1$. Since $\widetilde{\Phi}_0(0) \neq 0$ and $\alpha_n \neq 0$ for every n, then $\widetilde{\Phi}_n(0) \neq 0$ and therefore we get (iii, Corollary 2).
iv) From (ii, Theorem 1)

$$\frac{-c_1}{\| \widetilde{\Phi}_n \|^2_{s_1}} = \frac{-c_1}{1 + \frac{n^2}{\lambda} - \frac{|c_1|^2}{\|\widetilde{\Phi}_{n-1}\|^2_{s_1}}}$$

and using (2.1) we obtain (iv, Corollary 2). ∎

Corollary 3.

i) $$\lim_{n\to\infty} \| \widetilde{\Phi}_n(z) - z^n \|^2_{s_1} = 0.$$

ii) $$\lim_{n\to\infty} \| \widetilde{\Phi}_n(z) - z^n \|^2_{\mu_1} = 0 \ \textit{ and } \ \lim_{n\to\infty} \| \widetilde{\Phi}'_n(z) - nz^{n-1} \|^2_{\theta} = 0.$$

Proof.

i) $$\| \widetilde{\Phi}_n(z) - z^n \|^2_{s_1} = \langle \widetilde{\Phi}_n(z) - z^n, \widetilde{\Phi}_n(z) - z^n \rangle_{s_1} =$$
$$= \| z^n \|^2_{s_1} - \| \widetilde{\Phi}_n(z) \|^2_{s_1} = 1 + \frac{n^2}{\lambda} - \| \widetilde{\Phi}_n(z) \|^2_{s_1}.$$

Applying (iv, in Theorem 2) we get i).
ii) It is immediate from i). ∎

Next we see that we can reduce the study of all orthogonal families with the same $|c_1|$ to only one.
Let $c_1 = |c_1|e^{i\theta}$. If we denote the sequence α_n in (i, Theorem 1) by α_{n,c_1}, then

Lemma 1.

$$\alpha_{n,c_1} = e^{i\theta} \alpha_{n,|c_1|}.$$

Proof. We prove by induction that $\alpha_n = -c_1 f(n, |c_1|)$ where f is a function which depends only on n and $|c_1|$.

For $n = 1$, $\alpha_1 = -c_1$ then $f(n, |c_1|) = 1$. If we assume that the result is valid for $n - 1$, by using (iv, Corollary 2) we obtain

$$\begin{aligned}\alpha_{n,c_1} &= \frac{-c_1}{1 + \alpha_{n-1,c_1}\overline{c_1} + \frac{(n-1)^2}{\lambda}} = \\ &= \frac{-c_1}{1 - |c_1|^2 f(n-1, |c_1|) + \frac{(n-1)^2}{\lambda}} = -c_1 f(n, |c_1|) = e^{i\theta}\alpha_{n,|c_1|}. \quad \square\end{aligned}$$

By using the previous lemma we prove that the families of orthogonal polynomials corresponding to different values of c_1 with the same modulus can be related in a very simple way.

Theorem 3. *If we denote* $\widetilde{\Phi}_n(z) = \widetilde{\Phi}_{n,c_1}(z)$ *then it holds*

$$\widetilde{\Phi}_{n,c_1}(z) = e^{in\theta}\widetilde{\Phi}_{n,|c_1|}\Big(\frac{z}{e^{i\theta}}\Big).$$

Proof. From Corollary 1 and taking into account Lemma 1 we get

$$\begin{aligned}\widetilde{\Phi}_{n,c_1}(z) &= z^n + \sum_{k=0}^{n-1}\left(\prod_{j=k+1}^{n}\alpha_{j,c_1}\right) z^k = \\ &= e^{in\theta}\left(\Big(\frac{z}{e^{i\theta}}\Big)^n + \sum_{k=0}^{n-1}\left(\prod_{j=k+1}^{n}\frac{\alpha_{j,c_1}}{e^{i\theta}}\right)\Big(\frac{z}{e^{i\theta}}\Big)^k\right) = \\ &= e^{in\theta}\left(\Big(\frac{z}{e^{i\theta}}\Big)^n + \sum_{k=0}^{n-1}\left(\prod_{j=k+1}^{n}\alpha_{j,|c_1|}\right)\Big(\frac{z}{e^{i\theta}}\Big)^k\right) = e^{in\theta}\widetilde{\Phi}_{n,|c_1|}\Big(\frac{z}{e^{i\theta}}\Big). \quad \square\end{aligned}$$

Corollary 4.

i) $\widetilde{\Phi}_{n,c_1}(z_0) = 0$ *if and only if* $\widetilde{\Phi}_{n,|c_1|}(\frac{z_0}{e^{i\theta}}) = 0$.

ii) *The zeros of* $\widetilde{\Phi}_{n,|c_1|}$ *are symmetric with respect to the real line.*

iii) *The zeros of* $\widetilde{\Phi}_{n,c_1}$ *are symmetric with respect to the straight line segment from* 0 *to* c_1.

Proof. i) It is immediate from Theorem 3 and it means that the zeros of $\widetilde{\Phi}_{n,c_1}$ can be obtained from the zeros of $\widetilde{\Phi}_{n,|c_1|}$ making a rotation $e^{i\theta}$.
ii) Since $\widetilde{\Phi}_{n,|c_1|}$ is a polynomial with real coefficients we get the result.
iii) It is a consequence of i) and ii). □

Theorem 4.
i) *$\widetilde{\Phi}_n$ and $\widetilde{\Phi}_{n-1}$ do not have any common zero.*
ii) *For i and j large enough $\widetilde{\Phi}_i$ and $\widetilde{\Phi}_j$ do not have any common zero.*

Proof. i) Let β be a common zero of $\widetilde{\Phi}_n$ and $\widetilde{\Phi}_{n+1}$. Then, if we use (i, Theorem 1) we have $\widetilde{\Phi}_{n+1}(\beta) = \beta^{n+1} + \alpha_{n+1}\widetilde{\Phi}_n(\beta)$, which implies $\beta = 0$. This yields a contradiction if we take into account iii) in Corollary 2.
ii) It suffices to prove for $\widetilde{\Phi}_{i,-|c_1|}$ and $\widetilde{\Phi}_{j,-|c_1|}$ which are polynomials with positive coefficients. We assume, without loss of generality, $i > j$.

Next we use the following result due to Eneström (see [11]), concerning the boundness of the zeros for polynomials with positive coefficients:

"Let $P(z) = \sum_{k=0}^{n} a_k z^k$, $n \geq 1$ be a polynomial with $a_k > 0$ for all $0 \leq k \leq n$. Setting $a = \min_{0\leq k<n}(\frac{a_k}{a_{k+1}})$ and $b = \max_{0\leq k<n}(\frac{a_k}{a_{k+1}})$, then all zeros of $P(z)$ are contained in the annulus $a \leq |z| \leq b$."

Since in our case $a[\widetilde{\Phi}_j] = \min_{1\leq k\leq j}\{\alpha_k\}$ and $b[\widetilde{\Phi}_j] = \max_{1\leq k\leq j}\{\alpha_k\}$, the zeros of $\widetilde{\Phi}_j$ are in the annulus $a[\widetilde{\Phi}_j] \leq |z| \leq b[\widetilde{\Phi}_j]$.

Applying Corollary 2 we know that for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geq N$ $|\alpha_n| < \varepsilon$ and $\{\alpha_n\}_{n\geq N}$ is a decreasing sequence. Let $\varepsilon_1 > 0$ such that $\varepsilon_1 < \min_{i=1,\ldots,N}\{\alpha_i\}$. Then there exists $j \geq N$ such that for $n \geq j$ $|\alpha_n| < \varepsilon_1$, $\{\alpha_n\}_{n\geq j}$ is a decreasing sequence and $\alpha_j < \alpha_k$ $k = 1,\ldots,j-1$. If we take j in this way then $a[\widetilde{\Phi}_j] = \alpha_j$ and therefore the zeros of $\widetilde{\Phi}_j$ are in $\alpha_j \leq |z| \leq \max_{k=1,\ldots,j}\{\alpha_k\}$.

Now assume there exists β such that $\widetilde{\Phi}_i(\beta) = \widetilde{\Phi}_j(\beta) = 0$ which implies $\alpha_j \leq |\beta|$.

Since

$$\widetilde{\Phi}_i(z)=p(z)+\left(\prod_{k=i}^{j+1}\alpha_k\right)\widetilde{\Phi}_j(z) \quad \text{with} \quad p(z)=z^i+\alpha_i z^{i-1}+\ldots+\alpha_i\ldots\alpha_{j+2}z^{j+1}$$

then $p(\beta) = 0$. Therefore 0 is a zero of $p(z)$ and the other zeros are in the annulus $\min\{\alpha_i,\ldots,\alpha_{j+2}\} \leq |z| \leq \max\{\alpha_i,\ldots,\alpha_{j+2}\}$. This implies $\beta = 0$ or $|\beta| \leq \alpha_{j+2}$, which leads to a contradiction with $\alpha_j \leq |\beta|$. □

Theorem 5. *The zeros of* $\widetilde{\Phi}_n$ *are in the annulus* $\min_{i=1,\dots,n} |\alpha_i| \leq |z| \leq \max_{i=1,\dots,n} |\alpha_i|$.

Proof. Following the same ideas of the previous theorem we obtain the result.

Besides for n large enough $\min_{i=1,\dots,n} |\alpha_i| = |\alpha_n| = \frac{|c_1|}{\|\widetilde{\Phi}_{n-1}\|_{s_1}^2} > \frac{|c_1|}{1+\frac{(n-1)^2}{\lambda}}$, which gives a lower bound for the zeros.

Note that for $\lambda \leq 1$ and $n \geq 2$

$$|\alpha_n| = \frac{|c_1|}{\| \widetilde{\Phi}_{n-1} \|_{s_1}^2} \leq \frac{|c_1|}{\| \widetilde{\Phi}_{n-1} \|_{\mu}^2 + \frac{(n-1)^2}{\lambda}} \leq |c_1|.$$

Since $|\alpha_1| = |c_1| < 1$, then in this particular case the zeros are in the unit disk $D = \{z : |z| < 1\}$. ∎

Theorem 6. *Let* $\delta > 0$, *then* $\lim_{n\to\infty} \frac{\widetilde{\Phi}_n(z)}{z^n} = 1$ *uniformly for* $|z| \geq \delta$.

Proof. From (2.1)

$$|\alpha_n| = \frac{|c_1|}{\| \widetilde{\Phi}_{n-1} \|_{s_1}^2} \leq \frac{|c_1|}{\| \widetilde{\Phi}_{n-1} \|_{\mu}^2 + \frac{(n-1)^2}{\lambda}} < \frac{\lambda |c_1|}{(n-1)^2} \tag{2.2}$$

and for $|z| \geq \delta$

$$\frac{\lambda |c_1|}{|z|} \leq \frac{\lambda |c_1|}{\delta} = \gamma. \tag{2.3}$$

Therefore from Corollary 1 and applying (2.2) and (2.3) we get

$$\Big|\frac{\widetilde{\Phi}_n(z)}{z^n} - 1\Big| = \Big|\sum_{k=0}^{n-1} \prod_{j=k+1}^{n} \Big(\frac{\alpha_j}{z^{n-k}}\Big)\Big| <$$

$$< \frac{\lambda |c_1|}{|z|(n-1)^2} + \frac{\lambda^2 |c_1|^2}{|z|^2 (n-1)^2 (n-2)^2} + \dots$$

$$+\frac{\lambda^{n-1}|c_1|^{n-1}}{|z|^{n-1}(n-1)^2(n-2)^2\dots 1^2}+\frac{\lambda^{n-1}|c_1|^{n}}{|z|^{n}(n-1)^2(n-2)^2\dots 1^2}<$$

$$<\frac{\gamma}{(n-1)^2}+\frac{\gamma^2}{(n-1)^2(n-2)^2}+\ldots+\frac{\gamma^{n-1}}{(n-1)^2(n-2)^2\dots 1^2}+$$

$$+\frac{\gamma^n}{\lambda(n-1)^2(n-2)^2\dots 1^2}.$$

Let K be a natural number such that $(K-1)^2 \le \gamma < K^2$. Then for $K \le i \le n-1$, $\frac{\gamma}{(n-1)^2} \le \frac{\gamma}{i^2} \le \frac{\gamma}{K^2} < 1$ and so we have that $\frac{\gamma}{(n-1)^2}$ is a bound for the first $n-K$ terms. For the last $K-1$ terms $\frac{\gamma^n}{((n-1)!)^2}$ is a bound. Therefore

$$\sum_{k=K}^{n-1}\left(\prod_{j=k}^{n-1}\frac{\gamma^{n-k}}{(n-1)^2\dots k^2}\right)+\sum_{k=1}^{K-1}\left(\prod_{j=k}^{n-1}\frac{\gamma^{n-k}}{(n-1)^2\dots k^2}\right)+$$

$$+\frac{\gamma^n}{\lambda(n-1)^2\dots 1^2}<\frac{(n-K)\gamma}{(n-1)^2}+\frac{K\tau\gamma^n}{((n-1)!)^2}$$

with $\tau=\max\{1,\frac{1}{\lambda}\}$. Since these two last sequences converge to zero, we get the result. $\square$

Corollary 5. *Let $\delta > 0$. For n large enough the zeros of $\widetilde{\Phi}_n$ are in $|z| \le \delta$.*

Proof. It is a consequence of Hurwitz theorem (see [3]). $\square$

3 – Orthogonal polynomials related to $\langle\, ,\rangle_{s_2}$.

In order to study the Sobolev orthogonal polynomials with respect to the inner product $\langle\, ,\rangle_{s_2}$ we recall the following results:
1) Since $d\mu_2(\theta)=\frac{d\theta}{2\pi}+d\delta_a(\theta)$ then

$$\langle z^n,z^m\rangle_{\mu_2}=\langle z^n,z^m\rangle_\theta+a^{n-m}=\begin{cases}2 & \text{for } n=m\\ a^{n-m} & \text{for } n\ne m\end{cases}$$

and the MOPS(μ_2) $\{\Phi_n(z)\}$ is given by

$$\Phi_n(z)=z^n-\frac{a^n}{1+K_{n-1}(a,a)}K_{n-1}(z,a)$$

where $K_n(z,y)$ are the kernels with respect to the normalized Lebesgue measure and $\| \Phi_n \|_{\mu_2}^2 = \frac{1+K_n(a,a)}{1+K_{n-1}(a,a)}$, (see [2]).
2) Let us consider

$$\langle z^n, z^m \rangle_S = \langle z^n, z^m \rangle_\theta + \frac{1}{\lambda} nm \langle z^{n-1}, z^{m-1} \rangle_\theta = \begin{cases} 1 + \frac{n^2}{\lambda} & \text{for } n = m \\ 0 & \text{for } n \neq m \end{cases}$$

Then it holds that $\{z^n\}$ is the MOPS($\langle \, , \rangle_S$) and $\| z^n \|_S^2 = 1 + \frac{n^2}{\lambda}$. Taking into account these facts the inner product $\langle \, , \rangle_{s_2}$ can be written as

$$\langle z^n, z^m \rangle_{s_2} = \langle z^n, z^m \rangle_{\mu_2} + \frac{1}{\lambda} nm \langle z^{n-1}, z^{m-1} \rangle_\theta = \langle z^n, z^m \rangle_S + a^{n-m}$$

and it holds

THEOREM 7. *Let* $\{\widetilde{\Psi}_n\}$ *be the* MOPS($\langle \, , \rangle_{s_2}$). *Then*

i)
$$\widetilde{\Psi}_n(z) = z^n - \frac{a^n}{1 + H_{n-1}(a,a)} H_{n-1}(z,a),$$

with $H_{n-1}(z,y)$ *the kernel related to* $\langle \, , \rangle_S$, *that is* $H_{n-1}(z,y) = \sum_{k=0}^{n-1} \frac{z^k \overline{y}^k}{1+\frac{k^2}{\lambda}}$.

ii)
$$\| \widetilde{\Psi}_n \|_{s_2}^2 = 1 + \frac{n^2}{\lambda} + \frac{1}{1 + H_{n-1}(a,a)}.$$

PROOF. i) It follows from 1) and 2) above.

ii)
$$\begin{aligned} \| \widetilde{\Psi}_n \|_{s_2}^2 &= \langle \widetilde{\Psi}_n, z^n \rangle_{s_2} = \Big\langle z^n - \frac{a^n}{1 + H_{n-1}(a,a)} H_{n-1}(z,a), z^n \Big\rangle_{s_2} = \\ &= \Big\langle z^n - \frac{a^n}{1 + H_{n-1}(a,a)} H_{n-1}(z,a), z^n \Big\rangle_{\mu_2} + \\ &+ \frac{1}{\lambda} \Big\langle n z^{n-1} - \frac{a^n}{1 + H_{n-1}(a,a)} H_{n-1}^{(1,0)}(z,a), n z^{n-1} \Big\rangle_\theta = \\ &= \Big\langle z^n - \frac{a^n}{1 + H_{n-1}(a,a)} H_{n-1}(z,a), z^n \Big\rangle_\theta + \\ &+ |a|^{2n} \Big(1 - \frac{H_{n-1}(a,a)}{1 + H_{n-1}(a,a)} \Big) + \frac{n^2}{\lambda} = \\ &= 1 + \frac{n^2}{\lambda} + \frac{1}{1 + H_{n-1}(a,a)}. \end{aligned}$$

We denote $H_{n-1}^{(1,0)}(z,a) = \frac{\partial}{\partial z} H_{n-1}(z,a)$. □

Corollary 6. *Let* $\{\widetilde{\Psi}_n\}$ *be the* MOPS$(\langle\,,\rangle_{s_2})$. *Then*

i) $$\widetilde{\Psi}_n(z) = z^{n-1}(z-a) + \beta_{n-1}\widetilde{\Psi}_{n-1}(z) \ \ \textit{for} \ \ n \geq 2,$$

with $\beta_{n-1} = a(\frac{1+H_{n-2}(a,a)}{1+H_{n-1}(a,a)})$ *and therefore* $\beta_{n-1} \neq 0$ *for* $n \geq 2$.

ii) $$\widetilde{\Psi}_n(z) = (z+\beta_{n-1})\widetilde{\Psi}_{n-1}(z) - \beta_{n-2}z\widetilde{\Psi}_{n-2}(z) \ \ \textit{for} \ \ n \geq 3.$$

iii) $$\widetilde{\Psi}_n(z) = \sum_{k=1}^{n-1}\left(\prod_{j=k+1}^{n-1}\beta_k\right) z^k(z-a) + \prod_{j=1}^{n-1}\beta_j\Big(z-\frac{a}{2}\Big) \ \ \textit{for} \ \ n \geq 2.$$

iv) $$\widetilde{\Psi}_n(z) = z^n - \frac{\lambda}{1+H_{n-1}(a,a)}\sum_{k=1}^{n}\frac{a^k z^{n-k}}{\lambda+(n-k)^2}.$$

Proof. i) Let us consider (i, Theorem 7) for n and $n-1$

$$\begin{aligned}(1+H_{n-1}(a,a))(\widetilde{\Psi}_n(z)-z^n) &= -a^n H_{n-1}(z,a),\\ a(1+H_{n-2}(a,a))(\widetilde{\Psi}_{n-1}(z)-z^{n-1}) &= -a^n H_{n-2}(z,a).\end{aligned}$$

If we substract we get

$$(1+H_{n-1}(a,a))(\widetilde{\Psi}_n(z)-z^n) - a(1+H_{n-2}(a,a))(\widetilde{\Psi}_{n-1}(z)-z^{n-1}) = \\ = \frac{-az^{n-1}}{1+\frac{(n-1)^2}{\lambda}}$$

and then

$$(1+H_{n-1}(a,a))\widetilde{\Psi}_n(z) - a(1+H_{n-2}(a,a))\widetilde{\Psi}_{n-1}(z) + \\ -(1+H_{n-1}(a,a))(z^n-az^{n-1}) = 0,$$

which implies (i, Corollary 6).
ii) From (i, Corollary 6) $z\widetilde{\Psi}_{n-1}(z) = z^{n-1}(z-a)+\beta_{n-2}z\widetilde{\Psi}_{n-2}(z)$ and substracting from (i, Corollary 6) we have $\widetilde{\Psi}_n(z)-z\widetilde{\Psi}_{n-1}(z) = \beta_{n-1}\widetilde{\Psi}_{n-1}(z)-\beta_{n-2}z\widetilde{\Psi}_{n-2}(z)$, that is (ii, Corollary 6).

iii) (iii, Corollary 6) follows from (i, Theorem 7) and (i, Corollary 6) proceeding by induction on n. Indeed $\widetilde{\Psi}_2(z) = z(z-a) + \beta_1 \widetilde{\Psi}_1(z)$ with $\widetilde{\Psi}_1(z) = z - \frac{a}{2}$. Then if we suppose (iii, Corollary 6) is true for $n-1$ and take into account (i, Corollary 6) we obtain the result for n.
iv) Operating in (iii, Corollary 6) we obtain

$$\begin{aligned}\widetilde{\Psi}_n(z) &= z^n + (\beta_{n-1} - a)z^{n-1} + \beta_{n-1}(\beta_{n-2} - a)z^{n-2} + \dots \\ &+ \beta_{n-1} \dots \beta_2(\beta_1 - a)z - \beta_{n-1} \dots \beta_1 \frac{a}{2}.\end{aligned}$$

Taking into account that

$$\beta_{n-1} \dots \beta_{j+1}(\beta_j - a) = -\frac{\lambda a^{n-j}}{(\lambda + j^2)(1 + H_{n-1}(a,a))} \quad j = 1, \dots, n-2\,,$$

$$\beta_{n-1} - a = -\frac{\lambda a}{(\lambda + (n-1)^2)(1 + H_{n-1}(a,a))}$$

and

$$\beta_{n-1} \dots \beta_1 \frac{a}{2} = \frac{a^n}{1 + H_{n-1}(a,a)},$$

we deduce (iv, Corollary 6). □

Corollary 7.

i) $$\lim_{n\to\infty} \| \widetilde{\Psi}_n \|_{s_2}^2 = \infty \quad \textit{and} \quad \lim_{n\to\infty} \frac{\| \widetilde{\Psi}_n \|_{s_2}^2}{n^2} = \frac{1}{\lambda}.$$

ii) *There exists N such that for $n \geq N$ $\{\| \widetilde{\Psi}_n \|_{s_2}^2\}_{n\geq N}$ is increasing.*

Proof. Although the result in Theorem 2 is also valid in this particular case, it is easy to obtain i) directly as follows. Since

$$\lim_{n\to\infty} H_{n-1}(a,a) = \lim_{n\to\infty} \lambda \sum_{k=0}^{n-1} \frac{1}{\lambda + k^2} = H \in \mathbb{R}^+,$$

then if we take limits in (ii, Theorem 7) we get

$$\begin{aligned}\lim_{n\to\infty} \| \widetilde{\Psi}_n \|_{s_2}^2 = \infty \text{ and } \lim_{n\to\infty} \frac{\| \widetilde{\Psi}_n \|_{s_2}^2}{n^2} = \lim_{n\to\infty} \Big(\frac{1}{n^2} + \frac{1}{\lambda} + \frac{1}{H_{n-1}(a,a)n^2}\Big) = \\ = \frac{1}{\lambda}.\end{aligned}$$ □

THEOREM 8. *Let* $\{\widetilde{\Psi}_n\}$ *be the* MOPS $(\langle\, ,\rangle_{s_2})$ *and let* $\{\beta_n\}$ *be the sequence defined in Corollary* 6. *Then*

i) $$\widetilde{\Psi}_n(a) = -\widetilde{\Psi}_n(0) = \frac{a^n}{1 + H_{n-1}(a,a)}.$$

ii) $$\beta_n = \frac{\widetilde{\Psi}_{n+1}(a)}{\widetilde{\Psi}_n(a)} = \frac{\widetilde{\Psi}_{n+1}(0)}{\widetilde{\Psi}_n(0)}.$$

iii) $|\beta_n| < 1$ *for* $n \geq 1, \{|\beta_n|\}_{n\geq 1}$ *is increasing and* $\lim_{n\to\infty} \beta_n = a.$

PROOF. i) In (i, Theorem 7) taking $z = a$ and $z = 0$ we obtain $\widetilde{\Psi}_n(a) = \frac{a^n}{1+H_{n-1}(a,a)}$ and $\widetilde{\Psi}_n(0) = -\frac{a^n}{1+H_{n-1}(a,a)} H_{n-1}(0,a) = -\widetilde{\Psi}_n(a)$.

ii) From i) $\widetilde{\Psi}_n(a) \neq 0 \;\; n \geq 0$ and from (i, Corollary 6) we get ii).

iii) From Corollary 6 $|\beta_n| = |a|\frac{|1+H_{n-1}(a,a)|}{|1+H_n(a,a)|} < |a|$. Moreover $|\beta_{n-1}| < |\beta_n|$ if and only if $\frac{1+H_{n-2}(a,a)}{1+H_{n-1}(a,a)} < \frac{1+H_{n-1}(a,a)}{1+H_n(a,a)}$, and this last inequality is equivalent to $(\lambda + n^2)(1 + H_n(a,a)) > (\lambda + (n-1)^2)(1 + H_{n-1}(a,a))$, which is true. □

In order to obtain a result about the situation of the zeros of $\widetilde{\Psi}_n$ we first prove that we can reduce the study to the case in which $a = 1$.

Let us denote $\widetilde{\Psi}_n$ by $\widetilde{\Psi}_{n,a}$. If $a = 1$ we write $\widetilde{\Psi}_{n,1}$.Then

THEOREM 9.

i) $\widetilde{\Psi}_{n,a}(z) = a^n \widetilde{\Psi}_{n,1}(\frac{z}{a})$.

ii) *If* β *is a zero of* $\widetilde{\Psi}_{n,1}$ *then* $a\beta$ *is a zero of* $\widetilde{\Psi}_{n,a}$.

iii) *The zeros of* $\widetilde{\Psi}_{n,a}$ *are symmetric with respect to the straight line segment from* 0 *to* a.

iv) $\widetilde{\Psi}_{n,a}$ *and* $\widetilde{\Psi}_{n-1,a}$ *do not have any common zero.*

Proof. i) Applying (iv, Corollary 6)

$$\widetilde{\Psi}_{n,a}(z) = a^n\left(\left(\frac{z}{a}\right)^n - \frac{\lambda}{1+H_{n-1}(1,1)}\sum_{k=1}^{n}\frac{1}{\lambda+(n-k)^2}\left(\frac{z}{a}\right)^{n-k}\right) = \\ = a^n\widetilde{\Psi}_{n,1}\left(\frac{z}{a}\right)$$

ii) It is an immediate consequence of i).
iii) Since $\widetilde{\Psi}_{n,1}$ has real coefficients, its zeros are symmetric with respect to the real line and therefore from ii) we get iii).
iv) Assume β is a common zero of $\widetilde{\Psi}_{n,a}$ and $\widetilde{\Psi}_{n-1,a}$. Then taking $z=\beta$ in (i, Corollary 6) we deduce $\beta^{n-1}(\beta-a)=0$ which implies $\beta=0$ or $\beta=a$, but this is impossible because $\widetilde{\Psi}_n(a) = -\widetilde{\Psi}_n(0) \neq 0$ for every n. ∎

Theorem 10. *The zeros of $\widetilde{\Psi}_n$ are in $|z|<2$.*

Proof. First we recall the following result:

"If $P(z)=\sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n ($n \geq 1$) such that $a_k - \Lambda a_{k-1} \geq 0$ ($k=1,\dots,n$) for some $\Lambda > 0$, then $P(z)$ has all its zeros in the disk $|z| \leq \frac{a_n - a_0\Lambda^n + |a_0|\Lambda^n}{\Lambda|a_n|}$."

We apply the preceding result to $\widetilde{\Psi}_{n,1}$ which has real coefficients. If we take $\Lambda = 1$ we see that $a_k - a_{k-1} \geq 0$ for $k=1,\dots,n$:
$1+\frac{\lambda}{(1+H_{n-1}(a,a))(\lambda+(n-1)^2)} > 0$ and $-\frac{1}{\lambda+(n-i)^2}+\frac{1}{\lambda+(n-(i+1)^2)} > 0$ for $i = 1,\dots,n-1$. Besides $\frac{a_n-a_0+|a_0|}{|a_n|} = 1+\frac{2}{1+H_{n-1}(1,1)} < 2$ because $H_{n-1}(1,1) = \lambda\sum_{k=0}^{n-1}\frac{1}{\lambda+k^2} > 1$. ∎

Finally we study the asymptotic behavior of $\widetilde{\Psi}_n$.

Theorem 11.

$$\lim_{n\to\infty}\frac{\widetilde{\Psi}_n(z)}{z^n} = 1$$

uniformly on compact subsets of $|z|>1$.

PROOF. Let r and R be positive numbers such that $r > 1$, $R > 1$ and take z such that $r \le |z| \le R$. Given $\varepsilon' > 0$ take $\varepsilon > 0$ such that $\frac{\varepsilon}{r-1} < \frac{\varepsilon'}{2}$. Since $\{\beta_n\} \to a$ there exists $k \in \mathbb{N}$ such that for $n \ge k$ $|\beta_n - a| < \varepsilon$.

Thus, applying Theorem 8

$$\Big|\frac{(\beta_{n-1}-a)}{z} + \frac{\beta_{n-1}(\beta_{n-2}-a)}{z^2} + \ldots + \frac{\beta_{n-1}\ldots\beta_{k+1}(\beta_k - a)}{z^{n-k}}\Big| <$$
$$< \sum_{j=1}^{n-k} \frac{\varepsilon}{|z|^j} < \frac{\varepsilon}{r-1} < \frac{\varepsilon'}{2}.$$

On the other hand there exists $m \in \mathbb{N}$ such that for $n \ge m$ $\frac{1}{r^n} < \frac{\varepsilon'}{4k}$ and so $\frac{1}{|z|^n} < \frac{\varepsilon'}{4k}$ for z such that $r \le |z| \le R$. Thus

$$\Big|\frac{\beta_{n-1}\ldots\beta_k(\beta_{k-1}-a)}{z^{n-k+1}} + \ldots + \frac{\beta_{n-1}\ldots\beta_2(\beta_1 - a)}{z^{n-1}} + \frac{\beta_{n-1}\ldots\beta_1\frac{a}{2}}{z^n}\Big| \le$$
$$\le 2\sum_{j=1}^{k-1}\Big|\frac{1}{z}\Big|^{n-j} + \frac{1}{2}\Big|\frac{1}{z}\Big|^n < \frac{\varepsilon'}{2} \text{ for } n \ge m+k-1.$$

Therefore given $\varepsilon' > 0$ there exists $N = m + k - 1$ such that for $n \ge N$ $|\frac{\widetilde{\Psi}_n(z)}{z^n} - 1| < \varepsilon'$ for $r \le |z| \le R$. □

COROLLARY 8.

i)
$$\lim_{n\to\infty} \frac{\widetilde{\Psi}_n(z)}{z^{n-1}} = z$$

uniformly on compact subsets of $|z| > 1$.

ii)
$$\lim_{n\to\infty} \frac{\widetilde{\Psi}_n(z)}{z^{n-1}(z-a)} = \frac{z}{z-a}$$

uniformly on compact subsets of $|z| > 1$.

PROOF. It is straightforward from Theorem 11 □

COROLLARY 9. *Let* $\varepsilon > 0$. *For* n *large enough the zeros of* $\widetilde{\Psi}_n$ *are in* $|z| < 1 + \varepsilon$.

PROOF. It is a consequence of Hurwitz Theorem. □

Remark 1. As a conclusion we want to remark that there are some important differences between the cases studied:

Although the Sobolev polynomials behave like the orthogonal polynomials with respect to the second measure outside the unit disk; in the first case the result remains true for $|z| \geq \delta$ with $\delta > 0$ (Theorem 6), and it is interesting to note that the Sobolev norm of the difference tends to zero (Corollary 3). In the second case this last result is not true and the asymptotic behavior cannot be extended into the disk.

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