Rendiconti di Matematica, Serie VII
Volume 19, Roma (1999), 177-193

Asymptotic behaviour for the nonlinear beam equation in a time-dependent domain

J. FERREIRA – R. BENABIDALLAH

J.E. MUÑOZ RIVERA

RIASSUNTO: *In questo lavoro si dimostra l'esistenza di soluzioni forti per l'equazione non lineare delle travi (*) dove Q_t è un dominio non cilindrico di $\mathbb{R}^2$, ν è una costante positiva e $M(\lambda)$ è una funzione reale tale che $M(\lambda) \geq -m_0$, con m_0 costante positiva. Si riconosce inoltre che l'energia ha un decadimento esponenziale.*

ABSTRACT: *In this paper we prove existence of strong solutions as well as the exponential decay of the energy to the mixed problem for the nonlinear beam equation*

$$(*) \qquad u_{tt} + u_{xxxx} - M\Big(\int_{I_t} |u_x|^2 dx\Big) u_{xx} + \nu u_t = 0 \quad \text{in } Q_t\,,$$

where Q_t is a non-cylindrical domain of $\mathbb{R}^2$. By ν we are denoting a positive constant. Here $M(\lambda)$ is a real function such that $M(\lambda) \geq -m_0$, for all $\lambda \geq 0$, where $m_0 > 0$.

1 – Introduction

In this work we will study the existence of strong solution as well as the exponential decay of the energy to the nonlinear beam equation of

---

KEY WORDS AND PHRASES: *Beam equation – Non cylindrical domain – Asymptotic behaviour – Strong solutions.*
A.M.S. CLASSIFICATION: 35F30, 35K55

Kirchhoff type given by,

$$(1.1)\quad u_{tt}+u_{xxxx}-M\Big(\int_{I_t}|u_x|^2dx\Big)u_{xx}+\nu u_t=0 \quad \text{in } Q_t\ ,$$
$$(1.2)\quad u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x) \qquad \text{in } \alpha(0)<x<\beta(0),$$
$$(1.3)\quad u(x,t)=0,\quad u_x(x,t)=0 \qquad \text{on } \Sigma_t.$$

By Q_t we are denoting a non-cylindrical domain of $\mathbb{R}^2$ defined by

$$(1.4)\qquad Q_t=\{(x,t)\in\mathbb{R}^2|\alpha(t)<x<\beta(t),\quad 0<t<T\}$$

where $\alpha(\cdot)$, $\beta(\cdot)$ are C^3-functions such that

$$\alpha(t)<\beta(t)\quad \text{for all } 0\le t\le T\,.$$

The lateral boundary Σ_t of Q_t is given by

$$\Sigma_t=\bigcup_{0<t<T}(\alpha(t)\times\{t\})\cup(\beta(t)\times\{t\})\,.$$

The system (1.1)-(1.3) describes the transverse deflection $u(x,t)$ of a beam which changes its configuration at each instant of time, increasing its deformation and hence increasing its tension. This model was proposed by Woinowski [17] for the case of cylindrical domain, see also Eisley [7] and Burgreen [5] for the physical justification and background of the model.

The existence, uniqueness and regularity of solutions of this equation with some modifications was studied by Dickey [6], Ball [1], Medeiros [9], Pereira [13], Menzala [10], Rivera [16], Ramos [14], among others. While the exponential decay to zero in a cylinder domain was studied by Brito [4], Biler [3], Ball [2], Pereira [13], see also the references therein. The exponential decay over non cylindrical domains was studied by Nakao and Narazaki [11] for the nonlinear wave equation. In this work the authors, using the penalty method, were able to show the exponential decay of weaks solutions. The principal shortcoming of the above method is because it is not possible to show the existence of regular solution.

In this paper we deal with the nonlinear beam equation of Kirchhoff type over a non cylindrical domain. We show the existence and uniqueness of strong solutions to the system. The method we use to prove the result of existence and uniqueness is based on transform the system (1.1)-(1.3) into another initial boundary-value problem defined over a cylindrical domain whose sections are not time-dependent. This is done using a suitable change of variable. Then we show the existence and uniquenes for this new system. Our existence result on noncylindrical domains will follows using the inverse of the transformation. That is, using the diffeomorphism $h : Q_t \longrightarrow Q$ defined by

$$h(x,t) = (y,t) = \Big(\frac{x-\alpha}{\gamma}, t\Big) \quad \text{for} \quad (x,t) \in Q_t \quad \gamma = \beta - \alpha \tag{1.5}$$

and $h^{-1} : Q \longrightarrow Q_t$ defined by

$$h^{-1}(y,t) = (x,t) = (\alpha(t) + \gamma(t)y, t)\,. \tag{1.6}$$

Denoting by v the function

$$v(y,t) = u \circ h^{-1}(y,t) = u(\alpha(t) + \gamma(t)y, t)\,, \tag{1.7}$$

the eq. (1.1)-(1.3) becomes

$$\begin{aligned} v_{tt} + \frac{1}{\gamma^4} v_{yyyy} - \frac{1}{\gamma^2} M\Big(\frac{1}{\gamma}\int_0^1 (v_y)^2 dy\Big) v_{yy} + \nu v_t + \\ + a_1 v_{yy} + a_2 v_{ty} + a_3 v_y = 0 \quad \text{in } Q\,, \end{aligned} \tag{1.8}$$

$$v(t,0) = v(t,1) = 0 \qquad \text{on }]0,T[\,, \tag{1.9}$$

$$v_y(t,0) = v_y(t,1) = 0 \qquad \text{on }]0,T[\,, \tag{1.10}$$

$$v|_{t=0} = v_0, \quad v_t|_{t=0} = v_1 \quad \text{in }]0,1[\,, \tag{1.11}$$

where

$$\begin{cases} a_1(y,t) = \Big(\dfrac{\alpha' + \gamma' y}{\gamma}\Big)^2, \quad a_2(y,t) = -2\Big(\dfrac{\alpha' + \gamma' y}{\gamma}\Big), \\ a_3(y,t) = -\Big(\dfrac{\alpha'' + \gamma'' y}{\gamma}\Big) + (\gamma' - \nu\gamma)\Big(\dfrac{\alpha' + \gamma' y}{\gamma^2}\Big)\,. \end{cases} \tag{1.12}$$

Let us denote by $\widehat{M}$ the function

$$\widehat{M}(\lambda) = \int_0^\lambda M(s)ds\,. \tag{1.13}$$

To show the existence of strong solution we will use the following hypotheses

$$\alpha' \leq 0, \quad \beta' \geq 0\,, \tag{1.14}$$

$$\gamma = \beta - \alpha \; \in L^\infty(0,\infty), \quad \operatorname{ess}\inf_{0\leq t<\infty} \gamma(t) = \gamma_0 > 0 \tag{1.15}$$

$$\alpha',\; \beta' \in W^{2,\infty}(0,\infty) \cap W^{2,1}(0,\infty)\,. \tag{1.16}$$

Note that assumption (1.14) means that Q_t is increasing in the sense that when $t_1 < t_2$ then $I_{t_1} = [\alpha(t_1), \beta(t_1)] \subset I_{t_2}$.
The above method was introduced by R. Dal Passo and M. Ughi [12] to study certain class of parabolic equations in non cylindrical domains. On the contrary to the penalty method, the exponential decay of the solution in our case is not a simple task. The main difference is that in our case the first order energy is not a decreasing function. To see the dissipative properties of the system we have to construct a suitable functional whose derivative is negative and is equivalent to the first order energy. This functional is obtained using the multiplicative technique following Rivera's method in [16].

Concerning the function $M \in C^1([0,\infty[)$, we assume that

$$M(r) \geq -m_0, \quad M(r)r \geq \widehat{M}(r) \quad \forall\, r \geq 0\,, \tag{1.17}$$

where

$$0 \leq m_0 < \lambda_1 \|\gamma\|_{L^\infty}^{-2} \tag{1.18}$$

here λ_1 is the first eigenvalue of the Dirichlet problem

$$\begin{cases} w_{xxxx} = \lambda_1 w = \quad \text{in }]0,1[\\ w(0) = w(1) = 0, \quad w_x(0) = w_x(1) = 0\,. \end{cases}$$

We recall also the classical inequality

$$\|w_{xx}\|^2_{L^2(0,1)} \geq \lambda_1 \|w_x\|^2_{L^2(0,1)}\,. \tag{1.19}$$

The remaining part of this work is organized as follows. In the following Section 2 we show the existence and uniqueness of strong solutions. Finally in Section 3 we will show the exponential decay.

2 – Global solution

Let us denote by A the operator

$$Aw = w_{xxxx}, \quad D(A) = H^4(\Omega) \cap H_0^2(\Omega)\,.$$

where Ω is a bounded domain in $\mathbb{R}$. It is well known that A is a positive self adjoint operator in the Hilbert space $L^2(\Omega)$ for which there exist sequences $\{w_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ of eigenfunctions and eigenvalues of A such that the set of linear combinations of $\{w_n\}_{n\in\mathbb{N}}$ is dense in $D(A)$ and $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \to \infty$ as $n \to \infty$. Let us denote by

$$v_0^{(m)} = \sum_{j=1}^{m} (v_0, w_j) w_j, \quad v_1^{(m)} = \sum_{j=1}^{m} (v_1, w_j) w_j\,,$$

Note that for any $(v_0, v_1) \in D(A) \times H_0^2(0,1)$, we have

$$(v_0^{(m)}, v_1^{(m)}) \to (v_0, v_1) \quad \text{strong in} \quad D(A) \times H_0^2(\Omega)\,.$$

Let us denote by V_m the space generated by $w_1, \ldots, w_m$. Standard results on ordinary differential equations imply the existence of a local solution $v^{(m)}$ of the form

$$v^{(m)}(t) = \sum_{j=1}^{m} g_{jm}(t) w_j\,,$$

to the system

$$\begin{aligned} &(v_{tt}^{(m)}, w_j) + \mu(v_t^{(m)}, w_j) + \frac{1}{\gamma^2} M(\gamma^{-1}\|v_y^{(m)}\|_{L^2}^2)(v_{yy}^{(m)}, w_j) + \\ &\quad + \frac{1}{\gamma^4}(v_{yyyy}^{(m)}, w_j) + (a_1 v_{yy}^{(m)}, w_j) + \\ &\quad + (a_2 v_{yt}^{(m)}, w_j) + (a_3 v_y^{(m)}, w_j) = 0 \quad (j = 1, \ldots, m)\,, \end{aligned} \tag{2.1}$$

$$v^{(m)}(y,0) = v_0^{(m)}, \quad v_t^{(m)}(y,0) = v_1^{(m)}\,. \tag{2.2}$$

To obtain the corresponding estimates of the approximated solutions v_m we will define the following energy functions associated to the eq. (1.8)

$$\begin{aligned} E_1(t,v) &= \|v_t\|_{L^2}^2 + \|v_{yy}\|_{L^2}^2 \\ E_2(t,v) &= \|v_{ty}\|_{L^2}^2 + \|v_{yyy}\|_{L^2}^2, \\ E_3(t,v) &= \|v_{tt}\|_{L^2}^2 + \|v_{tyy}\|_{L^2}^2\,. \end{aligned}$$

We will denote by

$$E_i^{(m)}(t) = E_i(t, v^{(m)}) \quad (\text{i} = 1,\, 2,\, 3)\,.$$

To show the existence of strong solutions we will prove that the above energies are bounded. In fact, multiplying eq. (2.1) by $g'_{jm}(t)$, and summing up the product result we obtain the inequality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\mathcal{L}_1^{(m)}(t) + \nu\|v_t^{(m)}\|_{L^2}^2 + \frac{\gamma'}{\gamma^2}[\gamma^{-1}\|v_y^{(m)}\|_{L^2}^2 M(\gamma^{-1}\|v_y^{(m)}\|_{L^2}^2) + \\ &\qquad - \widehat{M}(\gamma^{-1}\|v_y^{(m)}\|_{L^2}^2)] \le c(a+b)(\mathcal{L}_1^{(m)}(t) + \|v_y^{(m)}\|_{L^2}^2)\,, \end{aligned} \tag{2.3}$$

where

$$a = a(t) = |\alpha'| + |\alpha''|, \quad b = b(t) = |\beta'| + |\beta''|\,, \tag{2.4}$$

and

$$\mathcal{L}_1^{(m)}(t) = \|v_t^{(m)}\|_{L^2}^2 + \gamma^{-4}\|v_{yy}^{(m)}\|_{L^2}^2 + \gamma^{-1}\widehat{M}(\gamma^{-1}\|v_y^{(m)}\|_{L^2}^2)\,. \tag{2.5}$$

From (1.17)-(1.19) it follows that

$$\gamma^{-4}\|v_{yy}^{(m)}\|_{L^2}^2 + \gamma^{-1}\widehat{M}(\gamma^{-1}\|v_y^{(m)}\|_{L^2}^2)) \ge \frac{m_1}{\|\gamma\|_{L^\infty}^2}\|v_y^{(m)}\|_{L^2}^2 \tag{2.6}$$

where

$$m_1 = \Big(\frac{\lambda_1}{\|\gamma\|_{L^\infty}^2} - m_0\Big) > 0\,. \tag{2.7}$$

From (1.17), it follows that the inequality (2.3) can be written as

$$\mathcal{L}_1^{(m)}(t) + \nu \int_0^t \|v_s^{(m)}\|_{L^2}^2 ds \le c(\|v_1\|_{L^2}^2 + \|v_0\|_{H^2}^2) + \tag{2.8}$$
$$+ c \int_0^t (a(s) + b(s)) \mathcal{L}_1^{(m)}(s) ds \,.$$

Using Gronwall's inequality and taking into account (1.16) we get

$$\mathcal{L}_1^{(m)}(t) + \int_0^t \|v_s^{(m)}\|_{L^2}^2 ds \le c(\|v_1\|_{L^2}^2 + \|v_0\|_{H^2}^2) \tag{2.9}$$

from (2.6) it follows that

$$\|v_y^{(m)}\|_{L^2}^2 \le c(\|v_1\|_{L^2}^2 + \|v_0\|_{H^2}^2) \,. \tag{2.10}$$

Therefore (2.8) implies

$$E_1^{(m)}(t) = \|v_t^{(m)}\|_{L^2}^2 + \|v_{yy}^{(m)}\|_{L^2}^2 \le c(\|v_1\|_{L^2}^2 + \|v_0\|_{H^2}^2) \,. \tag{2.11}$$

Multiplying eq. (2.1) by $g_{jm}(t)$, and summing up the product result we obtain the following inequality

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_2^{(m)}(t) - \|v_t^{(m)}\|_{L^2}^2 + \gamma^{-4}\|v_{yy}^{(m)}\|_{L^2}^2 + \tag{2.12}$$
$$+ \gamma^{-2} M(\gamma^{-1}\|v_y^{(m)}\|_{L^2}^2)\|v_y^{(m)}\|_{L^2}^2 \le$$
$$\le c(a(t) + b(t))(\|v_{yy}^{(m)}\|_{L^2}^2 + \|v_t^{(m)}\|_{L^2}^2) \,,$$

where

$$\mathcal{L}_2^{(m)}(t) = 2 \int_0^1 v^{(m)} v_t^{(m)} dy + \nu \|v^{(m)}\|_{L^2}^2 \,.$$

In virtue of (2.6) and (2.9), it follows from (2.12) that

$$\int_0^t \|v_y^{(m)}\|_{L^2}^2 ds \le c(\|v_1\|_{L^2}^2 + \|v_0\|_{H^2}^2) \,. \tag{2.13}$$

Multiplying eqs. (2.1) by $\sqrt{\lambda_j} g'_{jm}(t)$ and summing up in $j = 1, \dots, m$, we get after some calculations

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_3^{(m)}(t) + \frac{\nu}{2}\|v_{ty}^{(m)}\|_{L^2}^2 \le c(a(t)+b(t))\|v_{yy}^{(m)}\|_{L^2}^2 + \\ + c\|v_y^{(m)}\|_{L^2}^2 + c(a(t)+b(t))\mathcal{L}_3^{(m)}(t)\,, \tag{2.14}$$

where

$$\mathcal{L}_3^{(m)}(t) = \|v_{ty}^{(m)}\|_{L^2}^2 + \gamma^{-4}\|v_{yyy}^{(m)}\|_{L^2}^2 + \\ + \gamma^{-2}M(\gamma^{-1}\|v_y^{(m)}\|_{L^2}^2)\|v_{yy}^{(m)}\|_{L^2}^2)\,. \tag{2.15}$$

from (1.17)-(1.19), we have

$$\gamma^{-4}\|v_{yyy}^{(m)}\|_{L^2}^2 + \gamma^{-2}M(\gamma^{-1}\|v_y^{(m)}\|_{L^2}^2))\|v_{yy}^{(m)}\|_{L^2}^2 \ge \frac{m_1}{\|\gamma\|_{L^\infty}^2}\|v_{yy}^{(m)}\|_{L^2}^2\,. \tag{2.16}$$

Using Gronwall's inequality, relation (2.14) and taking into account (2.11), we get

$$\mathcal{L}_3^{(m)}(t) + \int_0^t \|v_{sy}^{(m)}\|_{L^2}^2 ds \le c(\|v_1\|_{H^1}^2 + \|v_0\|_{H^3}^2)\,. \tag{2.17}$$

Since

$$E_2^{(m)}(t) = \|v_{ty}^{(m)}\|_{L^2}^2 + \|v_{yyy}^{(m)}\|_{L^2}^2 \le c\mathcal{L}_3^{(m)}(t) + \\ + c\max_{0\le t\le\infty}|M(\gamma^{-1}\|v_y^{(m)}\|_{L^2}^2)|\|v_{yy}^{(m)}\|_{L^2}^2\,,$$

from (2.17), (2.10)-(2.11) it follows that there exists a positive constant c such that

$$E_2^{(m)}(t) = \|v_{ty}^{(m)}\|_{L^2}^2 + \|v_{yyy}^{(m)}\|_{L^2}^2 \le c(\|v_1\|_{H^1}^2 + \|v_0\|_{H^3}^2)\,. \tag{2.18}$$

Finally, differentiating (2.1) with respect t, multiplying by $g''_{jm}(t)$ and using similar arguments as in (2.14), we obtain

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_4^{(m)}(t) + \nu\|v_{tt}^{(m)}\|_{L^2}^2 \le \\ \le c(a(t)+|\alpha'''|+b(t)+|\beta'''|)(\|v_{yy}^{(m)}\|_{L^2}^2 + \|v_{ty}^{(m)}\|_{L^2}^2) + \\ + c\|v_{ty}^{(m)}\|_{L^2}^2 + c(a(t)+|\alpha'''|+b(t)+|\beta'''||)\mathcal{L}_4^{(m)}(t)\,, \tag{2.19}$$

where

$$\mathcal{L}_4^{(m)}(t) = \|v_{tt}^{(m)}\|_{L^2}^2 + \gamma^{-4}\|v_{tyy}^{(m)}\|_{L^2}^2 + \gamma^{-2}M(\gamma^{-1}\|v_y^{(m)}\|_{L^2}^2)\|v_{ty}^{(m)}\|_{L^2}^2) \,.$$

From eq. (2.1) we get

$$\|v_{tt}^{(m)}(0)\|_{L^2}^2 \leq c(\|v_0\|_{H^4}^2 + \|v_1\|_{H^2}^2) \,.$$

Using the Gronwall's inequality, relations (2.11), (2.17) and (2.19) we get

$$\mathcal{L}_4^{(m)}(t) + \int_0^t \|v_{tt}^{(m)}\|_{L^2}^2 ds \leq c(\|v_0\|_{H^4}^2 + \|v_1\|_{H^2}^2) \,.$$

From where it follows

$$E_3^{(m)}(t) = \|v_{tt}^{(m)}\|_{L^2}^2 + \|v_{tyy}^{(m)}\|_{L^2}^2 \leq c(\|v_1\|_{H^2}^2 + \|v_0\|_{H^4}^2) \,. \tag{2.20}$$

From estimates (2.11), (2.18) and (2.20), it follows that $\{v^{(m')}\}$ converges strongly at v in $L^2_{\text{loc}}(0,\infty; H^1_0(0,1))$. Moreover, since $M \in C^1(0,\infty)$ and $v_y^{(m)}$ is bounded in $L^\infty(0,\infty, L^2(0,1)) \cap L^2(0,\infty; L^2(0,1))$, we have for any $t > 0$

$$\begin{aligned} \int_0^t |M(\gamma^{-1}\|v_y^{(m')}(t)\|_{L^2}^2) - M(\gamma^{-1}\|v_y(t)\|_{L^2}^2)|dt &\leq \\ &\leq c\|v^{(m')} - v\|_{L^2(0,t;H^1)} \end{aligned} \tag{2.21}$$

where c is a positive constant independent of m' and t, so that

$$M(\gamma^{-1}\|v_y^{(m')}(t)\|_{L^2}^2)(v_{yy}^{(m')}, w_j) \longrightarrow M(\gamma^{-1}\|v_y(t)\|_{L^2}^2)(v_{yy}, w_j) \,. \tag{2.22}$$

Therefore we have that v satisfies

$$\begin{cases} v \in L^\infty(0,\infty; H_0^2(0,1) \cap H^3(0,1)) \,, \\ v_t \in L^\infty(0,\infty; H_0^2(0,1)) \,, \\ v_{tt} \in L^\infty(0,\infty; L^2(0,1)) \,, \end{cases} \tag{2.23}$$

and

$$(2.24)\qquad \begin{aligned}(v_{tt},w^j)+\nu(v_t,w^j)+\gamma^{-2}M\Big(\gamma^{-1}\int_0^1|v_y|^2dy\Big)(v_{yy},w^j)+\\ +\gamma^{-4}(v_{yy},w^j_{yy})+(a_1v_{yy},w^j)+\\ +(a_2v_{ty},w^j)+(a_3v_y,w^j)=0\,,\end{aligned}$$

$$(2.25)\qquad v(y,0)=v_0,\quad v_t(y,0)=v_1\,.$$

For any $w^j\in V_m$. Letting $m\to\infty$ we conclude that v satisfies eq. (1.8) in the sense of $L^\infty(0,\infty;L^2(0,1))$ therefore we have that

$$(2.26)\qquad v\in L^\infty(0,\infty;H_0^2(0,1)\cap H^4(0,1))\,.$$

The Uniqueness follows by using standard arguments. □

Thus we have the following result

Theorem 2.1. *Let us take* $v_0\in H_0^2(0,1)\cap H^4(0,1)$, $v_1\in H_0^2(0,1)$ *and let us suppose that assumptions* (1.14)-(1.18) *holds. Then there exists a unique solution* v *of the problem* (1.8)-(1.11) *in the class* (2.26) *satisfying the eq.* (1.8) *in the sense of* $L^\infty(0,\infty;L^2(0,1))$

To show the existence in non cylindrical domains, we return to our original problem by using the change variable given in (1.5) by $(y,t)=h(x,t)$, $(x,t)\in Q_t$.

Let v the solution obtained from Theorem 2.1 and u defined by (1.7), then u belong to the class

$$(2.27)\qquad \begin{cases}u\in L^\infty(0,\infty;H_0^2(I_t)\cap H^4(I_t)),\\ u_t\in L^\infty(0,\infty;H^2(I_t)),\\ u_{tt}\in L^\infty(0,\infty;L^2(I_t))\,,\end{cases}$$

where $I_t=]\alpha(t),\beta(t)[$ for any $t\geq 0$. Denoting by

$$u(x,t)=v(y,t)=(v\circ h)(x,t)\,,$$

then from (1.8) it is easy to see that u satisfies the eq. (1.1) in the sense $L^\infty(0,\infty;L^2(I_t))$. The uniqueness of u follows from the uniqueness of v. Therefore, we have the following result.

Theorem 2.2. *Let us take* $u_0 \in H_0^2(I_0) \cap H^4(I_0)$, $u_1 \in H_0^2(I_0)$ *and let us suppose that assumptions* (1.14)-(1.18) *holds. Then there exists a unique solution* u *of the initial boundary value problem* (1.1)-(1.3) *satisfying* (2.27) *and the eq.* (1.1) *in the sense of* $L^\infty(0,\infty;L^2(I_t))$.

3 – Asymptotic behaviour

In this section we will show the exponential decay of the solution given by the Theorem 2.2. To do this we assume that M satisfies the condition

$$M(r)r \geq \widehat{M}(r), \quad M(r) \geq -m_0 \quad \forall r \geq 0 \tag{3.1}$$

with m_0 such that

$$0 \leq m_0 < \frac{\lambda_1}{\|\gamma\|^2_{L^\infty}}\,. \tag{3.2}$$

Additionally, we assume that the functions β and α satisfies the conditions

$$\max_{0\leq t<\infty} |\beta'(t)| \leq \varepsilon, \quad \max_{0\leq t<\infty} |\alpha'(t)| \leq \varepsilon\,, \tag{3.3}$$

where

$$\varepsilon = \delta_0 \min(r_1, -r_0)$$

and the numbers δ_0, r_1 and r_0 are given by (3.6) and (3.20).

Then we have the following result.

Theorem 3.1. *Let us take initial datas* $u_0 \in H_0^2(I_0)$, $u_1 \in L^2(I_0)$ *and let us suppose that assumptions* (3.1)-(3.3) *hold. Then any regular solution of* (1.1) *satisfies the inequality*

$$E(t) \leq Ce^{-\omega t}$$

where C and ω are positive constants and

$$E(t) = \|u_t\|^2_{L^2(I_t)} + \|u\|^2_{H^2(I_t)}$$

is the energy associated to the eq. (1.1).

PROOF. Multiplying (1.1) by u_t and performing an integration by parts over I_t we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\mathcal{L}_1(t) + \nu\|u_t\|^2_{L^2(I_t)} - \frac{1}{2}\beta'(t)(|u_t(\beta(t),t)|^2 - |u_{xx}(\beta(t),t)|^2) + \\ + \frac{1}{2}\alpha'(t)(|u_t(\alpha(t),t)|^2 - |u_{xx}(\alpha(t),t)|^2) = 0 \end{aligned} \tag{3.4}$$

where

$$\mathcal{L}_1(t) = \int_{I_t} (|u_t|^2 + |u_{xx}|^2 + \widehat{M}(\|u_x\|^2_{L^2(I_t)}))dx \tag{3.5}$$

with $\widehat{M}(\cdot)$ given by (1.13). Let us take $r \in C^2([0,1])$ such that

$$r(1) = r_1 > 0, \quad r(0) = r_0 < 0\,. \tag{3.6}$$

For $x \in I_t$ $(t \in [0,\infty[)$, we put

$$q(x) = (r \circ h)(x) = r\Big(\frac{x-\alpha(t)}{\gamma(t)}\Big) = r(y)\,, \tag{3.7}$$

then $q \in C^2(\overline{I}_t)$ and we also have

$$\|q\|_{C^2(\overline{I}_t))} = (1 + \gamma_0^{-1} + \gamma_0^{-2})\|r\|_{C^2([0,1])} = S < \infty\,. \tag{3.8}$$

Let us introduce the functional

$$\mathcal{L}_2(t) = -2\int_{I_t} q u_x u_t\, dx\,.$$

Multiplying (1.1) by $-qu_x$ and integrating over I_t, we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{L}_2(t) + \frac{1}{2}q(\beta(t))(|u_{xx}(\beta(t),t)|^2 + |u_t(\beta(t),t)|^2) + \\ - \frac{1}{2}q(\alpha(t))(|u_{xx}(\alpha(t),t)|^2 + (|u_t(\alpha(t),t)|^2) = \sum_{i=1}^4 J_i\,, \end{aligned} \tag{3.9}$$

where

$$J_1 = \frac{1}{2} M\Big(\int_{I_t} |u_x|^2 dx\Big) \int_{I_t} q_x |u_x|^2 dx\,,$$

$$J_2 = \frac{1}{2} \int_{I_t} q_x |u_t|^2 dx, \quad J_3 = \nu \int_{I_t} q u_x u_t dx\,,$$

$$J_4 = -\int_{I_t} q_t u_x u_t dx\,,$$

$$J_5 = \int_{I_t} ((q_{xx})(u_{xx}) u_x + \tfrac{3}{2} q_x |u_{xx}|^2) dx\,.$$

Since $\|u_x\|_{L^\infty(0,\infty;L^2(I_t))} \le \Lambda$ and $M(\cdot) \in C^1([0,\infty[)$, it follows that

$$\max_{0\le\lambda\le\Lambda} |M(\lambda)| = R < \infty, \tag{3.10}$$

so that, taking into account (3.10), (3.8) and the Poincaré inequality, we have

$$|J_1| \le c_1 \|u_{xx}\|^2_{L^2(I_t)}\,,$$

where c_i $(i = 1, \dots, 6)$ denote positive constants which depend only on R, S and the norm $\|\gamma\|_{L^\infty}$. Moreover with the same arguments, we have

$$|J_2 + J_3| \le c_2 (\|u_t\|^2_{L^2(I_t)} + \|u_{xx}\|^2_{L^2(I_t)})\,,$$

$$|J_4 + J_5| \le c_3 \|u_{xx}\|^2_{L^2(I_t)}\,.$$

From (3.9) the above estimates of J_i $(i = 1, \dots, 5)$ and recalling that

$$q(\beta(t)) = r(1) = r_1, \quad q(\alpha(t)) = r(0) = r_0$$

we get

$$\begin{aligned} \frac{d}{dt}\mathcal{L}_2(t) + \frac{1}{2} r_1 (|u_{xx}(\beta(t),t)|^2 + |u_t(\beta(t),t)|^2) + \\ - \frac{1}{2} r_0 (|u_{xx}(\alpha(t),t)|^2 + (|u_t(\alpha(t),t)|^2) \\ \le c_2 \|u_t\|^2_{L^2(I_t)} + c_5 \|u_{xx}\|^2_{L^2(I_t)}\,. \end{aligned} \tag{3.11}$$

Using

$$\|u_{xx}\|^2_{L^2(I_t)} \le \|u_{xx}\|^2_{L^2(I_t)} + \widehat{M}(\|u_x\|^2_{L^2(I_t)}) + m_0 \|u_x\|^2_{L^2(I_t)}\,. \tag{3.12}$$

From (3.11) and (3.4), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\mathcal{L}_1(t)+\delta_0\mathcal{L}_2(t))+(\nu-c_2\delta_0)\|u_t\|^2_{L^2(I_t)}+\\
&\qquad+\frac{1}{2}(\delta_0 r_1+\beta'(t))|u_{xx}(\beta(t),t)|^2+\\
&\qquad+\frac{1}{2}(\delta_0 r_1-\beta'(t))|u_t(\beta(t),t)|^2+\\
&\qquad+\frac{1}{2}(\alpha'(t)-\delta_0 r_0))|u_t(\alpha(t),t)|^2-\\
&\qquad-\frac{1}{2}(\alpha'(t)+\delta_0 r_0))|u_{xx}(\alpha(t),t)|^2\leq\\
&\qquad\leq\delta_0 c_5(\|u_{xx}\|^2_{L^2(I_t)}+\widehat{M}(\|u_x\|^2_{L^2(I_t)}))+\\
&\qquad+c_5\delta_0 m_0\|u_x\|^2_{L^2(I_t)}\,,
\end{aligned}
\tag{3.13}
$$

where δ_0 is a positive constants to be fixed later. Let us denote by

$$
\mathcal{L}_3(t)=\int_{I_t}(u_t u+\frac{\nu}{2}|u|^2)dx\,.
$$

Multiplying (1.1) by u and integrating over I_t we obtain

$$
\frac{d}{dt}\mathcal{L}_3(t)-\|u_t\|^2_{L^2(I_t)}+\|u_{xx}\|^2_{L^2(I_t)}+M(\|u_x\|^2_{L^2(I_t)})\|u_x\|^2_{L^2(I_t)}=0\,. \tag{3.14}
$$

From (1.19), we have

$$
\|u_{xx}\|_{L^2(I_t)}\geq\frac{\sqrt{\lambda_1}}{\|\gamma\|^2_{L^\infty}}\|u_x\|_{L^2(I_t)}\,,
$$

so that (3.1) and (3.2) give us

$$
\|u_{xx}\|^2_{L^2(I_t)}+\widehat{M}(\|u_x\|^2_{L^2(I_t)})\geq m_1\|u_x\|^2_{L^2(I_t)}\,. \tag{3.15}
$$

From (3.2) and (3.14) it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\mathcal{L}_3(t)+\frac{1}{2}(\|u_{xx}\|^2_{L^2(I_t)}+\widehat{M}(\|u_x\|^2_{L^2(I_t)})+\\
&\qquad+\frac{m_1}{2}\|u_x\|^2_{L^2(I_t)}\leq\|u_t\|^2_{L^2(I_t)}\,.
\end{aligned}
\tag{3.16}
$$

Moreover, from (3.8) we have

$$2\delta_0\Big|\int_{\Omega_t}(qu_x)u_t dx\Big| \leq \frac{1}{2}\|u_t\|^2_{L^2(I_t)} + \delta_0^2 m_0 c_6\|u_x\|^2_{L^2(I_t)} + \\ + \delta_0^2 c_6(\|u_{xx}\|^2_{L^2(I_t)} + \widehat{M}(\|u_x\|^2_{L^2(I_t)}))\,.$$

Recalling the definition of $\mathcal{L}_1$ and $\mathcal{L}_2$ and using the above inequality we get

$$\mathcal{L}_1(t) + \delta_0\mathcal{L}_2(t) \geq \frac{1}{2}\|u_t\|^2_{L^2(\Omega_t)} + \Big(\frac{m_1}{2} - \delta_0^2 m_0 c_6\Big)\|u_x\|^2_{L^2(I_t)} + \\ + \Big(\frac{1}{2} - \delta_0^2 c_6\Big)(\|u_{xx}\|^2_{L^2(I_t)} + \widehat{M}(\|u_x\|^2_{L^2(I_t)}))\,. \tag{3.17}$$

Since

$$\mathcal{L}_3(t) \geq -\frac{2}{\nu}\|u_t\|^2_{L^2(I_t)} + \frac{\nu}{2}\|u\|^2_{L^2(I_t)}\,.$$

Finally we define

$$\mathcal{L}(t) = \mathcal{L}_1(t) + \delta_0\mathcal{L}_2(t) + \frac{\nu}{8}\mathcal{L}_3(t)\,.$$

From (3.17) it follows that

$$\mathcal{L}(t) \geq \frac{\nu}{16}\|u\|^2_{L^2(I_t)} + \frac{1}{4}\|u_t\|^2_{L^2(I_t)} + \Big(\frac{m_1}{2} - \delta_0^2 m_0 c_6\Big)\|u_x\|^2_{L^2(I_t)} + \\ + \Big(\frac{1}{2} - \delta_0^2 c_6\Big)(\|u_{xx}\|^2_{L^2(I_t)} + \widehat{M}(\|u_{xx}\|^2_{L^2(I_t)}))\,. \tag{3.18}$$

From (1.14) and (3.6) we get

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}(t) + \Big(\frac{7}{8}\nu - c_2\delta_0\Big)\|u_t\|^2_{L^2(I_t)} + \Big(\frac{\nu m_1}{16} - c_5\delta_0 m_0\Big)\|u_x\|^2_{L^2(I_t)} + \\ + \Big(\frac{\nu}{16} - c_5\delta_0\Big)(\|u_{xx}\|^2_{L^2(I_t)} + \widehat{M}(\|u_x\|^2_{L^2(I_t)}) + \\ + \frac{1}{2}(\delta_0 r_1 - \beta'(t))|u_t(\beta(t),t)|^2 - \frac{1}{2}(-\alpha'(t) + \\ + \delta_0 r_0|u_t(\alpha(t),t)|^2 \leq 0\,. \tag{3.19}$$

Now taking

$$\delta_0 = \min\Big(\sqrt{\frac{m_1}{2m_0c_6}}, \sqrt{\frac{1}{2c_6}}, \frac{7\nu}{8c_2}, \frac{m_1\nu}{16m_0c_5}, \frac{\nu}{16c_5}\Big)\,, \tag{3.20}$$

we deduce from (3.18) that

$$\mathcal{L}(t) \geq c_7[\|u_t\|^2_{L^2(\Omega_t)} + \|u_x\|^2_{L^2(\Omega_t)} + \|u_{xx}\|^2_{L^2(\Omega_t)} + \widehat{M}(\|u_x\|^2_{L^2(\Omega_t)})].$$

From (3.19) we arrive at

$$\frac{d}{dt}\mathcal{L}(t) + \alpha\mathcal{L}(t) \leq 0\,,$$

where α is a positive constante independent of t. Therefore we have that

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\alpha t}$$

which implies that

$$E(t) \leq Ce^{-\alpha t}$$

where C and α are positive constants independent of t and

$$E(t) = \|u_t\|^2_{L^2(\Omega_t)} + \|u\|^2_{H^2(\Omega_t)}\,.$$

The proof is now complete. □

REFERENCES

[1] J.M. Ball: *Initial boundary value problems for an extensible beam*, J. Math. Analysis Appl., **42** (1973), 61-90.

[2] J.M. Ball: *Stability theory for an extensible beam*, J. Diff. Eqns., **14** (1973), 399-418.

[3] J.M. Biller: *Remark on the decay for damped string and beam equations*, Nonlinear Analysis, **10** (1986), 839-842.

[4] E.H. Brito: *Decay estimates for genelized damped extensible string and beam equation*, Nonlinear Analysis, **8** (1984), 1489-1496.

[5] D. Burgreen: *Free vibrations of a pinended column with constant distance between pinendes*, J. Appl. Mech., **18** (1951), 135-139.

[6] R.W. Dickey: *Free vibrations and dynamics buckling of extensible beam*, J. Math. Analysis Appl., **29** (1970).

[7] J.G. Eisley: *Nonlinear vibrations of beams and rectangular plates*, Z. angew. Math. Phys., **15** (1964).

[8] J.L. Lions: *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéarires*, Dunod, Paris, 1969.

[9] L.A. Medeiros: *On a new class of nonlinear wave equations*, J. Math. Appl., **69** (1979), 252-262.

[10] G.P. Menzala: *On classical solutions of a quasilinear hyperbolic equation*, Nonlinear Analysis, **3** (1978), 613-627.

[11] M.L. Nakao – T. Narazaki: *Existence and decay of solutions of some nonlinear wave equations in noncylindrical domains*, Math. Rep., **11** (1978), 117-125.

[12] R.D. Passo – M. Ughi: *Problèmes de Dirichlet pour une classe d'équations paraboliques non linéaires dans des ouverts non cylindriques*, C. R. Acad. Sc. Paris., **308** (1989), 555-558.

[13] D.C. Pereira: *Existence, uniqueness and asymptotic behaviour for solutions of the nonlinear beam equation*, Nonlinear Analysis, **14** (8) (1990), 613-623.

[14] O.C. Ramos: *Regularity property for the nonlinear beam operator*, An. Acad. Bras. Ci., **61** (1) (1989), 15-24.

[15] J.E.M. Rivera: *Smoothness effect and decay on a class of non linear evolution equation*, Ann. Fac. Sc. Toulouse., **I** (20) (1992), 237-260.

[16] J.E.M. Rivera: *Existence and asymptotic behaviour in a class of nonlinear wave equation with thermal dissipation*, Nonlinear Analysis, **23** (10) (1994), 1255-1271.

[17] S.K. Woinowsky: *The effect of axial force on the vibration of hinged bars*, Appl. Mech., **17** (1950), 35-36.

Lavoro pervenuto alla redazione il 9 maggio 1997
modificato il 15 settembre 1998
ed accettato per la pubblicazione il 24 febbraio 1999.
Bozze licenziate il 20 aprile 1999

Indirizzo degli Autori:

J. Ferreira – Departamento de matemática – Universidade Estadual de Maringá Agencia Postal UEM 87020-900 – Maringá – PR, Brasil

R. Benabidallah – Dipartimento di Matematica – Università di Pisa – Via Buonarroti – 2, 56127 Pisa, Italy

J.E. Muñoz Rivera – Laboratório Nacional de computaçao Científica (LNCC-CNPq) – Rua Getulio Vargas 333 – Quitandinha 25651-070 – Petrópolis – RJ, Brazil IMUFRJ – P.O.Box 68530 – Rio de Janeiro – RJ, Brazil