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Regularity of minimizers of a class of anisotropic integral functionals

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RIASSUNTO: *In questo lavoro si considera un funzionale scalare del calcolo delle variazioni di tipo degenere ed a crescite anisotrope, il cui modello è del tipo* $(*)$ *dove si intende che* $2 \leq p < p_1 \leq p_2 \leq \cdots \leq p_n$. *Si dimostra un risultato di maggior sommabilità per il gradiente dei minimi di tale funzionale.*

Tale risultato consente poi di ottenere una regolarità più forte (*ad esempio lipschitziana*), *per mezzo di argomenti di iterazione standard.*

ABSTRACT: *In this paper we consider a scalar degenerate functional of the calculus of variations under anisotropic growth conditions, which model is of the type*

$$\int_\Omega \Big[|Du|^p + \sum_{\alpha=1}^n |D_\alpha u|^{p_\alpha}\Big]\,dx\,, \tag{$*$}$$

where we assume that $2 \leq p < p_1 \leq p_2 \leq \cdots \leq p_n$. *We will prove a higher integrability result for the gradient of the minimizers of such a functional.*

This result will allow us to recover higher regularity (*Lipschitz regularity for instance*), *by means of standard iteration arguments.*

1 – Introduction

In this paper we study the regularity properties of minimizers of an integral functional of the type

$$\int_\Omega \Big[|Du|^p + \sum_{\alpha=k}^n |D_\alpha u|^{p_\alpha}\Big]\,dx \tag{1.1}$$

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where $1 \leq k \leq n$ $(n \geq 2)$. To fix the ideas, from now on we assume that the exponents p_α are such that $p_n \geq p_{n-1} \geq \cdots \geq p_k > p \geq 2$. As usual, Ω denotes a bounded open subset of $\mathbb{R}^n$ and u is in the class of functions $W^{1,(p_\alpha)}(\Omega) = \{w \in W^{1,1}(\Omega) : D_\alpha w \in L^{p_\alpha}(\Omega), \forall \alpha = 1, \ldots, n\}$ (that is a Banach space with the natural norm $\|u\|_{W^{1,(p_\alpha)}} = \|u\|_{L^1} + \sum_\alpha \|D_\alpha u\|_{L^{p_\alpha}}$).

The study of this problem was started by a paper of Marcellini, where a non degenerate integral of the type

$$\int_\Omega \Big[(1 + |Du|^2) + \sum_{\alpha=1}^{n} (1 + |D_\alpha u|^{p_\alpha})\Big] dx\,,$$

is considered. In [17], it is proved that the minimizers of this functional are locally Lipschitz continuous, provided that $2 < p_\alpha < \frac{2n}{n-2}$ for every $\alpha \in \{1, \ldots, n\}$.

To get the regularity of the minimizers of functionals of the type (1.1), an upper bound for the exponents p_α is necessary. In fact, an example given in [16] and [9], shows that if $n \geq 6$, the minimizers of the functional

$$\int_\Omega \Big[\sum_{\alpha=1}^{n-1} |D_\alpha u|^2 + \frac{1}{2}|D_n u|^4\Big] dx$$

can even be unbounded. Hence, to get regularity, the upper exponent p_n cannot be much bigger than p. For instance, it has been proved in [7] (see also [14]) that if

$$\max_{k \leq \alpha \leq n} p_\alpha \leq \bar{p}^*\,, \quad \text{where} \quad \bar{p} = n\Big[\frac{k-1}{p} + \sum_{\alpha=k}^{n} \frac{1}{p_\alpha}\Big]^{-1} \tag{1.2}$$

and $\bar{p}^* = \frac{n\bar{p}}{n-\bar{p}}$ if $\bar{p} < n$, while $\bar{p}^*$ is any number larger than $\bar{p}$ otherwise, then any minimizer u of a functional of the type (1.1) is locally bounded. A counterexample given in [14], shows that the bound (1.2) is optimal. However it is not known if this bound is enough to ensure that u is locally Lipschitz, or even continuous.

In [18] the regularity of minimizers of a fairly general class of anisotropic functionals is considered. However these results, when applied to the model case

$$\int_\Omega \Big[(1 + |Du|^2)^{\frac{p}{2}} + \sum_{\alpha=k}^{n} |D_\alpha u|^{p_\alpha}\Big] dx$$

with $p \geq 2$, ensure that $Du \in L^{\infty}_{\mathrm{loc}}(\Omega)$, provided that the exponent p_n is strictly less than $\frac{n+2}{n}p$, a bound which is quite far from (1.2).

More general results for anisotropic functionals in the scalar case are given in [19] and for the vectorial case in [2] and [20].

In this paper we improve the regularity result of [17] and [18], by considering a degenerate functional of the type (1.1), where the exponents p_α satisfy the following bounds

$$
\begin{aligned}
&2 \leq p < p_1 \leq \cdots \leq p_n < \frac{2p_1}{n} + p &&\text{if } \; k = 1 \text{ and } p_1 < n \\
&2 \leq p < p_k \leq \cdots \leq p_n \leq \frac{np}{n-2}\,\frac{p_k - 2}{p_k} + 2 &&\text{for any } \; 1 \leq k \leq n\,.
\end{aligned}
$$

Indeed, under these assumptions we prove (see Theorems 1 and 2 below), that the minimizers are locally Lipschitz.

Instead of getting the boundedness of the gradient throughout a Moser type iteration argument, we first prove that if u is a minimizer such that $Du \in L^q(\Omega)$, with $p_n < q < \frac{2p_1}{n} + p$ and $p_1 < n$, then $u \in W^{1,\frac{np}{n-2}}_{\mathrm{loc}}(\Omega)$. The condition $Du \in L^q(\Omega)$ is then dropped by an interpolation and approximation argument.

Similarly, in the case $p_k \geq n$ the first step of the proof is to show that $Du \in L^{\frac{np}{n-2}}_{\mathrm{loc}}(\Omega)$. Once this summability property on the gradient is obtained, the boundedness of Du is achieved, arguing exactly as in the standard isotropic case.

2 – Statements and preliminary results

We consider a more general functional than (1.1), of the type

$$
\mathcal{F}(u,\Omega) = \int_\Omega \Big(F(Du) + \sum_{\alpha=1}^{n} c_\alpha F_\alpha(D_\alpha u)\Big) dx\,, \tag{2.1}
$$

where the numbers c_α may eventually vanish (it will happen for instance, that $c_1, \ldots, c_{k-1} = 0$ as we assumed for the model functional (1.1)). Moreover we assume that $F : \mathbb{R}^n \to \mathbb{R}$ and $F_\alpha : \mathbb{R} \to \mathbb{R}$ are C^2 functions satisfying, for some positive constants c, ν, L and, for any $\xi, \eta \in \mathbb{R}^n$ and

$\alpha = k, \ldots, n$, the following growth, coercivity and convexity conditions:

$$\text{(H1)} \qquad \frac{1}{c}|\xi|^p \leq F(\xi) \leq c(1 + |\xi|^p)$$

$$\frac{1}{c}|\xi_\alpha|^{p_\alpha} \leq F_\alpha(\xi_\alpha) \leq c(1 + |\xi_\alpha|^{p_\alpha})$$

$$\text{(H2)} \qquad |D^2F(\xi)| \leq L(1 + |\xi|^{p-2})$$

$$0 \leq F''_\alpha(\xi_\alpha) \leq L(1 + |\xi_\alpha|^{p_\alpha - 2})$$

$$\text{(H3)} \qquad \langle D^2F(\xi)\eta, \eta\rangle \geq \nu|\xi|^{p-2}|\eta|^2$$

We prove the following result

THEOREM 1. *Let $u \in W^{1,(p_\alpha)}(\Omega)$ be a minimizer of functional* (2.1), *satisfying growth conditions* (H1), (H2), (H3). *Then $u \in W^{1,\frac{np}{n-2}}_{\mathrm{loc}}(\Omega)$, for the exponents p_α satisfying*

$$\text{(2.2)} \quad 2 \leq p < p_1 \leq \cdots \leq p_n < \frac{2p_1}{n} + p \qquad \textit{if } k = 1 \textit{ and } p_1 < n$$

$$\text{(2.3)} \quad 2 \leq p < p_k \leq \cdots \leq p_n \leq \frac{np}{n-2}\frac{p_k - 2}{p_k} + 2 \quad \textit{for any } 1 \leq k \leq n.$$

REMARK 1. Notice that both conditions (2.2) and (2.3) give an improvement of the upper bound obtained in [18].

It follows from the higher integrability result that we obtain in Theorem 1, by mean of Moser's iteration technique, that the minimizers of functional (2.1) are Lipschitz functions. More precisely the following result holds

THEOREM 2. *Let $u \in W^{1,\frac{np}{n-2}}_{\mathrm{loc}}(\Omega)$ be a local minimizer of functional* (2.1) *and let us suppose that assumptions* (H1), (H2), (H3), (2.2) *or* (2.3) *hold. Then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.*

Fix $h > 0$ and a direction in $\mathbb{R}^n$, e_s. Let us define the difference quotient

$$\Delta_h^s v = \frac{v(x + he_s) - v(x)}{h}$$

(we shall write simply $\Delta_h v$, if no confusion arises). If a function v is defined in Ω, then $\Delta_h v$ is defined in $\Omega_h = \{x \in \Omega : d(x, \partial\Omega) < h\}$ and, obviously, if $v \in W^{1,p}(\Omega)$, then $\Delta_h v \in W^{1,p}(\Omega_h)$ and for every $i = 1, \ldots, n$, $D_i(\Delta_h v) = \Delta_h(D_i v)$.

Some properties of the difference quotients are stated in the next lemma. Their proof can be found in [18].

Lemma 1. *If $\Omega' \subset\subset \Omega$ and $h_0 = d(\Omega', \partial\Omega)$, for all $h < h_0$ the following properties hold*

(i) *If f or g are supported in Ω_h, then*

$$\int_\Omega f(\Delta_h g)dx = -\int_\Omega (\Delta_{-h} f) g dx\,.$$

(ii) *If $v \in W^{1,(p_s)}(\Omega)$, then*

$$\int_{\Omega'} |\Delta_h v|^{p_s} dx \le \int_\Omega |D_s v|^{p_s} dx \qquad \text{for all} \qquad s = 1, \ldots, n\,.$$

(iii) *If $v \in L^{p_s}(\Omega)$, and if there exists a constant $c > 0$ (independent on h) such that $\|\Delta_h v\|_{L^{p_s}(\Omega')} \le c$, then $D_s v \in L^{p_s}(\Omega')$ and*

$$\|D_s v\|_{L^{p_s}(\Omega')} \le c\,.$$

(iv) *If $v \in W^{1,(p_s)}(\Omega)$, then for all $s = 1, \ldots, n$, $\Delta_h v \to D_s v$ strongly in $L^{p_s}(\Omega')$.*

We recall the following technical lemma.

Lemma 2. *If $\delta \ge 0$, then for all $\xi, \eta \in \mathbb{R}^n$ there exist two constant c_0 and c_1 depending only on δ, such that*

$$c_0 \le \frac{\int_0^1 |t\xi + (1-t)\eta|^\delta dt}{(|\xi|^2 + |\eta|^2)^{\frac{\delta}{2}}} \le c_1\,. \tag{2.4}$$

PROOF. See for example [10]. ∎

Let us fix $0 < \mu < 1$ and q such that $p_n < q < \frac{np}{n-2}$. With $\mathcal{F}_\mu$ we denote the functional

$$\mathcal{F}_\mu(v, B_R) = \int_{B_R} G_\mu(Dv)dx = \mathcal{F}(v, B_R) + \mu \int_{B_R} |Dv|^q dx\,. \tag{2.5}$$

By hypothesis (H1), (H2) and (H3), it is easy to check that $G_\mu \in C^2(\mathbb{R}^n)$ and satisfies the following conditions

$$\frac{1}{c}|\xi|^p + \mu|\xi|^q \le G_\mu(\xi) \le \tilde{c}(1 + |\xi|^q) \tag{H4}$$

$$|D^2 G_\mu(\xi)| \le \tilde{L}(1 + |\xi|^{q-2}) \tag{H5}$$

$$\langle D^2 G_\mu(\xi)\eta, \eta\rangle \ge \nu|\xi|^{p-2}|\eta|^2 + \mu|\xi|^{q-2}|\eta|^2 \tag{H6}$$

for some constants $\tilde{c}, \tilde{L}$ depending on $c, L, n, p, q, p_k, \ldots, p_n$ but not on μ. Let us prove the following higher integrability result about the minimizers of (2.5), assuming for now that $k = 1$.

PROPOSITION 1. *Let $v \in W^{1,q}_{\mathrm{loc}}(\Omega)$ be a minimizer of functional* (2.5), *with q such that*

$$p_n < q < \frac{2p_1}{n} + p\,,$$

and let us assume that conditions (H4), (H5), (H6) *hold. Then $v \in W^{1,\frac{np}{n-2}}_{\mathrm{loc}}(\Omega)$ and for all $r < R$ such that $B_{2R} \subset\subset \Omega$ there exist two constants $\beta = \beta(n, p, q, p_1) > 1$ and $c = c(p, q, p_1, \ldots, p_n, r, R) > 0$ such that*

$$\|Dv\|_{L^{\frac{np}{n-2}}(B_r)} \le c(1 + \mathcal{F}(v, B_R))^{\frac{\beta}{p}}\,. \tag{2.6}$$

Proof. Let us start with the Euler equation

$$\int_\Omega \sum_{i=1}^n G_{\mu\xi_i}(Dv)D_i\varphi dx = 0 \qquad \forall\,\varphi \in W_0^{1,q}(\Omega)\,. \tag{2.7}$$

Let us fix $\Omega' \subset\subset \Omega$ and $h < d(\Omega', \Omega)$. Moreover let us consider a smooth function $\eta \in C_0^1(\Omega')$, such that $\eta \geq 0$, and let us take as a test function in the Euler equation (2.7)

$$\varphi(x) = \Delta_{-h}(\eta^2\Delta_h v)\,.$$

Then we have

$$\int_\Omega \sum_{i=1}^n \Delta_h G_{\mu\xi_i}(Dv)(2\eta D_i\eta\Delta_h v + \eta^2\Delta_h(D_i v))\; dx = 0$$

and since we can write

$$\Delta_h G_{\mu\xi_i}(Dv) = \frac{1}{h}\int_0^1 \sum_{j=1}^n G_{\mu\xi_i\xi_j}(Dv + th\Delta_h(Dv))dt\Delta_h(D_j v)\,,$$

we get

$$\begin{aligned}
&\int_0^1\int_\Omega \eta^2 \sum_{i,j=1}^n G_{\mu\xi_i\xi_j}(Dv + th\Delta_h(Dv))\Delta_h(D_i v)\Delta_h(D_j v)dx\,dt = \\
&\quad = -2\int_0^1\int_\Omega \eta\Delta_h v \sum_{i,j=1}^n G_{\mu\xi_i\xi_j}(Dv + th\Delta_h(Dv))D_i\eta\Delta_h(D_j v)dx\,dt \leq \\
&\quad \leq c\int_0^1\int_\Omega \Big(\eta^2 \sum_{i,j=1}^n G_{\mu\xi_i\xi_j}(Dv + th\Delta_h(Dv))\Delta_h(D_i v)\Delta_h(D_j v)\Big)^{\frac{1}{2}} \times \\
&\qquad \times \Big(|\Delta_h v|^2 \sum_{i,j=1}^n G_{\mu\xi_i\xi_j}(Dv + th\Delta_h(Dv))D_i\eta D_j\eta\Big)^{\frac{1}{2}} dx\,dt
\end{aligned}$$

by Cauchy-Schwartz inequality. Then, applying Young inequality we have

$$\begin{aligned}
&\int_0^1\int_\Omega \eta^2 \sum_{i,j=1}^n G_{\mu\xi_i\xi_j}(Dv + th\Delta_h(Dv))\Delta_h(D_i v)\Delta_h(D_j v)dx\,dt \leq \\
&\quad \leq \frac{c}{\epsilon}\int_0^1\int_\Omega |\Delta_h v|^2 \sum_{i,j=1}^n G_{\mu\xi_i\xi_j}(Dv + th\Delta_h(Dv))D_i\eta D_j\eta dx\,dt + \\
&\qquad + c\epsilon\int_0^1\int_\Omega \eta^2 \sum_{i,j=1}^n G_{\mu\xi_i\xi_j}(Dv + th\Delta_h(Dv))\Delta_h(D_i v)\Delta_h(D_j v)dx\,dt\,.
\end{aligned}$$

Subtracting the last integral on the right hand side and applying (H5) and (H6) we easily get

$$\int_0^1 \int_\Omega \eta^2 |Dv + th\Delta_h(Dv)|^{p-2} |\Delta_h(Dv)|^2 dx\,dt \leq$$
$$\leq c \int_0^1 \int_\Omega |D\eta|^2 |\Delta_h v|^2 (1 + |Dv + th\Delta_h(Dv)|^{q-2}) dx\,dt\,.$$

Finally, by Lemma 2 we have

$$\tag{2.8} \begin{aligned} &\int_\Omega \eta^2 (|Dv(x)|^2 + |Dv(x+he_s)|^2)^{\frac{p-2}{2}} |\Delta_h(Dv)|^2 dx \leq \\ &\leq c \int_\Omega |D\eta|^2 |\Delta_h v|^2 (1 + |Dv(x)|^{q-2} + |Dv(x+he_s)|^{q-2}) dx\,. \end{aligned}$$

Let us fix $R > 0$ such that $B_{2R} \subset\subset \Omega$ and for every $r < R$, let η be a standard *cut-off* function between $B_{r'}$ and $B_{R'}$, where $r < r' < R' < R$, i.e. $\eta \in C^1(B_{R'})$, $0 \leq \eta \leq 1$, $\eta = 1$ in $B_{r'}$ and $|D\eta| \leq \frac{c}{R'-r'}$. With this choice of η, from (2.8) we get

$$\tag{2.9} \begin{aligned} &\int_{B_{r'}} (|Dv(x)|^2 + |Dv(x+he_s)|^2)^{\frac{p-2}{2}} |\Delta_h(Dv)|^2 dx \leq \\ &\leq \frac{c_1}{(R'-r')^2} \int_{B_{R'} \setminus B_{r'}} |\Delta_h v|^2 (1 + |Dv(x)|^{q-2} + |Dv(x+he_s)|^{q-2}) dx\,. \end{aligned}$$

Since $v \in W^{1,q}$, by Lemma 1 it follows that the right hand side of (2.9) is finite. Thus letting $h \to 0$, we get that $|D(|D_s v|^{\frac{p}{2}})|^2 \leq c(p)|Dv|^{p-2}|D_s(Dv)|^2$ belongs to $L^1(B_{r'})$. Therefore, by Sobolev embedding theorem it follows that $D_s v \in L^{\frac{np}{n-2}}(B_{r'})$ for every $s = 1, \ldots, n$. Adding up over s we have (by Lemma 2.9 in [18])

$$\tag{2.10} \Big(\int_{B_{r'}} |Dv|^{\frac{np}{n-2}} dx \Big)^{\frac{n-2}{n}} \leq \frac{c_2}{(R'-r')^2} \int_{B_{R'} \setminus B_{r'}} (1 + |Dv|^q) dx\,.$$

Now since $p_1 \leq p_n < q < \frac{np}{n-2}$, we can apply the interpolation inequality

$$\|Dv\|_{L^q} \leq \|Dv\|_{L^{p_1}}^{\vartheta} \|Dv\|_{L^{\frac{np}{n-2}}}^{1-\vartheta}$$

where the number $\vartheta \in (0,1)$ can be computed directly from the relation

$$\frac{1}{q} = \frac{\vartheta}{p_1} + \frac{(1-\vartheta)(n-2)}{np},$$

obtaining that

$$\vartheta = \frac{p_1\Big(\dfrac{np}{n-2} - q\Big)}{q\Big(\dfrac{np}{n-2} - p_1\Big)}.$$

Therefore we have

$$\int_{B_{R'}\setminus B_{r'}} |Dv|^q dx \le \Big(\int_{B_{R'}} |Dv|^{p_1} dx\Big)^{\frac{\vartheta q}{p_1}} \times \times \Big(\int_{B_{R'}\setminus B_{r'}} |Dv|^{\frac{np}{n-2}} dx\Big)^{\frac{q(1-\vartheta)(n-2)}{np}}. \tag{2.11}$$

Next step in the proof will be now to apply Young inequality to the right hand side of (2.11) and then to use the so called hole-filling argument on the integral over the annulus $B_{R'}\setminus B_{r'}$. To do this we need that $\frac{q(1-\vartheta)}{p} < 1$, that is, by the definition of ϑ,

$$q < p\Big(1 - \frac{p_1(n-2)}{np}\Big) + p_1 = \frac{2p_1}{n} + p.$$

It is easy to show that this number is strictly smaller than $\frac{np}{n-2}$. Then, since $p_n < \frac{2p_1}{n} + p$, such a choice for q is always possible, and then we are allowed to apply the interpolation argument showed before.

Let us set $\sigma = \frac{p}{q(1-\vartheta)} > 1$, and let us apply Young inequality in (2.11), with exponents σ and $\frac{\sigma}{\sigma-1}$. Then since

$$\beta = \frac{\vartheta q \sigma}{p_1(\sigma - 1)} = \frac{np - q(n-2)}{2p_1 - n(q-p)} > 1,$$

choosing $\delta > 0$ such that

$$\frac{1}{(R'-r')^\delta} = \max\Big\{\frac{1}{(R'-r')^2}, \frac{1}{(R'-r')^{\frac{2\sigma}{\sigma-1}}}\Big\},$$

by (2.10) and (2.11) we easily get

$$\left(\int_{B_{r'}} |Dv|^{\frac{np}{n-2}} dx\right)^{\frac{n-2}{n}} \leq \frac{c_3}{(R'-r')^\delta}\left(\int_{B_{R'}} (1+|Dv|^{p_1}) dx\right)^\beta + \\ + c_4\left(\int_{B_{R'}\setminus B_{r'}} |Dv|^{\frac{np}{n-2}} dx\right)^{\frac{n-2}{n}}. \tag{2.12}$$

for some constant c_3, c_4 depending of $n, p, q, p_1, \ldots, p_n$.

Now let us raise both sides of (2.12) at $\frac{n}{n-2}$. We *fill the hole*, adding up to both sides the quantity

$$c_4^{\frac{n}{n-2}} \int_{B_{r'}} |Dv|^{\frac{np}{n-2}} dx\,.$$

Then we raise again to power $\frac{n-2}{n}$ obtaining finally

$$\left(\int_{B_{r'}} |Dv|^{\frac{np}{n-2}} dx\right)^{\frac{n-2}{n}} \leq \frac{c_3}{(R'-r')^\delta}\int_{B_{R'}} (1+|Dv|^{p_1}) dx + \\ + \gamma\left(\int_{B_{R'}} |Dv|^{\frac{np}{n-2}} dx\right)^{\frac{n-2}{n}} \tag{2.13}$$

for some constants c_3, γ that depend only of $n, p, q, p_1, \ldots, p_n$, and in particular $\gamma = \frac{c_4}{c_4+1} < 1$. To proceed now we need Lemma 3 below.

By using this lemma in inequality (2.13), we easily obtain

$$\left(\int_{B_r} |Dv|^{\frac{np}{n-2}} dx\right)^{\frac{n-2}{n}} \leq c_5(1+\mathcal{F}(v, B_R))^\beta \tag{2.14}$$

where $c_5 = c_5(n, p, q, p_1, \ldots, p_n, r, R)$. This completes the proof. $\square$

Lemma 3. *Let $f : [r, R] \to \mathbb{R}$ be a bounded function such that for $\delta > 0$ and $0 \leq \gamma < 1$ the inequality*

$$f(s) \leq \gamma f(t) + \frac{A}{(t-s)^\delta} + B$$

(where A and B are constants) holds for all $r \leq s < t \leq R$. Then we have

$$f(r) \leq c\left[\frac{A}{(R-r)^\delta} + B\right] \quad \textit{with} \quad c = c(\gamma, \delta).$$

PROOF. See for example [13].

Let us now go back to the minimizers of functional (2.1).

PROPOSITION 2. *Let* $u \in W^{1,(p_\alpha)}(\Omega)$ *be a minimizer of functional* (2.1) *and let us suppose that conditions* (H1), (H2), (H3) *hold and that*

$$2 \leq p < p_k \leq \cdots \leq p_n \leq \frac{np}{n-2}\frac{p_k-2}{p_k} + 2\,.$$

Then $u \in W^{1,\frac{np}{n-2}}_{\mathrm{loc}}(\Omega)$.

PROOF. The Euler equation for functional (2.1) is

$$\int_\Omega \Big[\sum_{i=1}^n F_{\xi_i}(Du)D_i\varphi + \sum_{\alpha=k}^n c_\alpha F'_\alpha(D_\alpha u)D_\alpha\varphi\Big]dx = 0 \qquad \forall\, \varphi \in W_0^{1,(p_\alpha)}(\Omega)\,.$$

Arguing with the difference quotient method as before we easily reach, in the same way of (2.8), the following inequality

$$\begin{aligned}\int_\Omega \eta^2(|Du(x)|^2 + |Du(x+he_s)|^2)^{\frac{p-2}{2}}|\Delta_h(Du)|^2 dx \leq \\ \leq c\int_\Omega |D\eta|^2\Big((1+|Du|^p)+ \\ +|\Delta_h u|^2\sum_{\alpha=k}^n c_\alpha(1+|D_\alpha u(x)|^2+|D_\alpha u(x+he_s)|^2)^{\frac{p_\alpha-2}{2}}\Big)dx\,.\end{aligned} \tag{2.15}$$

We change the previous approach, supposing first that $s=n$. Then

$$\begin{aligned}\int_{B_R} |\Delta_h^{(n)}u|^2(1+|D_\alpha u(x)|^2+|D_\alpha u(x+he_n)|^2)^{\frac{p_\alpha-2}{2}}dx \leq \\ \leq c_1\int_{B_R}|D_n u|^{p_\alpha}dx + c_2\int_{B_R}(1+|D_\alpha u|^{p_\alpha})dx < \infty\end{aligned}$$

for all $\alpha \leq n$ obviously. Then letting $h\to 0$ we get that $D_n(|Du|^{\frac{p}{2}}) \in L^2(B_r)$ and thus that $D_n u \in L^{\frac{np}{n-2}}(B_r)$. We are going to find conditions under which it is possible to iterate this argument. Namely let us suppose that for some $s>k$, $D_{s+1}u,\ldots,D_n u \in L^{\frac{np}{n-2}}(B_r)$. Then we have for $\alpha > s$

(since it is easy to check that all the terms in the right hand side of (2.15) with $k \le \alpha \le s$ are finite)

$$\int_{B_R} |\Delta_h u|^2 (1 + |D_\alpha u(x)|^2 + |D_\alpha u(x + he_s)|^2)^{\frac{p_\alpha - 2}{2}} dx \le$$
$$\le \Big(\int_{B_R} (1 + |D_\alpha u|^{\frac{np}{n-2}}) dx \Big)^{\frac{(n-2)(p_\alpha - 2)}{np}} \times$$
$$\times \Big(\int_{B_R} |\Delta_h u|^{\frac{2np}{np - (p_\alpha - 2)(n-2)}} dx \Big)^{\frac{np - (p_\alpha - 2)(n-2)}{np}} .$$

The right hand side is finite if

$$\frac{2np}{np - (p_\alpha - 2)(n-2)} \le p_s \,.$$

This inequality is satisfied if

$$p_n \le \frac{np}{n-2} \frac{p_k - 2}{p_k} + 2 \,.$$

Then letting $h \to 0$ in (2.15) we obtain that

$$\Big(\int_{B_r} |D_n u|^{\frac{np}{n-2}} dx \Big)^{\frac{n-2}{n}} \le \frac{c}{(R-r)^2} \Big[\mathcal{F}(u, B_R) + \sum_{\alpha=k}^{n} \int_{B_R} (1 + |D_n u|^{p_\alpha}) dx \Big] < \infty$$

and, by induction, for all $s = k, \ldots, n-1$

$$\begin{aligned} \Big(\int_{B_r} |D_s u|^{\frac{np}{n-2}} dx \Big)^{\frac{n-2}{n}} \le \frac{c}{(R-r)^2} \Big[& \mathcal{F}(u, B_R) + \\ & + \sum_{\alpha=k}^{s} \int_{B_R} (1 + |D_s u|^{p_\alpha}) dx + \\ & + \sum_{\alpha=s+1}^{n} \int_{B_R} (1 + |D_\alpha u|^{\frac{np}{n-2}} + |D_s u|^{q_\alpha}) dx \Big] \end{aligned} \tag{2.16}$$

where $q_\alpha = \frac{2np}{np - (p_\alpha - 2)(n-2)} \le p_k$. This concludes the proof. $\square$

Remark 2. If we compare the limitations for the exponents p_α given in the statements of propositions 1 and 2, we find that

$$\frac{2p_k}{n} + p \leq \frac{np}{n-2}\frac{p_k - 2}{p_k} + 2$$

if and only if $p_k \geq n$. This is obviously the meaning of this choice in the statement of Theorem 1.

Remark 3. In the special case of $n = 2$, (2.15) becomes (we suppose in this case that $k = 1$, so that $p < p_1 \leq p_2$)

$$\begin{aligned} &\int_{B_r} |\Delta_h^{(s)}(|Du|^{\frac{p}{2}})|^2 dx \leq \frac{c}{(R-r)^2}\int_{B_R} \times \\ &\times \Big[(1+|Du|^p)+|\Delta_h^{(s)}u|^2(1+|D_1u(x)|^2+|D_1u(x+he_s)|^2)^{\frac{p_1-2}{2}}+ \\ &+ |\Delta_h^{(s)}u|^2(1+|D_2u(x)|^2+|D_2u(x+he_s)|^2)^{\frac{p_2-2}{2}}\Big]dx \end{aligned} \tag{2.17}$$

for $s = 1, 2$. Using the method seen in the proof of Proposition 2, we obtain for $s = 2$, that $D(|D_2u|^{\frac{p}{2}}) \in L^2_{\text{loc}}(\Omega)$. Then by Sobolev imbedding theorem $|D_2u|^{\frac{p}{2}} \in L^\chi_{\text{loc}}(\Omega)$ for any $\chi > 2$.

If we now consider (2.17) with $s = 1$, the only term of the right hand side that is not trivially finite is

$$\begin{aligned} &\int_{B_R} |\Delta_h^{(1)}u|^2(1+|D_2u(x)|^2+|D_2u(x+he_s)|^2)^{\frac{p_2-2}{2}}dx \\ &\leq \Big(\int_{B_R} |\Delta_h^{(1)}u|^{\frac{2\chi p}{\chi p - 2(p_2-2)}}dx\Big)^{\frac{\chi p - 2(p_2-2)}{\chi p}}\Big(\int_{B_R}(1+|D_2u|^{\frac{\chi p}{2}})dx\Big)^{\frac{2(p_2-2)}{\chi p}} \end{aligned}$$

that however is finite if $\frac{2\chi p}{\chi p - 2(p_2-2)} \leq p_1$ that is if $\chi \geq \frac{2p_1(p_2-2)}{p(p_1-2)} > 2$.

By the arbitrarity of χ, we conclude that in the case of $n = 2$, we have regularity for the minimizer u of functional (2.1) without further conditions on the exponents p_α.

3 – Proof of Theorem 1

In this section we will use the results obtained in Section 2, to get the proof of Theorem 1. We observe that if condition (2.3) is satisfied, the result follows immediately from Proposition 2. To conclude the proof we shall restrict to the case that condition (2.2) holds.

We will use an approximation argument as in [6]. Let $u \in W^{1,(p_\alpha)}(\Omega)$ be a minimizer of functional (2.1). Fix $R > 0$ such that $B_{2R} \subset\subset \Omega$. Then for $0 < \varepsilon < \min\{1, R\}$, we can introduce a sequence of smooth functions u_ε, obtained by mollifying u, so that $u_\varepsilon \to u$ strongly in $W^{1,(p_\alpha)}(B_R)$. In particular we can suppose that $u_\varepsilon \in W^{1,q}_{\mathrm{loc}}(\Omega)$. We fix $0 < \mu < 1$ and consider, as in Section 2, the functional $\mathcal{F}_\mu(v, B_R)$.

Notice that by (H4), (H5) and (H6) it follows that functional $\mathcal{F}_\mu$ has the same growth q from above and from below. Thus let us denote by $v_{\varepsilon,\mu} \in u_\varepsilon + W^{1,q}_0(B_R)$ the solution of the Dirichlet problem

$$\min\Big\{ \int_{B_R} G_\mu(Dw)dx : \quad w \in u_\varepsilon + W^{1,q}(B_R)\Big\}.$$

We can now apply Proposition 1 to the minimizers $v_{\varepsilon,\sigma}$ of functional $\mathcal{F}_\mu$ thus obtaining

$$\begin{aligned}\Big(\int_{B_r} |Dv_{\varepsilon,\mu}|^{\frac{np}{n-2}}dx\Big)^{\frac{n-2}{n}} &\le c_6(1 + \mathcal{F}(v_{\varepsilon,\mu}, B_R))^\beta \le \\ &\le c_6(1 + \mathcal{F}_\mu(v_{\varepsilon,\mu}, B_R))^\beta \le c_6(1 + \mathcal{F}_\mu(u_\varepsilon, B_R))^\beta \le \\ &\le c_7\Big(1 + \mathcal{F}(u, B_{R+\varepsilon}) + \mu \int_{B_R} |Du_\varepsilon|^q\Big)^\beta.\end{aligned} \tag{3.1}$$

Since $\mu < 1$, we have that $v_{\varepsilon,\mu}$ is uniformly bounded with respect to μ in $W^{1,\frac{np}{n-2}}$ and then we can say that up to a subsequence

$$v_{\varepsilon,\mu} \rightharpoonup v_\varepsilon \qquad \text{weakly in} \quad W^{1,(p_\alpha)}(B_R)$$

for some $v_\varepsilon \in u_\varepsilon + W^{1,(p_\alpha)}_0(B_R)$.

Since the functional $\mathcal{F}$ is lower semicontinuous with respect to the weak topology of $W^{1,(p_\alpha)}(B_R)$, we can let $\mu \to 0$ in (3.1) obtaining

$$\Big(\int_{B_r} |Dv_\varepsilon|^{\frac{np}{n-2}}dx\Big)^{\frac{n-2}{n}} \le c_7(1 + \mathcal{F}(u, B_{R+\varepsilon}))^\beta. \tag{3.2}$$

These estimates are uniform with respect to ε. But $u_\varepsilon \to u$ strongly in $W^{1,(p_\alpha)}(B_R)$ and then by (3.2) we deduce that up to a subsequence

$$v_\varepsilon \rightharpoonup v \qquad \text{weakly in} \quad W^{1,(p_\alpha)}(B_R),$$

for some $v \in u + W^{1,(p_\alpha)}_0(B_R)$. Then letting $\varepsilon \to 0$ in (3.2) we get

$$\Big(\int_{B_r} |Dv|^{\frac{np}{n-2}}dx\Big)^{\frac{n-2}{n}} \le c_7(1 + \mathcal{F}(u, B_R))^\beta. \tag{3.3}$$

In particular by the lower semicontinuity of $\mathcal{F}$ in the weak topology of $W^{1,(p_\alpha)}$ we have

$$\begin{aligned}(3.4)\qquad \mathcal{F}(v,B_R) &\le \liminf_{\varepsilon\to0}\mathcal{F}(v_\varepsilon,B_R)\le \liminf_{\varepsilon\to0}\liminf_{\mu\to0}\mathcal{F}_\mu(v_{\varepsilon,\mu},B_R)\le\\ &\le \liminf_{\varepsilon\to0}\liminf_{\mu\to0}\mathcal{F}_\mu(u_\varepsilon,B_R)=\liminf_{\varepsilon\to0}\mathcal{F}(u_\varepsilon,B_R)\le\mathcal{F}(u,B_R).\end{aligned}$$

By (3.4) and the minimality of u, we have finally that $v=u$ in B_R since they are both solutions to the same Dirichlet problem

$$\min\left\{\mathcal{F}(w,B_R):\quad w\in u+W_0^{1,(p_\alpha)}(B_R)\right\}.$$

Then in particular $u\in W^{1,\frac{np}{n-2}}_{\mathrm{loc}}(\Omega)$ and so Theorem 1 is proved. $\square$

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