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Logic programming and ultrametric spaces

S. PRIESS-CRAMPE – P. RIBENBOIM

RIASSUNTO: *Con questo articolo di carattere espositorio vogliamo richiamare l'attenzione sugli spazi ultrametrici e le loro applicazioni alla programmazione logica. Presentiamo gli elementi essenziali sulla programmazione logica e diamo un'introduzione alla teoria degli spazi ultrametrici. Per questi ultimi dimostriamo un teorema del punto fisso ed anche un teorema del punto fisso per una mappa a molti valori. Il teorema del punto fisso è usato per derivare un criterio per l'esistenza di un modello di Herbrand per un programma che non è necessariamente positivo.*

ABSTRACT: *By this expository paper we would like to call the attention to ultrametric spaces and their applications to logic programming. We present the essentials of logic programming and give an introduction to the theory of ultrametric spaces. For these, we prove a fixed point theorem and also a multivalued fixed point theorem. The fixed point theorem is used to derive a criterion for the existence of a Herbrand model for a program which is not assumed to be positive.*

– Introduction

This expository paper was written to call the attention to generalized ultrametric spaces and their applications to logic programming.

We aim at two kinds of readers. Those familiar with ultrametric spaces will benefit reading the first four sections devoted to the essentials

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of logic programming. The readers knowledgeable about logic programming may go directly to the fifth section.

We begin with a succinct presentation of the required concepts of first-order language models, truth in models and programs, mostly following [1].

Then we give a leisurely introduction to the theory of generalized ultrametric spaces, including proofs for the convenience of the reader who wishes to learn this new theory. (For more details see [10] or [6] and [7].)

In the final parts of the paper we apply the fixed point theorems for ultrametric spaces in order to derive a criterion for the existence of a fixed valuation for the immediate consequence operator of a program which is not assumed to be positive.

We conclude with a brief look at very recent developments, which refer to stratified and locally stratified programs, as well as to the so-called disjunctive programs.

1 – First-order languages

1.1 – First-order languages: definitions

A *first-order language* is a set L of sequences, defined recursively, of the following kinds of symbols:

(1) Symbols of punctuation: parentheses, commas.
(2) Logical symbols: $\neg$ (negation), $\vee$ (disjunction), $\wedge$ (conjunction), $\rightarrow$ (implication), $\leftrightarrow$ (equivalence), $\exists$ (existential quantifier), $\forall$ (universal quantifier).
(3) Symbols from an infinite set $\mathbb{V}$, whose elements are called *variables* (denoted by x, y, ...).
(4) Symbols from a set $\mathbb{R} = \amalg_{n=1}^{\infty}\mathbb{R}_n$ (where each $\mathbb{R}_n$ is a set which may be empty); each $\underline{r} \in \mathbb{R}_n$ is called a *n-ary relation symbol.*
(5) Symbols from a set $\mathbb{F} = \amalg_{n=1}^{\infty}\mathbb{F}_n$ (where each $\mathbb{F}_n$ is a set which may be empty); each element $\underline{f} \in \mathbb{F}_n$ is called a *function symbol of n arguments.*
(6) Symbols from a set $\mathbb{C}$ whose elements are called *constant symbols.*

The sets $\mathbb{V}$, $\mathbb{R}$, $\mathbb{F}$, $\mathbb{C}$ are assumed pairwise disjoint.

Each element of $\mathbb{L}$ is called an *expression*; certain expressions are called terms, the others are formulas. We indicate how to define recursively the terms.

Terms:

(1) Each variable is a term, each constant symbol is a term.
(2) If $\underline{f} \in \mathbb{F}_n$ and $t_1, \dots, t_n$ are terms, then the sequence $\underline{f}(t_1, \dots, t_n)$ is a term.

Let $\mathbb{T}$ denote the set of terms of $\mathbb{L}$. We define recursively the set of variables $\mathrm{Var}(t)$ of a term t:

(1) $\mathrm{Var}(x) = \{x\}$ for each variable x, $\mathrm{Var}(\underline{c}) = \emptyset$ for each constant symbol $\underline{c}$.
(2) If $\underline{f} \in \mathbb{F}_n$ then $\mathrm{Var}\left(\underline{f}(t_1, \dots, t_n)\right) = \mathrm{Var}(t_1) \cup \ldots \cup \mathrm{Var}(t_n)$.

If $\mathrm{Var}(t) = \emptyset$ then t is called a *ground term*. The set of ground terms is denoted by $\mathrm{G}\mathbb{T}$.

Formulas:

(1) *Atoms* or *atomic formulas*: the sequences $\underline{r}(t_1, \dots, t_n)$, where $t_1, \dots, t_n \in \mathbb{T}$ and $\underline{r} \in \mathbb{R}_n$.
(2) If φ, ψ are formulas, then $\neg\varphi, \varphi\vee\psi, \varphi\wedge\psi, \varphi \to \psi, \varphi \leftrightarrow \psi$ are formulas.

Atomic formulas are also called *positive literals*; formulas $\neg\varphi$, where φ is an atomic formula, are called *negative literals*.

The formulas defined in (1) and (2) are said to be *quantifier-free* formulas. For each quantifier-free formula φ the set of variables $\mathrm{Var}(\varphi)$ is defined recursively:

$\mathrm{Var}\left(\underline{r}(t_1, \dots, t_n)\right) = \mathrm{Var}(t_1) \cup \ldots \cup \mathrm{Var}(t_n), \mathrm{Var}(\neg\varphi) = \mathrm{Var}(\varphi)$, and for $\varphi \vee \psi, \varphi \wedge \psi, \varphi \to \psi, \varphi \leftrightarrow \psi$ the set of variables is $\mathrm{Var}(\varphi) \cup \mathrm{Var}(\psi)$.

The set of atoms is denoted by $\mathbb{A}$, the set of literals is denoted by $\mathbb{L}\mathrm{i}$, the set of quantifier-free formulas is denoted by $\mathrm{QF}\mathbb{F}\mathrm{o}$, so $\mathbb{A} \subset \mathbb{L}\mathrm{i} \subset \mathrm{QF}\mathbb{F}\mathrm{o}$.

Next we define the quantified formulas:

(3) If φ is a formula and $x \in \mathrm{Var}(\varphi)$, $\exists x\varphi$ and $\forall x\varphi$ are *quantified formulas.*

In each formula $\exists x\varphi$, $\forall x\varphi$, the variable x is said to be *bound.*

We define $\mathrm{Var}(\exists x\varphi) = \mathrm{Var}(\varphi)\setminus\{x\}$, $\mathrm{Var}(\forall x\varphi) = \mathrm{Var}(\varphi)\setminus\{x\}$. The variables of $\mathrm{Var}(\exists x\varphi)$ and $\mathrm{Var}(\forall x\varphi)$ are said to be *free variables.*

The set of quantified formulas is denoted by $\mathrm{Q}\mathbb{F}\mathrm{o}$. A formula φ with $\mathrm{Var}(\varphi) = \emptyset$ is called a *sentence*. The set of sentences is denoted by $\mathbb{S}$. An atom which is a sentence is called a *ground atom*. The set of ground atoms is denoted by $\mathrm{G}\mathbb{A}$. $\mathbb{F}\mathrm{o} = \mathrm{QF}\mathbb{F}\mathrm{o} \cup \mathrm{Q}\mathbb{F}\mathrm{o}$ is the set of *formulas* of the language. The language is the set $\mathbb{L} = \mathbb{T} \cup \mathbb{F}\mathrm{o}$.

A sentence $\forall x_1(\forall x_2 \dots (\forall x_n \varphi) \dots)$ (where $\varphi \in \mathbb{F}\mathrm{o}$, $\mathrm{Var}(\varphi) = \{x_1, \dots, x_n\}$) is called a *universal sentence* and it is said to be a *universal closure* of the formula φ. Note that distinct permutations of the set $\mathrm{Var}(\varphi)$ give rise to distinct universal closures of φ.

Similarly, a sentence $\exists x_1(\exists x_2 \dots (\exists x_n \varphi) \dots)$ is called an *existential sentence* and it is said to be an *existential closure* of φ. Distinct permutations of $\mathrm{Var}(\varphi)$ give rise to distinct existential closures of φ.

If $\mathbb{R}$, $\mathbb{F}$, $\mathbb{C}$ are the sets of relation symbols, function symbols, constant symbols of a language $\mathbb{L}$, we write $\mathbb{L} = (\mathbb{R}, \mathbb{F}, \mathbb{C})$. If $\mathbb{R}$, $\mathbb{F}$, $\mathbb{C}$ are empty, the language is *trivial*. If $\mathbb{R} = \emptyset$ the language is said to be *algebraic*.

1.2 – Substitutions and instances

Let $\mathbb{L}$ be a first-order language. A *substitution* is a map $\theta : \mathbb{V} \to \mathbb{T}$ such that the set $\{x \mid x\theta \neq x\}$ is finite. θ is a *ground substitution* if $x\theta \in \mathrm{G}\mathbb{T}$ for all $x \in \mathbb{V}$ such that $x\theta \neq x$.

The substitution θ may be extended recursively to $\mathbb{T} \cup \mathbb{F}\mathrm{o} = \mathbb{L}$ as follows:

- $\underline{c}\theta = \underline{c}$ for every $\underline{c} \in \mathbb{C}$
- if $\underline{f} \in \mathbb{F}_n$ and $t_1, \dots, t_n \in \mathbb{T}$ then $\Big(\underline{f}(t_1, \dots, t_n)\Big)\,\theta = \underline{f}(t_1\theta, \dots, t_n\theta)$.
- if $\underline{r} \in \mathbb{R}_n$ and $t_1, \dots, t_n \in \mathbb{T}$ then $(\underline{r}(t_1, \dots, t_n))\,\theta = \underline{r}(t_1\theta, \dots, t_n\theta)$.
- $(\neg\varphi)\theta = \neg(\varphi\theta)$
- $(\varphi \vee \psi)\theta = \varphi\theta \vee \psi\theta$
- $(\varphi \wedge \psi)\theta = \varphi\theta \wedge \psi\theta$
- $(\varphi \to \psi)\theta = \varphi\theta \to \psi\theta$
- $(\varphi \leftrightarrow \psi)\theta = \varphi\theta \leftrightarrow \psi\theta$
- $(\forall x\varphi)\theta = \forall x(\varphi\theta_x)$
- $(\exists x\varphi)\theta = \exists x(\varphi\theta_x)$ where $\theta_x : \mathbb{V} \to \mathbb{T}$ is the substitution such that $y\theta_x = y\theta$ for all $y \in \mathbb{V}$, $y \neq x$ and $x\theta_x = x$.

If $\varphi \in \mathbb{L}$ then $\varphi\theta$ is called the *θ-instance* of φ. If $\varphi \in \mathbb{L}$ the set $\{\varphi\theta \mid \theta$ is a substitution with $x\theta \in \mathrm{G}\mathbb{T}$ for all $x \in \mathrm{Var}(\varphi)\}$ is called the *Skolem* *transform* of φ and denoted by $Sk(\varphi)$. For any set P of formulas, we write $Sk(P) = \bigcup_{\varphi \in P} Sk(\varphi)$.

2 – Models of a first-order language

2.1 – Models

Let $\mathbb{L} = (\mathbb{R}, \mathbb{F}, \mathbb{C})$ be a first-order language. A *model* of $\mathbb{L}$ is a pair $(\mathcal{A}, \lambda)$ where:

(1) $\mathcal{A} = (A, \mathcal{R}, \mathcal{F}, \mathcal{C})$ where A is a non-empty set, $\mathcal{R}$ is a set of relations on A, $\mathcal{F}$ is a set of functions $f : A^n \to A$ (with $n \geq 1$, n depending on f), $\mathcal{C} \subseteq A$

(2) $\lambda : \mathbb{R} \amalg \mathbb{F} \amalg \mathbb{C} \to \mathcal{R} \amalg \mathcal{F} \amalg \mathcal{C}$ is a mapping such that $\lambda(\mathbb{R}) = \mathcal{R}$, $\lambda(\mathbb{F}) = \mathcal{F}$, $\lambda(\mathbb{C}) = \mathcal{C}$ and

   (a) If $\underline{r} \in \mathbb{R}_n$ then $\lambda(\underline{r})$ is a n-ary relation on A, that is $\lambda(\underline{r}) \subseteq A^n$.
   (b) If $\underline{f} \in \mathbb{F}_n$ then $\lambda(\underline{f})$ is a function with n arguments, $\lambda(\underline{f}) : A^n \to A$.

λ is called the *interpretation mapping* of the model $(\mathcal{A}, \lambda)$, A is the *universe* of the model $(\mathcal{A}, \lambda)$. The restrictions of λ to $\mathbb{R}$, $\mathbb{F}$, $\mathbb{C}$ are denoted respectively by $\lambda_{\mathbb{R}}$, $\lambda_{\mathbb{F}}$, $\lambda_{\mathbb{C}}$.

The interpretation mappings $\lambda_{\mathbb{F}} : \mathbb{F} \to \mathcal{F}, \lambda_{\mathbb{C}} : \mathbb{C} \to \mathcal{C}$ may be canonically extended to a mapping, still denoted λ, $\lambda : \mathrm{G}\mathbb{T} \to A$ which is defined recursively: if $\underline{f} \in \mathbb{F}_n$, if $t_1, \ldots, t_n \in \mathrm{G}\mathbb{T}$ then $\lambda\left(\underline{f}(t_1, \ldots, t_n)\right) = \lambda(\underline{f})\left(\lambda(t_1), \ldots, \lambda(t_n)\right) \in A$.

2.2 – Adjunction of the universe of a model to the constant symbols of the language

If $(\mathcal{A}, \lambda)$ is a model of the language $\mathbb{L} = (\mathbb{R}, \mathbb{F}, \mathbb{C})$, we define the first-order language $\mathbb{L}_{(\mathcal{A},\lambda)}$, written more simply $\mathbb{L}_A = (\mathbb{R}_A, \mathbb{F}_A, \mathbb{C}_A)$ as follows.

The set of variables of $\mathbb{L}_A$ is equal to the set of variables of $\mathbb{L}$, moreover $\mathbb{R}_A = \mathbb{R}$, $\mathbb{F}_A = \mathbb{F}$, $\mathbb{C}_A = \mathbb{C} \amalg A$. We denote by $\mathbb{T}_A$ the set of terms of $\mathbb{L}_A$ and use similar notation for atoms, literals, formulas, etc. ... $\mathbb{L}_A$ is said to be the first-order language obtained from $\mathbb{L}$ by *adjunction of the elements of A as constant symbols*.

We define the model $(\mathcal{A}_A, \lambda_A)$ of $\mathbb{L}_A$ as follows:

(1) $\mathcal{A}_A = (A, \mathcal{R}_A, \mathcal{F}_A, \mathcal{C}_A)$ with $\mathcal{R}_A = \mathcal{R}$, $\mathcal{F}_A = \mathcal{F}$, $\mathcal{C}_A = A$.

(2) $\lambda_A : \mathbb{R}_A \amalg \mathbb{F}_A \amalg \mathbb{C}_A \to \mathcal{R}_A \amalg \mathcal{F}_A \amalg \mathcal{C}_A$ is the mapping whose restrictions to $\mathbb{R}_A = \mathbb{R}$ coincides with $\lambda_{\mathbb{R}}$, whose restriction to $\mathbb{F}_A = \mathbb{F}$ coincides with $\lambda_{\mathbb{F}}$ and whose restriction to $\mathbb{C}_A = \mathbb{C} \amalg A$ is so defined: $\lambda_A(\underline{c}) = \lambda(\underline{c})$, $\lambda_A(a) = a$ for every $\underline{c} \in \mathbb{C}$, $a \in A$.

$(\mathcal{A}_A, \lambda_A)$ is the model of $\mathbb{L}_A$ obtained from $(\mathcal{A}, \lambda)$ by the adjunction of the elements of A as constant symbols to the language.

2.3 – Assignments

Let $(\mathcal{A}, \lambda)$ be a model of the language $\mathbb{L}$. An A-*assignment* is a mapping $\alpha : \mathbb{V} \amalg \mathbb{C} \amalg A \to \mathbb{C} \amalg A$ such that $\underline{c}\alpha = \underline{c}$ for every $\underline{c} \in \mathbb{C}$, $a\alpha = a$ for every $a \in A$, and $x\alpha \in A$ for every $x \in \mathbb{V}$. α may be recursively extended to $\mathbb{L}_A = \mathbb{T}_A \amalg \mathbb{F}\mathrm{o}_A$:

First we extend α to $\mathbb{T}_A$:

- If $\underline{f} \in \mathbb{F}_n$ and $t_1, \dots, t_n \in \mathbb{T}_A$ then $\Big(\underline{f}(t_1, \dots, t_n)\Big)\alpha = \underline{f}(t_1\alpha, \dots, t_n\alpha)$.

Next α may be recursively extended to all formulas of $\mathbb{L}_A$, as follows:

- If $\underline{r} \in \mathbb{R}_n$ and $t_1, \dots, t_n \in \mathbb{T}_A$ then $(\underline{r}(t_1, \dots, t_n))\,\alpha = \underline{r}(t_1\alpha, \dots, t_n\alpha)$
- $(\neg\varphi)\alpha = \neg(\varphi\alpha)$
- $(\varphi \vee \psi)\alpha = \varphi\alpha \vee \psi\alpha$
- $(\varphi \wedge \psi)\alpha = \varphi\alpha \wedge \psi\alpha$
- $(\varphi \to \psi)\alpha = \varphi\alpha \to \psi\alpha$
- $(\varphi \leftrightarrow \psi)\alpha = \varphi\alpha \leftrightarrow \psi\alpha$
- $(\forall x\varphi)\alpha = \forall x(\varphi\alpha_x)$
- $(\exists x\varphi)\alpha = \exists x(\varphi\alpha_x)$

where $\varphi\alpha_x$ is the formula obtained from φ by replacing each variable $y \in \mathrm{Var}(\varphi)$, $y \neq x$, by $y\alpha$.

2.4 – Herbrand models

A model $(\mathcal{H}, \gamma)$ is called a *Herbrand model* if $\mathcal{H} = (\mathrm{G}\mathbb{T}, \mathcal{R}, \mathcal{F}, \mathcal{C})$, where $\mathcal{C} = \mathbb{C}$, $\gamma_{\mathbb{C}}(\underline{c}) = \underline{c}$, if $\underline{f} \in \mathbb{F}_n$ then $\gamma_{\mathbb{F}}(\underline{f}) = f : \mathrm{G}\mathbb{T}^n \to \mathrm{G}\mathbb{T}$, where $f(t_1, \dots, t_n) = \underline{f}(t_1, \dots, t_n)$ for all $t_1, \dots, t_n \in \mathrm{G}\mathbb{T}$. It follows that $\gamma(t) = t$ for every $t \in \mathrm{G}\mathbb{T}$.

It is customary to denote $\mathrm{G}\mathbb{T}$ by HU and call this set the *Herbrand universe*.

(2.1) *There is a natural bijection between the set* $\mathcal{P}(\mathrm{G}\mathbb{A})$ *of all subsets of* $\mathrm{G}\mathbb{A}$ *and the set of Herbrand models of* $\mathbb{L}$.

PROOF. Let $(\mathcal{H}, \gamma)$ be a Herbrand model. Let $\Phi(\mathcal{H}, \gamma)$ be the set of all $\underline{r}(t_1 \ldots, t_n) \in \mathrm{G}\mathbb{A}$, where $\underline{r} \in \mathbb{R}_n$, $t_1, \ldots, t_n \in \mathrm{G}\mathbb{T}$ and $(t_1, \ldots, t_n) \in \gamma(\underline{r}) \subseteq \mathrm{G}\mathbb{T}^n$. It is straightforward to show that Φ is a bijection as required in the statement of the theorem. ∎

By means of the mapping Φ we define an order relation $\leq$ on the set of Herbrand models: $(\mathcal{H}, \gamma) \leq (\mathcal{H}', \gamma')$ whenever $\Phi(\mathcal{H}, \gamma) \subseteq \Phi(\mathcal{H}', \gamma')$.

If $\Phi(\mathcal{H}, \gamma) = \mathrm{G}\mathbb{A}$ then $(\mathcal{H}, \gamma)$ is the *trivial Herbrand model.*

It is convenient to use the notation $(\mathcal{H}_X, \gamma_X)$ for the Herbrand model which corresponds by Φ^{-1} to the subset X of $\mathrm{G}\mathbb{A}$. In the study of Herbrand models it is customary to call $\mathrm{G}\mathbb{A}$ the *Herbrand base* of $\mathbb{L}$, and to use the notation HB.

3 – Truth in a model

Let $\mathbb{L}$ be a first-order language, let $(\mathcal{A}, \lambda)$ be a model of $\mathbb{L}$. We consider the extended language $\mathbb{L}_A$ and define recursively a function $\tau = \tau_{(\mathcal{A},\lambda)} : \mathbb{S}_A \to \{0, 1\}$.

If $\underline{r}(t_1 \ldots, t_n)$ is a ground atom, then $\tau\,(\underline{r}(t_1, \ldots, t_n)) = 1$ if and only if $(\lambda_A(t_1), \ldots, \lambda_A(t_n)) \in \lambda(\underline{r}) \subseteq A^n$.

Next we define, for sentences φ, ψ in $\mathbb{S}_A$:

- $\tau(\neg\varphi) = 1$ if and only if $\tau(\varphi) = 0$;
- $\tau(\varphi \vee \psi) = 1$ if and only if $\tau(\varphi) = 1$ or $\tau(\psi) = 1$;
- $\tau(\varphi \wedge \psi) = 1$ if and only if $\tau(\varphi) = 1$ and $\tau(\psi) = 1$;
- $\tau(\varphi \to \psi) = 1$ if and only if $\tau(\varphi) = 0$ or $\tau(\psi) = 1$;
- $\tau(\varphi \leftrightarrow \psi) = 1$ if and only if $\tau(\varphi) = \tau(\psi)$.

If $\varphi \in \mathbb{F}\mathrm{o}$ with $\mathrm{Var}(\varphi) = \{x\}$, we define:

- $\tau(\exists x\varphi) = 1$ if and only if there exists $a \in A$ such that $\tau(\varphi_a) = 1$;
- $\tau(\forall x\varphi) = 1$ if and only if $\tau(\varphi_a) = 1$ for all $a \in A$
  ($\varphi_a \in \mathbb{S}_A$ denotes the sentence obtained from φ by replacing the free variable x by a).

Now we extend τ to $\mathbb{F}\mathrm{o}_A$ by defining $\tau(\varphi) = 1$ if and only if $\tau(\varphi\alpha) = 1$ for every A-assignment α duly extended to a map (still denoted α) from $\mathbb{F}\mathrm{o}_A$ to $\mathbb{S}_A$. $\tau = \tau_{(\mathcal{A},\lambda)}$ is called the *truth valuation* of the model $(\mathcal{A}, \lambda)$.

If $\varphi \in \mathbb{F}\mathrm{o}_A$ and $\tau(\varphi) = 1$ we say that φ is *satisfied* in $(\mathcal{A}_A, \lambda_A)$ and we write $(\mathcal{A}_A, \lambda_A) \models \varphi$. If $\varphi \in \mathbb{F}\mathrm{o} \subset \mathbb{F}\mathrm{o}_A$ and $\tau(\varphi) = 1$, we say that φ is *satisfied* in $(\mathcal{A}, \lambda)$ and write $(\mathcal{A}, \lambda) \models \varphi$.

If $P \subseteq \mathbb{F}\mathrm{o}_A$, we say that P is *satisfied in* $(\mathcal{A}_A, \lambda_A)$ when $(\mathcal{A}_A, \lambda_A) \models \varphi$ for every $\varphi \in P$. If $P \subseteq \mathbb{F}\mathrm{o} \subset \mathbb{F}\mathrm{o}_A$ and $(\mathcal{A}_A, \lambda_A) \models P$ then we say that P is *satisfied* in $(\mathcal{A}, \lambda)$ and we write $(\mathcal{A}, \lambda) \models P$. When $P \subseteq \mathbb{F}\mathrm{o}$ and $(\mathcal{A}, \lambda) \models P$ we say that $(\mathcal{A}, \lambda)$ is a *model of* P.

Let $P \subseteq Fo$. We say that P is *satisfiable* if P has a model, that is if there exists a model $(\mathcal{A}, \lambda)$ of $\mathbb{L}$ such that $(\mathcal{A}, \lambda) \models P$.

We define a pairing $\bar{\tau} : \mathbb{F}\mathrm{o} \times \mathcal{M} \to \{0, 1\}$ where $\mathcal{M}$ denotes the class of models of the language $\mathbb{L}$, in the following way: $\bar{\tau}(\varphi, (\mathcal{A}, \lambda)) = 1$ when $(\mathcal{A}, \lambda) \models \varphi$; that is $\tau_{(\mathcal{A},\lambda)}(\varphi) = 1$. This pairing leads naturally to the following concepts. The models $(\mathcal{A}, \lambda)$, $(\mathcal{A}', \lambda')$ of $\mathbb{L}$ are said to be *elementarily equivalent* when for any $\varphi \in \mathbb{F}\mathrm{o}$ $(\mathcal{A}, \lambda) \models \varphi$ if and only if $(\mathcal{A}', \lambda') \models \varphi$. The formulas φ, ψ are *logically equivalent* when for each model $(\mathcal{A}, \lambda)$ of $\mathbb{L}$, $(\mathcal{A}, \lambda) \models \varphi$ if and only if $(\mathcal{A}, \lambda) \models \psi$.

It is easy to see that if φ is any formula of $\mathbb{L}$, then φ is logically equivalent to any one of its universal closures.

The formula φ is a *logical consequence* of ψ when the following holds: if $(\mathcal{A}, \lambda)$ is any model of $\mathbb{L}$ such that $(\mathcal{A}, \lambda) \models \psi$ then $(\mathcal{A}, \lambda) \models \varphi$. We write $\psi \vdash \varphi$. More generally, if P is a set of formulas of $\mathbb{L}$, the formula φ is a *logical consequence* of P when the following holds: if $(\mathcal{A}, \lambda)$ is any model of $\mathbb{L}$ and $(\mathcal{A}, \lambda) \models P$ then $(\mathcal{A}, \lambda) \models \varphi$. We write $P \vdash \varphi$.

φ is a *tautology* when $(\mathcal{A}, \lambda) \models \varphi$ for every model $(\mathcal{A}, \lambda)$ of $\mathbb{L}$.

φ is *invalid* when $(\mathcal{A}, \lambda) \models \varphi$ is false for every model $(\mathcal{A}, \lambda)$ of $\mathbb{L}$.

(3.1) *Assume that $\mathbb{L}$ contains at least one constant symbol. Let P be a set of sentences of the form $\forall x_1 \ldots \forall x_n \varphi$, where Var$(\varphi) = \{x_1, \ldots, x_n\}$ with $n \geq 0$. Then P has a model if and only if P has a Herbrand model.*

PROOF. Let $(\mathcal{A}, \lambda)$ be a model of P. Let $\mathrm{Q} = \{\varphi \in \mathrm{G}\mathbb{A} \mid (\mathcal{A}, \lambda) \models \varphi\}$. By (2.1) there exists a Herbrand model $(\mathcal{H}, \gamma)$ such that $\Phi(\mathcal{H}, \gamma) = \mathrm{Q}$. Let $\varphi \in P$ so $(\mathcal{A}, \lambda) \models \varphi$. If φ is quantifier-free, it is easy to show by recursion that $(\mathcal{H}, \gamma) \models \varphi$. Using ground substitutions it may be shown that the above assertion holds for any universal formula $\forall x_1 \ldots \forall x_n \varphi$, where $\mathrm{Var}(\varphi) = \{x_1, \ldots, x_n\}$. $\square$

The following criterion is useful in order to verify that a set P of formulas is satisfied in a model.

(3.2) *Let* $(\mathcal{A}, \lambda)$ *be a model such that* $\lambda(\mathrm{G}\mathbb{T}) = A$ *and let* P *be a set of formulas. Then* $(\mathcal{A}, \lambda) \models P$ *if and only if* $(\mathcal{A}, \lambda) \models \mathrm{Sk}(P)$*.*

4 – Programs

A *program* is a finite set P of formulas, each of the type $C : \varphi \vee (\neg\psi_1) \vee \ldots \vee (\neg\psi_n)$ where $n \geq 0$, φ is an atom and $\psi_1, \ldots, \psi_n$ are literals. Each formula C is called a *clause* of the program. It is customary to write the clause C as follows: $\varphi \leftarrow (\psi_1, \ldots, \psi_n)$. If each ψ_i $(i = 1, \ldots, n)$ is an atom, the clause C is said to be a *positive clause*. If all clauses of the program P are positive, then P is said to be a *positive program*. We note, as already said, that each clause C is logically equivalent to any of its universal closures.

Every program P has a Herbrand model, namely the trivial Herbrand model $(\mathcal{H}_{\mathrm{HB}}, \lambda_{\mathrm{HB}})$, which corresponds to HB. Indeed, if $C = \varphi \vee (\neg\psi_1) \vee \ldots \vee (\neg\psi_m) \in P$, since $(\mathcal{H}_{\mathrm{HB}}, \lambda_{\mathrm{HB}}) \models \varphi$ then also $(\mathcal{H}_{\mathrm{HB}}, \lambda_{\mathrm{HB}}) \models C$. A program P may have more than one Herbrand model.

Let $W = \{0,1\}^{HB}$ be the set of all mappings $w : \mathrm{HB} \to \{0,1\}$; each mapping w is called a *valuation*.

To each valuation w, it is canonically associated the subset $X = \{\varphi \in \mathrm{HB} \mid w(\varphi) = 1\}$ and therefore by (2.1), the Herbrand model $(\mathcal{H}_X, \lambda_X)$. If P is a program, $(\mathcal{H}_X, \lambda_X) \models P$ if and only if for every ground assignment α, $w(\varphi\alpha) = 1$, i. e. $\varphi\alpha \in X$ for all clauses $C = \varphi \vee (\neg\psi_1) \vee \ldots \vee (\neg\psi_n)$ of P.

To each program we associate the *immediate consequence operator* $T_P : \mathcal{P}(\mathrm{HB}) \to \mathcal{P}(\mathrm{HB})$, which is defined as follows: if $X \subseteq \mathrm{HB}$ then $T_P(X)$ is the set of all $\varphi \in \mathrm{HB}$ such that there exists a ground assignment of a clause $C \in P$ which is of the form $\varphi \vee (\neg\psi_1) \vee \ldots \vee (\neg\psi_m)$, $(m \geq 0)$, each ψ_i is a ground literal, moreover if $\psi_i \in \mathrm{HB}$ then $\psi_i \in X$ and if $\psi_i = \neg\psi_i'$ with $\psi_i' \in \mathrm{HB}$, then $\psi_i' \notin X$. Due to the natural isomorphism between $\mathcal{P}(\mathrm{HB})$ and $W = \{0,1\}^{\mathrm{HB}}$ (the set of valuations), the mapping T_P corresponds canonically to a mapping, still denoted by T_P, $T_P : W \to W$.

With this definition we have:

(4.1) *Let P be a program, let $(\mathcal{H}_X, \gamma_X)$ be a Herbrand model of $\mathbb{L}$. Then $(\mathcal{H}_X, \gamma_X) \models P$ if and only if $T_P(X) \subseteq X$.*

PROOF. Let $(\mathcal{H}_X, \lambda_X) \models P$ and $\varphi \in T_P(X)$ so there exists a ground formula $\pi = \varphi \vee (\neg\psi_1) \vee \ldots \vee (\neg\psi_n) \in Sk(P)$ (with $n \geq 0$) such that if $\psi_i \in \mathrm{HB}$ then $\psi_i \in X$ and if $\psi_j = \neg\psi'_j$, with $\psi'_j \in \mathrm{HB}$ then $\psi'_j \notin X$. Since $(\mathcal{H}_X, \lambda_X) \models \pi$ then necessarily $\varphi \in X$, showing that $T_P(X) \subseteq X$.

Conversely, it suffices to show that $(\mathcal{H}_X, \lambda_X) \models \mathrm{Sk}(P)$. If $\pi = \varphi \vee (\neg\psi_1) \vee \ldots \vee (\neg\psi_n) \in \mathrm{Sk}(P)$ (with $n \geq 0$), $\varphi \in \mathrm{HB}$, either $\psi_i \in \mathrm{HB}$ or $\psi_j = \neg\psi'_j$ with $\psi'_j \in \mathrm{HB}$. If some $\psi_i \notin X$ or some $\psi'_j \in X$ then $(\mathcal{H}_X, \lambda_X) \models \pi$. If $\psi_i \in X$ and $\psi'_j \notin X$ (for all i, j) then $\varphi \in T_P(X) \subseteq X$, hence $(\mathcal{H}_X, \lambda_X) \models \pi$. ∎

An interesting special case is when $T_P(X) = X$, that is X is fixed by the immediate consequence operator. We shall discuss the existence of models $(\mathcal{H}_X, \lambda_X)$ such that $T_P(X) = X$.

We shall assume now that P is a positive program.

(4.2) *If P is a positive program, then T_P is a monotone operator, that is if $X \subseteq Y \subseteq \mathrm{HB}$ then $T_P(X) \subseteq T_P(Y)$.*

PROOF. The proof is immediate. ∎

It should be noted that the assertion does not hold in general for a program which is not positive.

The existence of a fixed point $X \in \mathcal{P}(\mathrm{HB})$ for the immediate consequence operator T_P of a positive program is proved using the *fixed point theorem* of Knaster and Tarski. For the convenience of the reader we establish below this useful theorem.

(4.3) *Let L be a complete lattice, let $T : L \to L$ be a monotone mapping. Then there exists a unique minimal $x \in L$ such that $T(x) = x$.*

PROOF. Let u denote the largest element of L, so $T(u) \leq u$. Let $x = \inf\{y \in L \mid T(y) \leq y\}$ – note that x exists because L is a complete lattice. Also from $x \leq y$ for all $y \in L$ with $T(y) \leq y$, it follows that $T(x) \leq T(y) \leq y$ (since T is monotone), hence $T(x) \leq \inf\{y \in L \mid T(y) \leq y\} = x$. Again, $T(T(x)) \leq T(x)$, since T is monotone. By definition of x, $x \leq T(x)$, proving that $x = T(x)$.

If $y = T(y)$ then by definition of x, we have $x \leq y$. This shows that x is the unique minimal fixed point of T. ∎

We apply this theorem to the complete lattice $\mathcal{P}(\mathrm{HB})$ and the monotone operator T_P associated to a positive program P.

(4.4) *If P is a positive program, there exists a unique minimal subset Z of* HB *which is fixed by T_P.*

To the above set Z corresponds the unique minimal Herbrand model $(\mathcal{H}_Z, \gamma_Z)$ for the positive program P.

The unique minimal model may also be obtained as follows.

(4.5) *Let P be a positive program. With above notation, $Z = \langle P \rangle$ where $\langle P \rangle = \{\varphi \in \mathrm{HB} \mid P \vdash \varphi\}$.*

Proof. Clearly $(\mathcal{H}_{\langle P \rangle}, \lambda_{\langle P \rangle})$ is a Herbrand model for P, so $T_P(\langle P \rangle) \subseteq \langle P \rangle$ by (4.1). By the definition of Z (see proof of (4.3)), $Z \subseteq \langle P \rangle$. On the other hand, let $\varphi \in \langle P \rangle$, that is φ is a logical consequence of P. Since $(\mathcal{H}_Z, \gamma_Z) \models P$ then $(\mathcal{H}_Z, \gamma_Z) \models \varphi$ so the ground atom $\varphi \in Z$, proving the inclusion $\langle P \rangle \subseteq Z$. □

The situation for programs which are not positive is not so simple. We shall apply a fixed point theorem for ultrametric spaces to obtain a criterion for a fixed point for the immediate consequence operator of a program.

The next sections concern ultrametric spaces.

5 – Ultrametric spaces

5.1 – Metric spaces and Banach's fixed point theorem

Fitting [2] was able to prove the existence of stable models i. e. fixed points of the immediate consequence operator for certain non-positive programs. For this purpose he applied the classical Banach's fixed point theorem:

(5.1) *Let $(X, d, \mathbb{R}_{\geq 0})$ be a complete metric space, let $f : X \to X$ be a mapping. Assume that there exists a real number α, $0 < \alpha < 1$, such that $d(fx, fy) \leq \alpha\, d(x, y)$ for all x, $y \in X$. Then f has a unique fixed point x_0: $f(x_0) = x_0$.*

In the application of Fitting, it turned out that the metric space was actually an ultrametric space, in the sense of the definition below.

5.2 – Ultrametric spaces and the fixed point theorem

Let $(\Gamma, \leq)$ be a partially ordered set with first element denoted by 0. Let X be a non-empty set. A mapping $d : X \times X \to \Gamma$ is called an *ultrametric distance* on X with values in Γ, when the following conditions are satisfied (for any x, y, $z \in X$, $\gamma \in \Gamma$):

(d1) $d(x, y) = 0$ if and only if $x = y$,
(d2) $d(x, y) = d(y, x)$,
(d3) If $d(x, y) \leq \gamma$ and $d(y, z) \leq \gamma$ then $d(x, z) \leq \gamma$.

The triple (X, d, Γ) is called an *ultrametric space*. To discard the trivial case, we assume that $\Gamma \neq \{0\}$. If $0 < \gamma \in \Gamma$ and $a \in X$ let $B_\gamma(a) = \{x \in X \mid d(x, a) \leq \gamma\}$. These sets are called *balls*.

We note the following properties:

(5.2) *Let* $a, b \in X$, $\alpha, \beta \in \Gamma$ *with* $0 < \alpha \leq \beta$ *and* $a \in B_\beta(b)$. *Then* $B_\alpha(a) \subseteq B_\beta(b)$. *If* $B_\alpha(a)$ *is properly contained in* $B_\beta(b)$ *then* $\beta \not\leq \alpha$.

Proof. If $x \in X$ and $d(x, a) \leq \alpha \leq \beta$, from $d(a, b) \leq \beta$ it follows that $d(x, b) \leq \beta$. Hence, if $\beta \leq \alpha$ then $B_\alpha(a) \subset B_\beta(b) \subseteq B_\beta(a) \subseteq B_\alpha(a)$, and this is impossible. □

The ultrametric space (X, d, Γ) is said to be *spherically complete* when any non-empty chain of balls (with respect to inclusion) has a non-empty intersection.

A map $f : X \to X$ is said to be a *contracting map* when $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$. The mapping is *strictly contracting* when $d(f(x), f(y)) < d(x, y)$ for any $x, y \in X$ such that $x \neq y$. The mapping f is *strictly contracting on orbits* when $d(f(f(x)), f(x)) < d(f(x), x)$ for all $x \in X$ such that $f(x) \neq x$.

We are ready to state and prove the *fixed point theorem* (see [5] or [10]):

(5.3) *Let* (X, d, Γ) *be a spherically complete ultrametric space and* $f : X \to X$.

(1) *If* f *is a contracting map which is strictly contracting on orbits there exists a fixed point* $z \in X : f(z) = z$.
(2) *If* f *is a strictly contracting map then* f *has a unique fixed point.*

Proof. (1) Assume that f has no fixed point, so $\pi_x = d(x, f(x)) \neq 0$ for every $x \in X$. Let $\mathcal{B} = \{B_x \mid x \in X\}$ where $B_x = B_{\pi_x}(x)$. $\mathcal{B}$ is a partially ordered set be inclusion. By Zorn's lemma, there exists a maximal chain $\mathcal{C} \subseteq \mathcal{B}$. Since the space is spherically complete, there exists $z \in \bigcap_{B_x \in \mathcal{C}} B_x$ and we show that $B_z \subseteq B_x$ for all $B_x \in \mathcal{C}$. Indeed, $d(z,x) \leq \pi_x$ since $z \in B_x$, $d(x, f(x)) = \pi_x$, $d(f(x), f(z)) \leq d(x,z) \leq \pi_x$, hence $\pi_z = d(z, f(z)) \leq \pi_x$ and therefore $B_z \subseteq B_x$. This implies that $B_z \in \mathcal{C}$ because $\mathcal{C}$ is a maximal chain, and so B_z is the smallest element of $\mathcal{C}$.

Now $B_{f(z)} \subseteq B_z$ because $d(f(z), z) = \pi_z$ so $f(z) \in B_z$. From $d(f(f(z)), f(z)) < d(f(z), z)$ (since $f(z) \neq z$) then $B_{f(z)} \subseteq B_z$, but actually $z \notin B_{f(z)}$ so the above inclusion is proper. This is a contradiction, since $\mathcal{C}$ is a maximal chain in $\mathcal{B}$.

(2) By (1) there exists $z \in X$ such that $f(z) = z$. If $y \in X$, $y \neq z$, and $f(y) = y$ then $d(z,y) = d(f(z), f(y)) < d(z,y)$, which is impossible. □

In order to deal with programs of a more general kind (the so-called disjunctive programs) it became necessary to consider multi-valued mappings. Nadler [4] proved a fixed point theorem for multi-valued mappings in complete metric spaces. This was extended by Khamsi, Kreinovich and Misane [3] for certain types of metric and even generalized metric spaces, which in fact are ultrametric spaces.

Very recently, we have proved a multi-valued fixed point theorem, which generalizes (5.3).

For a non-empty set X, a mapping $f : X \to \mathcal{P}(X)$ is called a *multi-valued mapping*. It is said to be *non-empty* when $f(x) \neq \emptyset$ for all $x \in X$. If $f(x)$ has exactly one element for each $x \in X$, the mapping is called *single-valued* and in this case f is canonically identified to a mapping from X to X.

The notions of contracting map, strictly contracting map and maps strictly contracting on orbits are generalized as follows for non-empty multi-valued mappings.

The mapping $f : X \to \mathcal{P}(X)$ is said to be *contracting* when for all x, y, $a \in X$ such that $a \in f(x)$, there exists $b \in f(y)$ such that $d(a,b) \leq d(x,y)$.

The map f is *strictly contracting* when for all x, y, $a \in X$ such that $x \neq y$ and $a \in f(x)$, there exists $b \in f(y)$ such that $d(a,b) < d(x,y)$.

The map f is *strictly contracting on orbits* when for every $x \in X$ and $a \in f(x)$, with $a \neq x$, there exists $b \in f(a)$ such that $d(a,b) < d(a,x)$.

If $x \in f(x)$, we say that x is a *fixed point* of f. For single-valued maps these concepts coincide with the ones introduced above.

For each $x \in X$ let $\Pi_x = \{d(x,y) \mid y \in f(x)\}$ and let $\operatorname{Min}\Pi_x$ denote the set (which may be empty) of minimal elements of Π_x.

With these notations, we proved in [8] the *multi-valued fixed point theorem:*

(5.4) *Let (X, d, Γ) be a spherically complete ultrametric space, let $f : X \to \mathcal{P}(X)$ be a non-empty map which is contracting and strictly contracting on orbits. Assume:*

(∗) *For each $x \in X$ the set $\operatorname{Min}\Pi_x$ is finite and every element of Π_x has a lower bound in $\operatorname{Min}\Pi_x$.*

Then f has a fixed point.

(The proof is similar to the one of (5.3).)

We indicate conditions which imply that each set Π_x satisfies the assumption (∗) of the theorem. An ordered set $(\Delta, \leq)$ is said to be *narrow* if every trivially ordered subset is finite. It is said to be *artinian* if it does not contain any infinite strictly descending chain. If $(\Delta, \leq)$ is artinian, every element of Δ has a lower bound which is a minimal element of Δ.

Thus if each set Π_x in (5.4) is assumed to be artinian and narrow, then the hypothesis (∗) of (5.4) holds.

We may also show:

(5.5) *With above notations, assume that Γ is narrow and that for every $x \in X$ the set $f(x)$ is a spherically complete subset of (X, d, Γ). Then each set Π_x satisfies the assumption (∗) of (5.4).*

PROOF. This is trivial if $x \in f(x)$, since then $\operatorname{Min}\Pi_x = \{0\}$. Now we assume that $x \notin f(x)$. Let $\delta \in \Pi_x$ and consider a maximal chain Λ of the set $\{\gamma \in \Pi_x \mid \gamma \leq \delta\}$. Let K be the set of balls $\{B_\lambda(x) \mid \lambda \in \Lambda\}$. So K is a chain and $B_\lambda(x) \cap f(x) \neq \emptyset$ for every $\lambda \in \Lambda$. Since $f(x)$ is spherically complete there exists $z \in \bigcap_{\lambda\in\Lambda} (B_\lambda(x) \cap f(x))$. Then $d(z,x) \in \Pi_x$, $d(z,x) \leq \lambda$ for all $\lambda \in \Lambda$. Since Λ is maximal, then $d(z,x) \in \operatorname{Min}\Pi_x$, $d(z,x) \leq \delta$. Finally, since Γ is narrow, $\operatorname{Min}\Pi_x$ is finite. □

5.3 – Power sets as ultrametric spaces

For our considerations we shall be interested in the following class of ultrametric spaces.

Let U be a non-empty set and $\mathcal{P}(U)$ the set, ordered by inclusion, of all subsets of U. Let $\Gamma = \mathcal{P}(U)$; the empty set its smallest element. Let $d : \mathcal{P}(U) \times \mathcal{P}(U) \to \mathcal{P}(U)$ be defined by $d(R,S) = R \oplus S$ where $R \oplus S$ is the symmetric difference of R, S that is $R \oplus S = (R \cap S') \cup (R' \cap S)$. Then d is an ultrametric distance on $\mathcal{P}(U)$ with values in $\mathcal{P}(U)$; the verification is immediate.

If $R \neq \emptyset$ and S are subsets of U then $B_R(S) = \{X \subseteq U \mid X \oplus S \subseteq R\}$. We see that $B_R(S) = \{X \subseteq U \mid X \cap R' = S \cap R'\}$. Indeed, if $X \oplus S \subseteq R$ then $(X \oplus S) \cap R' = \emptyset$ so $(X \cap R') \oplus (S \cap R') = \emptyset$, so $X \cap R' = S \cap R'$. And conversely, it follows that $B_R(S) = B_R(S \cap R')$.

We note also that if $B_{R_1}(S_1) = B_{R_2}(S_2)$ with $S_1 \subseteq R'_1$ and $S_2 \subseteq R'_2$ then $R_1 = R_2$, $S_1 = S_2$. Indeed, first we show that $R_1 = R_2$. There exists X such that $d(X, S_1) = X \oplus S_1 = R_1$. So $X \in B_{R_2}(S_2)$ and $d(X, S_2) \subseteq R_2$, $d(S_1, S_2) \subseteq R_2$, hence $R_1 = d(X, S_1) \subseteq R_2$; the other inclusion also holds by symmetry. Since $S_1 \in B_{R_1}(S_2)$ then $S_1 = S_1 \cap R'_1 = S_2 \cap R'_1 = S_2$. If $S \subseteq R'$ then $B_R(S) = \{X \subseteq U \mid S \subseteq X \subseteq S \cup R\}$ and this is verified at once. Thus if $S_1 \subseteq R'_1$, $S_2 \subseteq R'_2$ then $B_{R_1}(S_1) \subseteq B_{R_2}(S_2)$ if and only if $S_2 \subseteq S_1 \subseteq S_1 \cup R_1 \subseteq S_2 \cup R_2$.

We are now ready to show:

(5.6) *The ultrametric space $(\mathcal{P}(U), d, \mathcal{P}(U))$ is spherically complete.*

PROOF. Let $\mathcal{C}$ be a chain of balls; by the above considerations, each ball is of the form $B_R(S)$ where $S \subseteq R'$ and S, R are uniquely defined by the ball. Let $S_0 \subseteq U$ be such that $B_{R_0}(S_0) \in \mathcal{C}$. Then $X = \bigcup\{S \mid B_R(S) \in \mathcal{C}\} \subseteq S_0 \cup R_0$ because if $B_{R_0}(S_0) \supseteq B_R(S)$ then $S_0 \subseteq S \subseteq S \cup R \subseteq S_0 \cup R_0$. Since this holds for every $B_{R_0}(S_0) \in \mathcal{C}$ then X belongs to the intersection of all balls in $\mathcal{C}$, proving the statement. ∎

If U is a non-empty set, let $W = \{0,1\}^U$ be the set of all maps from U to $\{0,1\}$. Let $d : W \times W \to \mathcal{P}(U)$ be defined by $d(w,v) = \{x \in U \mid w(x) \neq v(x)\}$. Again, it is easily seen that d is an ultrametric distance on W with values in $\mathcal{P}(U)$. Let $\theta : \mathcal{P}(U) \to W$ be defined by $\theta(R) = v$ where $v(x) = 1$ if and only if $x \in R$. Then θ is a bijection and

$d(\theta(R), \theta(S)) = d(R, S)$, so the mapping θ is an isomorphism from the ultrametric space $(\mathcal{P}(U), d, \mathcal{P}(U))$ to $(W, d, \mathcal{P}(U))$.

It follows from (5.6) that $(W, d, \mathcal{P}(U))$ is also spherically complete. We note also explicitly that the balls of $(W, d, \mathcal{P}(U))$ are the sets $B_R(w) = \{v \in W \mid \text{if } v(x) \neq w(x) \text{ then } x \in R\}$.

We have:

(5.7) *If $f : W \to W$ is a strictly contracting map, then $f(v) = f(w)$ for all $v, w \in W$.*

Proof. Suppose that $v, w \in W$ are such that $f(v) \neq f(w)$. Let $x \in U$ be such that $f(v)(x) \neq f(w)(x)$. Let $t \in W$ be defined by $t(x) = w(x)$, $t(y) = v(y)$ for all $y \in U$, $y \neq x$. Then $d(t, v) \subseteq \{x\}$. Since f is strictly contracting, then $d(f(t), f(v))$ is properly contained in $d(t, v)$, so $f(t) = f(v)$. Thus $d(f(w), f(v)) = d(f(w), f(t)) \subset d(w, t) \subseteq d(w, v) \setminus \{x\}$ hence $f(v)(x) \neq f(w)(x)$, which is a contradiction. ∎

5.4 – Level mappings

Let $(\Delta, \leq)$ be a totally ordered set having smallest element, denoted by 0, and largest element, denoted by 1. We assume that $0 \neq 1$ and also that $(\Delta, \leq)$ is noetherian, that is any non-empty subset of Δ has a maximal element, which is therefore unique. Let U be a non-empty set, let $W = \{0, 1\}^U$ and let Λ be the set of mappings $\lambda : U \to \Delta \setminus \{0\}$. Each mapping λ is called a *level mapping*. Associated to each $\lambda \in \Lambda$ we define $d_\lambda : W \times W \to \Delta$ as follows: $d_\lambda(v, v) = 0$; if $v \neq w$ let $d_\lambda(v, w) = \max\{\lambda(x) \mid v(x) \neq w(x)\}$.

We first show:

(5.8) (W, d_λ, Δ) *is an ultrametric space. If λ is surjective, for every $v \in W$ and $\delta \in \Delta \setminus \{0\}$, there exists $w \in W$ such that $d_\lambda(v, w) = \delta$.*

Proof. Let $d_\lambda(v, w) \leq \delta$ and $d_\lambda(w, t) \leq \delta$. If $x \in U$ and $v(x) \neq t(x)$ then either $v(x) \neq w(x)$ or $w(x) \neq t(x)$, hence $\lambda(x) \leq \delta$. Thus $d_\lambda(v, t) \leq \delta$ and d_λ is an ultrametric distance.

Let $\delta \in \Delta \setminus \{0\}$ and let $x \in U$ be such that $\lambda(x) = \delta$. Let $v \in W$. We define $w \in W$ by $w(x) \neq v(x)$ and $w(y) = v(y)$ for all $y \in U$, $y \neq x$. Then $d_\lambda(v, w) = \lambda(x) = \delta$. ∎

Let $\lambda \in \Lambda$, $v \in W$ and $\delta \in \Delta\backslash\{0\}$. We determine explicitly the ball $B_\delta^{(\lambda)}(v)$ of the ultrametric space (W, d_λ, Δ). We have $w \in B_\delta^{(\lambda)}(v)$ if and only if $d_\lambda(v, w) \leq \delta$; equivalently if $v(x) \neq w(x)$ then $\lambda(x) \leq \delta$. Let $Y = \{x \in U \mid \lambda(x) \leq \delta\}$. If $Y = \emptyset$ then $B_\delta^{(\lambda)}(v) = \{v\}$. If $Y \neq \emptyset$ then $B_\delta^{(\lambda)}(v) = B_Y(v)$ because $w \in B_\delta^{(\lambda)}(v)$ if and only if $d(w, v) \subseteq Y$.

We may now prove:

(5.9) *For every level mapping λ, the ultrametric space (W, d_λ, Δ) is spherically complete.*

Proof. Let $\mathcal{C}$ be a chain of balls of (W, d_λ, Δ). By the above considerations, either each ball of $\mathcal{C}$ is a ball of $(W, d, \mathcal{P}(U))$ or some ball of $\mathcal{C}$ is reduced to a set with only one element v. In the first case, by (5.6), the intersection of the balls of $\mathcal{C}$ is non-empty; in the second case, it is equal to $\{v\}$. ∎

We shall now compare contracting maps of $(W, d, \mathcal{P}(U))$ and of (W, d_λ, Δ). For each $x \in U$ let Λ_x be the set of all mappings $\lambda : U \to \Delta\backslash\{0\}$ such that $\lambda^{-1}(1) = \{x\}$.

We have:

(5.10) *Let $f : W \to W$. The following conditions are equivalent:*

(1) *f is a contracting map with respect to d.*
(2) *f is a contracting map with respect to d_λ, for each $\lambda \in \Lambda$.*
(3) *f is a contracting map with respect to d_λ, for each $\lambda \in \bigcup_{x \in U} \Lambda_x$.*
(4) *For each $x \in U$ there exists $\lambda \in \Lambda_x$ such that f is a contracting map with respect to d_λ.*

Proof. (1) $\Rightarrow$ (2): Let $\lambda \in \Lambda$, $v, w \in W$ and assume that $f(v) \neq f(w)$, otherwise it is trivial. Let $x \in U$ be such that $f(v)(x) \neq f(w)(x)$. Then $x \in d\,(f(v), f(w)) \subseteq d(v, w)$, so $v(x) \neq w(x)$. Thus $\lambda(x) \leq \delta = d_\lambda(v, w)$. This implies that $d_\lambda\,(f(v), f(w)) \leq d_\lambda(v, w)$.

(2) $\Rightarrow$ (3): This is trivial.

(3) $\Rightarrow$ (4): This is also trivial.

(4) $\Rightarrow$ (1): Let v, $w \in W$; we may assume that $f(v) \neq f(w)$, otherwise it is trivial. Let $x \in U$ be such that $f(v)(x) \neq f(w)(x)$. By hypothesis, there exists $\lambda \in \Lambda_x$ such that f is contracting with respect to d_λ. Since $\lambda(x) = 1$, then $1 = d_\lambda(f(v), f(w)) \leq d_\lambda(v, w)$, thus $1 = d_\lambda(v, w)$. From $\lambda^{-1}(1) = \{x\}$ then $v(x) \neq w(x)$. This shows that $d\,(f(v), f(w)) \subseteq d(v, w)$. ∎

Now we consider maps which are strictly contracting on orbits.

(5.11) *Let* $f : W \to W$*. The following conditions are equivalent:*

(1) f *is strictly contracting on orbits with respect to* d*.*

(2) (a) *For every* $\lambda \in \Lambda$ *and* $v \in W$*, we have*

$$d_\lambda(f^2(v), f(v)) \le d_\lambda(f(v), v)\,.$$

(b) *If* $v \in W$ *with* $f(v) \neq v$ *there exists* $x \in U$ *such that*

$$d_\lambda(f^2(v), f(v)) < d_\lambda(f(v), v) = 1 \textit{ for every } \lambda \in \Lambda_x\,.$$

(3) (a) *For every* $\lambda \in \Lambda$ *and* $v \in W$*, we have*

$$d_\lambda(f^2(v), f(v)) \le d_\lambda(f(v), v)\,.$$

(b) *If* $v \in W$ *with* $f(v) \neq v$ *there exists* $x \in U$ *and* $\lambda \in \Lambda_x$ *such that*

$$d_\lambda(f^2(v), f(v)) < d_\lambda(f(v), v)\,.$$

PROOF. (1) $\Rightarrow$ (2):

(a) Let $v \in W$, let $\lambda \in \Lambda$. It is trivial if $f(v) = v$, so let $f(v) \neq v$. If $x \in U$ is such that $f^2(v)(x) \neq f(v)(x)$, since $d(f^2(v), f(v)) \subset d(f(v), v)$ then $x \in d(f(v), v)$; so $\lambda(x) \le d_\lambda(f(v), v)$. This implies that $d_\lambda(f^2(v), f(v)) \le d_\lambda(f(v), v)$.

(b) Let $f(v) \neq v$, then $d(f^2(v), f(v)) \subset d(f(v), v)$. Let $x \in d(f(v), v)$ but $x \notin d(f^2(v), f(v))$. So $d_\lambda(f(v), v) = 1$ for every $\lambda \in \Lambda_x$. Since $\lambda^{-1}(1) = \{x\}$ and $f^2(v)(x) = f(v)(x)$ then $d_\lambda(f^2(v), f(v)) < 1$.

(2) $\Rightarrow$ (3): This is trivial.

(3) $\Rightarrow$ (1): Let $f(v) \neq v$, let $x \in d(f^2(v), f(v))$ and let $\lambda \in \Lambda_x$ be as in the statement of (3). If $f(v)(x) = v(x)$ then $d_\lambda(f^2(v), f(v)) = 1$. Since $\lambda^{-1}(1) = \{x\}$ then $d_\lambda(f(v), v) < 1$ which is contrary to the assumption (a). Thus $d(f^2(v), f(v)) \subseteq d(f(v), v)$. Let $f(v) \neq v$. By (b) there exists $x \in U$ and $\lambda \in \Lambda_x$ such that $d_\lambda(f^2(v), f(v)) < d_\lambda(f(v), v) = 1$. Since $\lambda^{-1}(1) = \{x\}$, then $x \in d(f(v), v)$, but $x \notin d(f^2(v), f(v))$. $\square$

We investigate conditions for $f: W \to W$ to be a strictly contracting map with respect to the distances associated to level mappings. By (5.7) f is a strictly contracting map with respect to d if and only if $f(v) = f(w)$ for all v, $w \in W$.

We have:

(5.12) *Let $f: W \to W$. The following conditions are equivalent:*

(1) *f is strictly contracting with respect to d.*
(2) *f is strictly contracting with respect to d_λ, for each $\lambda \in \Lambda$.*
(3) *f is strictly contracting with respect to d_λ, for each $\lambda \in \bigcup_{x \in U} \Lambda_x$.*
(4) *For each $x \in U$ there exists $\lambda \in \Lambda_x$ such that f is strictly contracting with respect to d_λ.*

Proof. (1) $\Rightarrow$ (2): By the above remark $f(v) = f(w)$ for all v, $w \in W$. If $v \neq w$ then $0 = d_\lambda(f(v), f(w)) < d_\lambda(v, w)$ for every $\lambda \in \Lambda$.

(2) $\Rightarrow$ (3): This is trivial.

(3) $\Rightarrow$ (4): This is also trivial.

(4) $\Rightarrow$ (1): Let v, $w \in W$, $v \neq w$, let $x \in U$ be such that $v(x) \neq w(x)$. By hypothesis there exists $\lambda \in \Lambda_x$ such that $d_\lambda(f(v), f(w)) < d_\lambda(v, w) = 1$, so $f(v)(x) = f(w)(x)$. By (5.10) $d(f(v), f(w)) \subseteq d(v, w)$, then $d(f(v), f(w)) \subset d(v, w)$. ∎

Similarly we may show:

(5.13) *Let $f: W \to W$. The following conditions are equivalent:*

(1) *$f^2 = f$.*
(2) *If $f(v) \neq v$ then $d_\lambda(f^2(v), f(v)) < d_\lambda(f(v), v)$ for each $\lambda \in \Lambda$.*
(3) *If $f(v) \neq v$ then $d_\lambda(f^2(v), f(v)) < d_\lambda(f(v), v)$ for each $\lambda \in \bigcup_{x \in U} \Lambda_x$.*
(4) *If $f(v) \neq v$, for every $x \in U$ there exists $\lambda \in \Lambda_x$ such that $d_\lambda(f^2(v), f(v)) < d_\lambda(f(v), v)$.*

Proof. It is trivial that (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (1): Assume that $f^2 \neq f$, so there exists $v \in W$ such that $f^2(v) \neq f(v)$, hence $f(v) \neq v$. Let $x \in U$ be such that $f^2(v)(x) \neq f(v)(x)$. By hypothesis, there exists $\lambda \in \Lambda_x$ such that $d_\lambda(f^2(v), f(v)) < d_\lambda(f(v), v)$. But $d_\lambda(f(v), v) \leq 1 = d_\lambda(f^2(v), f(v))$. ∎

6 – Criteria for the existence of fixed points for non-positive programs

Let $\mathbb{L}$ be a first-order language, let P be a program; we assume that P is not positive. With the notations of Sections 4, 5 $(\mathcal{P}(\mathrm{HB}), d, \mathcal{P}(\mathrm{HB}))$ is a spherically complete ultrametric space. It is isomorphic to the ultrametric space $(W, d, \mathcal{P}(\mathrm{HB}))$, where $W = \{0,1\}^{HB}$ is the set of valuations.

From the fixed point theorem (5.3) we obtain the following criterion:

(6.1) *Assume that the immediate consequence operator T_P of the program P is a contracting map which is strictly contracting on orbits. Then there exists $X \subseteq \mathrm{HB}$ such that $T_P(X) = X$.*

PROOF. This is an immediate consequence of (5.3) and (5.6). ∎

It is easy to provide examples of programs P such that T_P has no fixed point, or such that T_P has more than one fixed point. There are also examples of programs (even positive programs) with T_P not contracting, but having a fixed point.

The next results use level mappings $\lambda : W \to \Delta\backslash\{0\}$, where Δ is a noetherian totally ordered set with smallest element 0 and largest element 1.

(6.2) *Let P be a program. Assume that there exists a level mapping $\lambda : \mathrm{HB} \to \Delta\backslash\{0\}$ such that T_P is contracting and strictly contracting on orbits with respect to the distance d_λ on $\mathcal{P}(\mathrm{HB})$. Then there exists $X \subseteq \mathrm{HB}$ such that $T_P(X) = X$. If T_P is strictly contracting with respect to d_λ, there exists a unique $X \subseteq \mathrm{HB}$ such that $T_P(X) = X$.*

PROOF. By (5.9), (W, d_λ, Δ) is spherically complete. The result follows at once from the fixed point theorem and the natural bijection between W and $\mathcal{P}(\mathrm{HB})$. ∎

In view of (5.10), (5.11), (5.12), we see that (6.1) is a corollary of (6.2).

Interesting level mappings are obtained when $\Delta = \{0\} \cup \{\frac{1}{2^n} \mid n = 0, 1, 2, \ldots\}$; endowed with the usual order Δ is noetherian. As a corollary of (6.2), we mention the following result of Fitting [2]:

(6.3) *Let P be a program, let $\lambda : \mathrm{HB} \to \{\frac{1}{2^n} \mid n = 0, 1, 2, \ldots\}$ be a level mapping. Assume that there exists α, $0 < \alpha < 1$, such that $d_\lambda\,(T_P(v), T_P(w)) \le \alpha\, d_\lambda(v, w)$ for all $v, w \in W$. Then there exists a unique valuation $v \in W$ such that $T_P(v) = v$.*

Proof. From the hypothesis it follows that T_P is strictly contracting with respect to d_λ, hence the result follows from (6.2). □

The proof of this result by Fitting ran as follows: Since $d_\lambda(v,w) \in [0,1] \subseteq \mathbb{R}$, then (W, d_λ) is a metric space which is necessarily complete. By the Banach fixed point theorem, T_P has a unique fixed point.

The following types of levels (for the set $\mathrm{Lit} = \mathrm{HB} \cup \{\neg A \mid A \in \mathrm{HB}\}$ instead of HB) have been used.

Let $\Delta^* = \{1, 2, \ldots, n, \ldots\}$ endowed with the opposite order to the usual order, let $\Delta = \{0\} \cup \Delta^*$ with $0 < n$ for all $n \in \Delta^*$.

A level $\lambda : \mathrm{Lit} \to \Delta \setminus \{0\}$ which satisfies some additional properties (see [3] for details) is called a *stratification*. Similarly, let γ be any countable ordinal, let $\Delta^* = \{\alpha \mid 0 < \alpha < \gamma\}$, Δ^* with the order opposite to the usual order.

$\Delta = \{0\} \cup \Delta^*$, $0 < \alpha$ for all $\alpha \in \Delta^*$. A level mapping $\lambda : \mathrm{Lit} \to \Delta \setminus \{0\}$, which satisfies the properties mentioned above, is called a *local stratification*.

A program P, together with a stratification, respectively a local stratification, is called a *stratified program*, resp. a *locally stratified program*. Locally stratified programs were studied under a different guise already by Przymusinski [9]. Khamsi, Kreinovich and Misane [3] proved a multi-valued fixed point theorem (special case of (5.4)) for generalized metric spaces, which are associated to stratified disjunctive programs. These programs have clauses of the shape $\varphi_1 \vee \ldots \vee \varphi_n \leftarrow \varphi_{n+1} \wedge \ldots \wedge \varphi_{n+m} \wedge (\neg\varphi_{n+m+1}) \wedge \ldots \wedge (\neg\varphi_{n+m+k})$, where the $\varphi_1, \ldots, \varphi_{n+m+k}$ are literals. By means of their fixed point theorem they could prove the existence of answer sets for such programs. The generalized metric spaces which they study turn out to be special ultrametric spaces.

The idea of applying fixed point theorems to operators associated to programs of increasing generality is a very fruitful one. It has been the object of lively new research, as described for example in the paper of Seda and Hitzler [11].

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Indirizzo degli Autori:

Sibylla Priess-Crampe – Mathematisches Institut der Universität München – Theresienstraße 39 – 80333 München – Germany – e-mail: priess@rz.mathematik.uni-muenchen.de

Paulo Ribenboim – Department of Mathematics and Statistics – Queen’s University Kingston – Ontario – K7L3N6, Canada – e-mail: mathstat@mast.queensu.ca