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A semilinear heat equation with concave-convex nonlinearity

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RIASSUNTO: *In questo articolo, ci interessiamo all'equazione parabolica* $u_t - \Delta u = \lambda u^q + u^p$ *in un dominio limitato di* $\mathbb{R}^N$*, con la condizione al bordo di Dirichlet e parametri* $0 < q < 1 < p$ *e* $\lambda > 0$*. Studiamo il problema di Cauchy associato e il comportamento globale delle soluzioni positive. Ci interessiamo in particolar modo alle relazioni tra le soluzioni globali (in tempo) dell'equazione parabolica e le soluzioni del problema stazionario ellittico. In particolare, dimostiamo che esiste una soluzione globale se e solo se esiste una soluzione debole dell'equazione stazionaria.*

ABSTRACT: *In this paper, we are interested in the parabolic equation* $u_t - \Delta u = \lambda u^q + u^p$ *in a bounded domain of* $\mathbb{R}^N$*, with the Dirichlet boundary condition and the parameters* $0 < q < 1 < p$ *and* $\lambda > 0$*. We study the initial value problem and the global behavior of the the positive solutions. We are mainly interested in the relations between the global (in time) solutions of the parabolic equation and the solutions of the stationary, elliptic problem. We show in particular that there exists a global solution if and only if there exists a weak solution of the stationary equation.*

1 – Introduction

Let Ω be a bounded, smooth domain of $\mathbb{R}^N$, and consider the non-linear heat equation

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$$
\begin{cases} u_t - \Delta u = \lambda u^q + u^p & (t,x) \in (0,T) \times \Omega, \\ u = 0 & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega. \end{cases} \tag{1.1}
$$

Here, $0 < q < 1 < p$ and $\lambda > 0$, so that the nonlinearity on the right-hand side of (1.1) is the sum of a concave and a convex term. The nonlinearity is "singular" at 0 (in the sense that it is not Lipschitz) because $q < 1$. For the nonlinearity $g(u) = \lambda u^q$ with $0 < q < 1$, existence and uniqueness of positive solutions is proved in [8,9]. However, their methods do not apply immediately to the problem (1.1) because of the presence of the term u^p. For completeness, we study in the Appendix at the end of this paper the initial value problem for equations of the type (1.1). Even if we use essentially the methods of [8,9], some new ingredients are needed. We prove in particular the following result (see Theorem 6.2).

Theorem 1.1. *For all $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, there exists a unique, positive solution u of* (1.1) *defined on a maximal time interval* $[0, T_{\mathrm{m}})$, $u \in L^\infty((0,T) \times \Omega)$ *for all $T < T_{\mathrm{m}}$. Moreover, there is the blow up alternative: either $T_{\mathrm{m}} = +\infty$ or else $T_{\mathrm{m}} < \infty$ and $\|u(t)\|_{L^\infty} \underset{t \uparrow T_{\mathrm{m}}}{\longrightarrow} \infty$.*

The elliptic version of (1.1), i.e.

$$
\begin{cases} -\Delta u = \lambda u^q + u^p & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.2}
$$

was studied in particular by Boccardo, Escobedo and Peral [3], Ambrosetti, Brezis and Cerami [1], Bartsch and Willem [2], Cabré and Majer [6]. It is known that there exists a critical value $0 < \lambda^* < \infty$ of the parameter λ, such that for $\lambda \in (0, \lambda^*)$ there exists a minimal, positive solution $u_\lambda \in L^\infty(\Omega)$ of (1.2). u_λ is minimal in the sense that if $u \geq 0$, $u \not\equiv 0$ is any solution of (1.2), then $u \geq u_\lambda$. For $\lambda = \lambda^*$, there exists a weak, positive solution u_{λ^*} of (1.2), $u_{\lambda^*} \in H_0^1(\Omega) \cap L^{p+1}(\Omega)$. u_{λ^*} is obtained as the increasing limit of u_λ as $\lambda \uparrow \lambda^*$. For $\lambda > \lambda^*$, there does not exist any solution $u \geq 0$, $u \not\equiv 0$ of (1.2) in $L^\infty(\Omega)$.

Our main results are the following.

Theorem 1.2. *Let $u_0 = 0$ and let u be the unique, positive solution of* (1.1) *defined on the maximal interval* $(0, T_{\mathrm{m}})$.

(i) *If* $0 < \lambda < \lambda^*$, *then* $T_{\mathrm{m}} = \infty$, $0 < u < u_\lambda$ *and* $u(t)$ *converges to* u_λ *in* $L^\infty(\Omega)$ *as* $t \to \infty$.

(ii) *If* $\lambda = \lambda^*$, *then* $T_{\mathrm{m}} = \infty$, $0 < u < u_{\lambda^*}$ *and* $u(t)$ *converges to* u_{λ^*} *in* $L^{p+1}(\Omega)$ *as* $t \to \infty$. *Moreover, there exists a constant* C *such that* $\|u(t)\|_{L^\infty} \leq C e^{\frac{\lambda_1}{p-1}t}$, *where* λ_1 *is the first eigenvalue of* $-\Delta$ *in* $H_0^1(\Omega)$.

(iii) *If* $\lambda > \lambda^*$, *then* $T_{\mathrm{m}} < \infty$.

Corollary 1.3. *Let $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ and let u be the unique, positive solution of (1.1) defined on the maximal interval* $(0, T_{\mathrm{m}})$.

(i) *If* $0 < \lambda < \lambda^*$ *and if* $u_0 \leq u_\lambda$, *then* $T_{\mathrm{m}} = \infty$, $0 < u \leq u_\lambda$ *and* $u(t)$ *converges to* u_λ *in* $L^\infty(\Omega)$ *as* $t \to \infty$.

(ii) *If* $\lambda = \lambda^*$ *and if* $u_0 \leq u_{\lambda^*}$, *then* $T_{\mathrm{m}} = \infty$, $0 < u \leq u_{\lambda^*}$ *and* $u(t)$ *converges to* u_{λ^*} *in* $L^{p+1}(\Omega)$ *as* $t \to \infty$. *Moreover, there exists a constant* C *such that* $\|u(t)\|_{L^\infty} \leq C e^{\frac{\lambda_1}{p-1}t}$, *where* λ_1 *is the first eigenvalue of* $-\Delta$ *in* $H_0^1(\Omega)$.

(iii) *If* $\lambda > \lambda^*$, *then* $T_{\mathrm{m}} < \infty$.

The proof of Theorem 1.2 relies on the arguments introduced in [5]. Several modifications are necessary because the nonlinearity $\lambda u^q + u^p$ is convex only for u large (see also [6]).

The paper is organized as follows. In Section 2, we study the problem (1.2) and show in particular that it has no positive weak solution for $\lambda > \lambda^*$. In Section 3, we prove Theorem 1.2 while in Section 4 we prove some further results including Corollary 1.3. In Section 5, we compare some aspects of the cases $q < 1$ and $q = 1$ and we study in particular the behavior of the branch $(u_\lambda)_{0<\lambda<\lambda^*}$ as $q \uparrow 1$. Finally, Section 6 is an appendix devoted to the study of the initial value problem for equations of the type (1.1).

Throughout the paper, $(T(t))_{t\geq 0}$ is the heat semigroup, i.e. $T(t) = e^{t\Delta}$ and we denote by λ_1 the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and by φ_1 the corresponding eigenvector such that $\int_\Omega \varphi_1 = 1$. We set

$$\mathrm{d}_\Omega(x) \equiv \mathrm{dist}(x, \partial\Omega),$$

and we recall that there exist two constants $c_0, C_0 > 0$ such that $c_0\varphi_1 < \mathrm{d}_\Omega < C_0\varphi_1$ in Ω.

2 – Weak solutions of (1.2)

We begin with a definition.

Definition 2.1. Consider a continuous function $g : [0,\infty) \to [0,\infty)$. A weak solution of the equation

$$\begin{cases} -\Delta u = g(u) & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

is a function $u \in L^1(\Omega)$, $u \geq 0$ such that $g(u)\mathrm{d}_\Omega \in L^1(\Omega)$ and

$$\int_\Omega u(-\Delta\xi) = \int_\Omega g(u)\xi,$$

for all $\xi \in C^2(\overline{\Omega})$, $\xi_{|\partial\Omega} = 0$. (Note that the definition makes sense, since $|\xi| \leq C\mathrm{d}_\Omega$.)

Remark 2.2. Here are some comments on Definition 2.1.

(i) One can define similarly the notion of weak supersolution, i.e.

$$\int_\Omega u(-\Delta\xi) \geq \int_\Omega g(u)\xi,$$

for all $\xi \in C^2(\overline{\Omega})$, $\xi_{|\partial\Omega} = 0$, $\xi \geq 0$ in Ω. Subsolutions are defined accordingly.

(ii) It is clear that if u is a weak supersolution, if $f \in L^\infty(\Omega)$ satisfies $f \leq g(u)$ and if w is a subsolution of the equation

$$\begin{cases} -\Delta w = f & \text{in} \quad \Omega, \\ w = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

then $u \geq w$ a.e. in Ω. Indeed,

$$\int_\Omega (u - w)(-\Delta\xi) \geq 0,$$

for all $\xi \in C^2(\overline{\Omega})$, $\xi_{|\partial\Omega} = 0$, $\xi \geq 0$ in Ω, which implies that $u - w \geq 0$.

(iii) It follows in particular from (ii) and the maximum principle that if u is a weak supersolution and if $g(u) \not\equiv 0$, then there exists $\delta > 0$ such that $u \geq \delta \mathrm{d}_\Omega$.

As recalled before, there is no nontrivial solution of (1.2) in $L^\infty(\Omega)$ for $\lambda > \lambda^*$. The next result shows that there is no nontrivial weak solution either.

Proposition 2.3. *Suppose $\lambda > \lambda^*$. If u is a weak solution of* (1.2), *then $u \equiv 0$.*

The proof of Proposition 2.3 is based on the following variant of Kato's inequality for weak solutions of (1.2) (see Lemma 2 of [5]).

Lemma 2.4. *Let $\Phi \in C(\mathbb{R})$ and $\Psi \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Assume that $\Phi(0) = 0$, Φ is concave, $\Phi' \in L^\infty(\mathbb{R})$ and $\Phi'(s) \geq \Psi(s)$ for a.a. $s \in \mathbb{R}$. Let $f \geq 0$ such that $f \mathrm{d}_\Omega \in L^1(\Omega)$ and let $u \in L^1(\Omega)$ satisfy*

$$\int_\Omega u(-\Delta\xi) = \int_\Omega f\xi, \tag{2.1}$$

for all $\xi \in C^2(\overline{\Omega})$, $\xi_{|\partial\Omega} = 0$. It follows that

$$\int_\Omega \Phi(u)(-\Delta\xi) \geq \int_\Omega \Psi(u) f\xi, \tag{2.2}$$

for all $\xi \in C^2(\overline{\Omega})$, $\xi_{|\partial\Omega} = 0$, $\xi \geq 0$.

Proof. We proceed in three steps.

Step 1. The case $\Phi \in C^2(\mathbb{R})$ and $f \in C_\mathrm{c}^\infty(\Omega)$. Note that in this case $u \in C^2(\overline{\Omega})$, so that

$$-\Delta\Phi(u) = \Phi'(u)(-\Delta u) - \Phi''(u)|\nabla u|^2 \geq \Phi'(u)(-\Delta u) = \Phi'(u)f \geq \Psi(u)f.$$

Multiplying the above inequality by $\xi \in C^2(\overline{\Omega})$, $\xi_{|\partial\Omega} = 0$, $\xi \geq 0$ and integrating by parts, we deduce (2.2).

Step 2. The case $\Phi \in C^2(\mathbb{R})$ and $f\mathrm{d}_\Omega \in L^1(\Omega)$. Let $(f_n)_{n\geq 0} \subset C_\mathrm{c}^\infty(\Omega)$, $f_n \geq 0$, be such that $f_n \mathrm{d}_\Omega \to f\mathrm{d}_\Omega$ in $L^1(\Omega)$, and let $(u_n)_{n\geq 0}$ be the corresponding solutions of (2.1). It follows (see Lemma 1 of [5]) that $u_n \to u$ in $L^1(\Omega)$. Since by Step 1,

$$\int_\Omega \Phi(u_n)(-\Delta\xi) \geq \int_\Omega \Psi(u_n) f_n \xi,$$

for all $\xi \in C^2(\overline{\Omega})$, $\xi_{|\partial\Omega} = 0$, $\xi \geq 0$, we deduce (2.2) by letting $n \to \infty$.

STEP 3. The general case. Let $(\rho_n)_{n\geq 0}$ be a sequence of nonnegative mollifiers, and set $\Phi_n = \rho_n \star \Phi$, $\Psi_n = \rho_n \star \Psi$. It follows that $\Phi_n, \Psi_n \in C^\infty(\mathbb{R})$ and that $\Phi_n \to \Phi$ and $\Psi_n \to \Psi$ uniformly on bounded sets. In addition, Φ_n is concave, $\Phi_n' \in L^\infty(\mathbb{R})$, $\Psi_n \in L^\infty(\mathbb{R})$ and $\Phi_n'(s) \geq \Psi_n(s)$ for a.a. $s \in \mathbb{R}$. Therefore, we deduce from Step 2 that

$$\int_\Omega (\Phi_n(u) - \Phi_n(0))(-\Delta\xi) \geq \int_\Omega \Psi_n(u) f \xi,$$

for all $\xi \in C^2(\overline{\Omega})$, $\xi_{|\partial\Omega} = 0$, $\xi \geq 0$. Using the fact that Φ_n' and Ψ_n are uniformly bounded in $L^\infty(\mathbb{R})$, we may pass to the limit as $n \to \infty$ and we obtain (2.2). This completes the proof. □

COROLLARY 2.5. *Let $\lambda > 0$ and let $u \not\equiv 0$ be a weak solution of* (1.2). *It follows that for every $\widetilde{\lambda} \in (0, \lambda)$, there exists a corresponding solution $\widetilde{u} \in L^\infty(\Omega)$, $\widetilde{u} > 0$ of* (1.2) (*with λ replaced by $\widetilde{\lambda}$*).

PROOF. The result follows by applying Lemma 2.4 with appropriate functions Φ and Ψ, which we now construct. Set $g(u) = \lambda u^q + u^p$ and $\widetilde{g}(u) = \widetilde{\lambda} u^q + u^p$ for $u \geq 0$. Note that $\widetilde{g} \leq g$, $\widetilde{g}' \leq g'$ and that there exists $a > 0$ such that g is convex on (a, ∞). We now define the functions Φ and Ψ as follows. Set

$$H(x) = \int_0^x \frac{ds}{g(s+a)}, \qquad \widetilde{H}(x) = \int_0^x \frac{ds}{\widetilde{g}(s+a)},$$

and let

$$\Phi(x) = \begin{cases} x & \text{for} \quad 0 \leq x \leq a, \\ a + \widetilde{H}^{-1}(H(x-a)) & \text{for} \quad x \geq a, \end{cases} \quad \text{and} \quad \Psi(x) = \frac{\widetilde{g}(\Phi(x))}{g(x)}.$$

It is clear that $\Phi, \Psi \in C(\mathbb{R})$. Next,

$$\lim_{x\to\infty} \widetilde{H}(x) = \int_0^\infty \frac{ds}{\widetilde{g}(s+a)} > \int_0^\infty \frac{ds}{g(s+a)} = \lim_{x\to\infty} H(x),$$

so that $\sup_{x\geq 0} \Phi(x) < \infty$. Also, since $\widetilde{H}(x) \geq H(x)$, we see that $\Phi(x) \leq x$. It follows in particular that $\Psi(x) \leq 1$. Furthermore,

$$\Phi'(x) = \begin{cases} 1 & \text{for} \quad 0 \leq x < a, \\ \frac{\widetilde{g}(\Phi(x))}{g(x)} & \text{for} \quad x > a. \end{cases}$$

Thus $1 \geq \Phi'(x) \geq \Psi(x)$ for all $x \geq 0$. In addition,

$$\Phi''(x) = \frac{\widetilde{g}(\Phi(x))}{g(x)^2}[\widetilde{g}'(\Phi(x)) - g'(x)] \leq \frac{\widetilde{g}(\Phi(x))}{g(x)^2}[g'(\Phi(x)) - g'(x)] \leq 0,$$

for $x \geq a$. It follows that Φ is concave; and so we may apply Lemma 2.4 with the functions Φ and Ψ defined above.

Let w be defined by $w(x) = \Phi(u(x))$. It follows from Lemma 2.4 that w is a bounded, weak supersolution of the equation

$$\begin{cases} -\Delta\widetilde{u} = \widetilde{g}(\widetilde{u}) & \text{in} \quad \Omega, \\ \widetilde{u} = 0 & \text{in} \quad \partial\Omega. \end{cases} \tag{2.3}$$

We now set $w_0 = w$ and we define the sequence $(w_n)_{n\geq 0}$ by

$$\begin{cases} -\Delta w_n = \widetilde{g}(w_{n-1}) & \text{in} \quad \Omega, \\ w_n = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

for $n \geq 1$. Since $g(w_0) \in L^\infty(\Omega)$, the sequence is well defined. In addition, since $w_0 \geq 0$, it follows that $w_n \geq 0$. We claim that $w_{n+1} \leq w_n$ for all $n \geq 0$. Indeed, it follows from Remark 2.2 (ii) that $w_0 \geq w_1$. The result for $n \geq 1$ now follows from the maximum principle. Therefore, w_n decreases to a solution $\widetilde{u} \in L^\infty(\Omega)$, $\widetilde{u} \geq 0$ of (2.3).

Finally, we show that $\widetilde{u} > 0$. Since $u \geq 0$, $u \not\equiv 0$, we see that $w \geq 0$, $w \not\equiv 0$. By the strong maximum principle, it follows that there exists $\eta > 0$ such that $w_1 \geq \eta\varphi_1$. Note also that $z = \mu\varphi_1$ is clearly a subsolution of (2.3) for $\mu > 0$ small enough. Fix such a μ with $\mu \le \eta$. Since $w_1 \ge \mu\varphi_1$, we have $-\Delta w_2 = \widetilde{g}(w_1) \ge \widetilde{g}(\mu\varphi_1) \ge -\Delta(\mu\varphi_1)$, so that $w_2 \ge \mu\varphi_1$ by the maximum principle. An obvious iteration argument shows that $w_n \ge \mu\varphi_1$, thus $\widetilde{u} \ge \mu\varphi_1 > 0$. This completes the proof. ∎

Proof of Proposition 2.3. Let $\lambda > \lambda^*$, let u be a weak solution of (1.2) and suppose by contradiction that $u \not\equiv 0$. Given $\widetilde{\lambda} \in (\lambda^*, \lambda)$, it follows from Corollary 2.5 that there exists a corresponding solution $\widetilde{u} \in L^\infty(\Omega)$, $\widetilde{u} > 0$ of (1.2). This contradicts the definition of λ^*. ∎

In the next section, we will also use the following application of Lemma 2.4.

Lemma 2.6. *Let $0 < \lambda \le \lambda^*$, $M > 0$ and set $g(u) = \lambda u^q + u^p$. There exist $\varepsilon_0 > 0$ and continuous functions $\Phi_\varepsilon : [0,\infty) \to [0,\infty)$ for $0 < \varepsilon \le \varepsilon_0$ with the following properties.*

(i) *$\Phi_\varepsilon(x) = x$ for $0 \le x \le M$ and $\Phi_\varepsilon(x) \le x$ for $x \ge M$.*

(ii) *$\sup_{x \ge 0} \Phi_\varepsilon(x) \le C\varepsilon^{-\frac{1}{p-1}}$ for some constant C independent of ε.*

(iii) *If $w \ge 0$ is a weak solution of (1.2), then $w_\varepsilon = \Phi_\varepsilon(w)$ satisfies*

$$\int_\Omega w_\varepsilon(-\Delta\xi) \ge \int_\Omega (g(w_\varepsilon) - \varepsilon)^+ \xi, \tag{2.4}$$

for all $\xi \in C^2(\overline{\Omega})$, $\xi \ge 0$, $\xi_{|\partial\Omega} = 0$.

Proof. Let $a > M$ be large enough so that g is convex on (a,∞). Given $0 < \varepsilon < g(a)$, set now

$$H(x) = \int_0^x \frac{ds}{g(s+a)}, \quad H_\varepsilon(x) = \int_0^x \frac{ds}{g(s+a) - \varepsilon},$$

and let

$$\Phi_\varepsilon(x) = \begin{cases} x & 0 \le x \le a, \\ a + H_\varepsilon^{-1}(H(x-a)) & x \ge a, \end{cases}$$

and

$$\Psi_\varepsilon(x) = \begin{cases} \dfrac{g(a) - \varepsilon}{g(a)} & 0 \le x \le a, \\[2mm] \dfrac{g(\Phi_\varepsilon(x)) - \varepsilon}{g(x)} & x \ge a. \end{cases}$$

It is clear that $\Phi_\varepsilon, \Psi_\varepsilon \in C([0,\infty))$. Since $H_\varepsilon \geq H$, we have $\Phi_\varepsilon(x) \leq x$ so that (i) is satisfied. Next, arguing as in the proof of Corollary 2.5, one shows easily that $\Phi'_\varepsilon \geq \Psi_\varepsilon$ and that Φ_ε is concave, so that we may apply Lemma 2.4 with the functions Φ_ε and Ψ_ε defined above. Let now $w \geq 0$ be a weak solution of (1.2) and, given $0 < \varepsilon < a$, set $w_\varepsilon = \Phi_\varepsilon(w)$. It follows from Lemma 2.4 that

$$\int_\Omega w_\varepsilon(-\Delta\xi) \geq \int_\Omega \Psi_\varepsilon(w)g(w)\xi = \int_\Omega h_\varepsilon(w_\varepsilon)\xi,$$

for every $\xi \in C^2(\overline{\Omega})$, $\xi_{|\partial\Omega} = 0$, $\xi \geq 0$, where

$$h_\varepsilon(x) = \Psi_\varepsilon(\Phi_\varepsilon^{-1}(x))g(\Phi_\varepsilon^{-1}(x)) = \begin{cases} \dfrac{g(a)-\varepsilon}{g(a)}g(x) & 0 \leq x \leq a \\ g(x)-\varepsilon & x \geq a. \end{cases}$$

For $0 \leq x \leq a$,

$$h_\varepsilon(x) = \frac{g(a)-\varepsilon}{g(a)}g(x) = g(x) - \varepsilon\frac{g(x)}{g(a)} \geq g(x) - \varepsilon.$$

It follows that $h_\varepsilon(x) \geq g(x) - \varepsilon$ for all $x \geq 0$; and since $h_\varepsilon(x) \geq 0$, we deduce that $h_\varepsilon(x) \geq (g(x)-\varepsilon)^+$. Thus w_ε satisfies (2.4), which proves (iii).

We finally show (ii). Setting $A_\varepsilon = \|\Phi_\varepsilon\|_{L^\infty} - a$, we have

$$\int_0^{A_\varepsilon} \frac{ds}{g(s+a)-\varepsilon} = \int_0^\infty \frac{ds}{g(s+a)},$$

which we write as

$$\varepsilon\int_0^{A_\varepsilon} \frac{ds}{g(s+a)(g(s+a)-\varepsilon)} = \int_{A_\varepsilon}^\infty \frac{ds}{g(s+a)}.$$

Note that $A_\varepsilon \to \infty$ as $\varepsilon \downarrow 0$; and so

$$\int_0^{A_\varepsilon} \frac{ds}{g(s+a)(g(s+a)-\varepsilon)} \underset{\varepsilon\downarrow 0}{\longrightarrow} \int_0^\infty \frac{ds}{g(s+a)^2}.$$

On the other hand, $g(s+a) \geq (s+a)^p$, so that

$$\int_{A_\varepsilon}^\infty \frac{ds}{g(s+a)} \leq \frac{(A_\varepsilon+a)^{-(p-1)}}{p-1}.$$

(ii) follows from the above estimates. $\square$

Remark 2.7. Arguing as in the proof of Corollary 2.5 above, one can show the existence of a solution $\widetilde{w}_\varepsilon \in L^\infty(\Omega)$ of the equation

$$\begin{cases} -\Delta \widetilde{w}_\varepsilon = h_\varepsilon(\widetilde{w}_\varepsilon) & \text{in} \quad \Omega, \\ \widetilde{w}_\varepsilon = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

such that $\mu \mathrm{d}_\Omega \le w_\varepsilon \le w$ and $\|w_\varepsilon\|_{L^\infty} \le C\varepsilon^{-\frac{1}{p-1}}$ for some constants $C, \mu > 0$ independent of ε.

3 – Proof of Theorem 1.2

For the proof of Property (i), we will use the following lemma.

Lemma 3.1. *Let $\lambda > 0$ and let u be the positive solution of* (1.1) *with the initial value $u_0 = 0$. If u is globally defined (i.e., if $T_{\mathrm{m}} = \infty$), then there exists a positive weak solution w of* (1.2) *such that $u(t) \uparrow w$ in $L^1(\Omega)$ as $t \to \infty$. If, in addition, $w \in L^r(\Omega)$ for some $1 < r \le \infty$, then $u(t) \to w$ in $L^r(\Omega)$.*

Proof. Let u be as above, so that by Proposition 6.12 u is increasing in t. Also, u satisfies the equation (1.1) in $L^2(\Omega)$ for almost all $t > 0$ (see Remark 6.1).

We begin by obtaining *a priori* estimates of u. They follow from the mere fact that u is a global solution. On multiplying the equation (1.1) by $\xi \in C^2(\overline{\Omega})$ such that $\xi_{|\partial\Omega} = 0$, we obtain

$$\frac{d}{dt}\int_\Omega u(t)\xi + \int_\Omega u(t)(-\Delta\xi) = \int_\Omega (\lambda u(t)^q + u(t)^p)\xi. \tag{3.1}$$

Letting $\xi = \varphi_1$ in (3.1), we find

$$\frac{d}{dt}\int_\Omega u(t)\varphi_1 + \lambda_1 \int_\Omega u(t)\varphi_1 \ge \int_\Omega u(t)^p \varphi_1 \ge \Big(\int_\Omega u(t)\varphi_1\Big)^p, \tag{3.2}$$

by Jensen's inequality; and so,

$$\frac{d}{dt}\int_\Omega u(t)\varphi_1 \ge \Big(\Big(\int_\Omega u(t)\varphi_1\Big)^{p-1} - \lambda_1\Big)\int_\Omega u(t)\varphi_1.$$

If $\int_\Omega u(t_0)\varphi_1 > \lambda_1^{\frac{1}{p-1}}$ for some $t_0 > 0$, then we deduce from the above differential inequality that $\int_\Omega u(t)\varphi_1$ blows up in finite time, which is absurd. Thus

$$\sup_{t\geq 0}\int_\Omega u(t)\varphi_1 \leq \lambda_1^{\frac{1}{p-1}}. \tag{3.3}$$

Integrating (3.2) on $(t, t+1)$ and applying (3.3), we now obtain

$$\int_t^{t+1}\int_\Omega u^p\varphi_1 \leq (1+\lambda_1)\lambda_1^{\frac{1}{p-1}}.$$

Since u is increasing in t, this implies

$$\sup_{t\geq 0}\int_\Omega u(t)^p\varphi_1 \leq (1+\lambda_1)\lambda_1^{\frac{1}{p-1}}. \tag{3.4}$$

Let now ξ_0 be the solution of

$$\begin{cases} -\Delta\xi_0 = 1 & \text{in} \quad , \\ \xi_0 = 0 & \text{in} \quad \partial\Omega. \end{cases}$$

Letting $\xi = \xi_0$ in (3.1) and integrating on $(t, t+1)$, we obtain

$$\int_\Omega u(t) \leq \int_t^{t+1}\int_\Omega u \leq \int_\Omega u(t)\xi_0 + \int_t^{t+1}\int_\Omega (\lambda u^q + u^p)\xi_0.$$

Since $\xi_0 \leq C\varphi_1$, we now deduce by applying (3.3), (3.4) and the inequality $u^q \leq c + cu^p$ that

$$\sup_{t\geq 0}\int_\Omega u(t) < \infty. \tag{3.5}$$

$u(t)$ being increasing, it follows from (3.4) and (3.5) that there exists a positive function w such that $u(t) \underset{t\to\infty}{\longrightarrow} w$ in $L^1(\Omega)$ and $\varphi_1 u(t)^p \underset{t\to\infty}{\longrightarrow} \varphi_1 w^p$ in $L^1(\Omega)$.

We claim that w is a weak solution of (1.2). Indeed, integrating (1.1) on $(t, t+1)\times\Omega$, we obtain

$$\int_\Omega u(t+1)\xi - \int_\Omega u(t)\xi + \int_t^{t+1}\int_\Omega u(-\Delta\xi) = \lambda\int_t^{t+1}\int_\Omega u^q\xi + \int_t^{t+1}\int_\Omega u^p\xi.$$

Letting $t \to \infty$, we find

$$\int_\Omega w(-\Delta\xi) = \lambda \int_\Omega w^q \xi + \int_\Omega w^p \xi.$$

Finally, it remains to show that $u(t) \to w$ in $L^r(\Omega)$ if $w \in L^r(\Omega)$. This is clear by monotone convergence if $r < \infty$. If $r = \infty$, note that $w \in C(\overline{\Omega})$. Since $u(t) \in C(\overline{\Omega})$ for all $t > 0$, we see that $z(t) = w - u(t) \in C(\overline{\Omega})$ for all $t > 0$ and $z(t,x)$ decreases to 0 as $t \to \infty$ for all $x \in \overline{\Omega}$. We easily deduce that $\|z(t)\|_{L^\infty} \underset{t\to\infty}{\longrightarrow} 0$, which completes the proof. ∎

Proof of Theorem 1.2 (i). Since $u_\lambda > 0$ by the strong maximum principle and since $v(t) \equiv u_\lambda$ is a solution of (1.1), it follows from the maximum principle (see Theorem 6.2 (ii)) that $u(t) \le u_\lambda$ for all $t \in (0, T_{\mathrm{m}})$. Furthermore, u is increasing, so that $0 < u(t) < u_\lambda$ for all $t \in (0, T_{\mathrm{m}})$. Because $u_\lambda \in L^\infty(\Omega)$, we deduce that $T_{\mathrm{m}} = \infty$. We now may apply Lemma 3.1 and it follows that $u(t)$ increases to a weak positive solution w of (1.2) as $t \to \infty$. We have in particular $w \le u_\lambda \in L^\infty(\Omega)$, so that $w = u_\lambda$ (recall that u_λ is the minimal positive solution of (1.2)). It then follows from Lemma 3.1 that $u(t) \underset{t\to\infty}{\longrightarrow} u_\lambda$ in $L^\infty(\Omega)$. ∎

For the proof of Theorem 1.2 (ii), we will use the following lemma.

Lemma 3.2. *Let $\lambda, \varepsilon > 0$ and set $g(u) = \lambda u^q + u^p$ and $g_\varepsilon(u) = (g(u) - \varepsilon)^+$ for all $u \ge 0$. Suppose $w_\varepsilon \in L^\infty(\Omega)$ is a weak supersolution of the equation*

$$\begin{cases} -\Delta w_\varepsilon = g_\varepsilon(w_\varepsilon) & \text{in} \quad \Omega, \\ w_\varepsilon = 0 & \text{in} \quad \partial\Omega. \end{cases} \tag{3.6}$$

Let $v_0 \in L^\infty(\Omega)$, $v_0 \ge 0$ and let v be the solution of

$$\begin{cases} v_t - \Delta v = g_\varepsilon(v) & \text{in} \quad (0,\infty) \times \Omega, \\ v = 0 & \text{in} \quad (0,\infty) \times \partial\Omega, \\ v(0) = v_0 & \text{in} \quad \Omega, \end{cases} \tag{3.7}$$

defined on the maximal interval $[0, S_{\mathrm{m}})$. If $v_0 \le w_\varepsilon$, then $S_{\mathrm{m}} = +\infty$ and $0 \le v(t) \le w_\varepsilon$ for all $t \ge 0$.

Proof. Note that the function $x \mapsto g_\varepsilon(|x|)$ is locally Lipschitz on $\mathbb{R}$ so that v is well defined and nonnegative on a maximal interval $[0, S_{\mathrm{m}})$. In particular $g_\varepsilon(|v|) = g_\varepsilon(v)$. Fix $0 < T < S_{\mathrm{m}}$ and set

$$a = \begin{cases} 0 & \text{if} \quad v \le w_\varepsilon, \\ \dfrac{g_\varepsilon(v) - g_\varepsilon(w_\varepsilon)}{v - w_\varepsilon} & \text{if} \quad v > w_\varepsilon. \end{cases}$$

In particular $a \in L^\infty((0,T) \times \Omega)$ and

$$g_\varepsilon(v) - g_\varepsilon(w_\varepsilon) \le a(v - w_\varepsilon). \tag{3.8}$$

Let now $h \in C_{\mathrm{c}}^\infty((0,T) \times \Omega)$, $h \ge 0$ and let ξ be the solution of

$$\begin{cases} -\xi_t - \Delta\xi - a\xi = h, \\ \xi_{|\partial\Omega} = 0, \\ \xi(T) = 0. \end{cases}$$

Since $a \in L^\infty((0,T) \times \Omega)$, we have in particular $\xi \in C([0,T], C^2(\overline{\Omega}) \cap C_0(\Omega))$ and $\xi \ge 0$. Multiplying (3.7) by ξ and integrating on $(0,T) \times \Omega$, we obtain

$$-\int_\Omega v_0 \xi(0) + \int_0^T \int_\Omega v(h + a\xi) = \int_0^T \int_\Omega g_\varepsilon(v)\xi.$$

w_ε being a weak supersolution of (3.6), we also have

$$-\int_\Omega w_\varepsilon \xi(0) + \int_0^T \int_\Omega w_\varepsilon(h + a\xi) \ge \int_0^T \int_\Omega g_\varepsilon(w_\varepsilon)\xi;$$

and so,

$$\int_0^T \int_\Omega (v - w_\varepsilon)h \le \int_\Omega (v_0 - w_\varepsilon)\xi(0) + \int_0^T \int_\Omega (g_\varepsilon(v) - g_\varepsilon(w_\varepsilon) - a(v - w_\varepsilon))\xi \le 0,$$

by (3.8). h being arbitrary, we deduce that $v \le w_\varepsilon$ on $(0,T) \times \Omega$ and the result follows. □

Proof of Theorem 1.2 (ii). Note that we cannot argue as in the proof of Theorem 1.2 (i), since u_{λ^*} may be unbounded. Instead, we use

Lemma 2.6 to obtain estimates of $u(t)$. We note that $u_{\lambda^*} \geq \mu \mathrm{d}_\Omega$ for some $\mu > 0$. Fix $M \geq \mu \|\mathrm{d}_\Omega\|_{L^\infty}$ and let $\varepsilon_0 > 0$ and $(\Phi_\varepsilon)_{0<\varepsilon\leq\varepsilon_0}$ be given by Lemma 2.6 and set $w_\varepsilon = \Phi_\varepsilon(u_{\lambda^*})$. The choice of M and Property (i) of Lemma 2.6 imply that

$$\mu \mathrm{d}_\Omega \leq w_\varepsilon \leq u_{\lambda^*}. \tag{3.9}$$

Let v be the maximal solution of the problem (3.7) with $v_0 = \mu \mathrm{d}_\Omega$. It follows from Lemma 3.2 that $0 \leq v(t) \leq w_\varepsilon$ for all $t < \infty$.

Fix $T < \infty$ and fix $0 < \varepsilon \leq \varepsilon_0$ small enough so that the solution Z of the linear equation

$$\begin{cases} Z_t - \Delta Z = -\varepsilon & \text{in} \quad (0,T) \times \Omega, \\ Z = 0 & \text{in} \quad (0,T) \times \partial\Omega, \\ Z(0) = \mu \mathrm{d}_\Omega & \text{in} \quad \Omega, \end{cases} \tag{3.10}$$

is nonnegative on $(0,T) \times \Omega$ (see for example Lemma 7 of [5]). It is not difficult to verify that one can choose ε of the form

$$\varepsilon = \alpha e^{-\lambda_1 T}, \tag{3.11}$$

where $\alpha > 0$ is a constant depending only on Ω and μ.

We now set $z(t) = u(t) + Z(t) \geq u(t) \geq 0$ for $0 \leq t < \min\{T, T_{\mathrm{m}}\}$. It follows that

$$\begin{cases} z_t - \Delta z = g(u) - \varepsilon \leq g(z) - \varepsilon \leq (g(z) - \varepsilon)^+ & \text{in} \quad (0, \min\{T, T_{\mathrm{m}}\}) \times \Omega, \\ z = 0 & \text{in} \quad (0, \min\{T, T_{\mathrm{m}}\}) \times \partial\Omega, \\ z(0) = \mu \mathrm{d}_\Omega & \text{in} \quad \Omega. \end{cases}$$

By the maximum principle (recall that $(g(x) - \varepsilon)^+$ is locally Lipschitz), we deduce that $z \leq v$; and so

$$u(t) \leq w_\varepsilon, \tag{3.12}$$

for $0 \leq t < \min\{T, T_{\mathrm{m}}\}$. Applying now (3.11) and Lemma 2.6 (ii), we see that $\|u(t)\|_{L^\infty} \leq C e^{\frac{\lambda_1}{p-1} T}$ for all $t < \min\{T, T_{\mathrm{m}}\}$. Since $T < \infty$ is arbitrary, the blow up alternative implies that $T_{\mathrm{m}} = \infty$; and so $\|u(t)\|_{L^\infty} \leq C e^{\frac{\lambda_1}{p-1} t}$ for all $t > 0$.

We finally show the convergence property. Note that it follows from (3.9) and (3.12) that $u(t) \le u_{\lambda^*}$ for all $t > 0$. On the other hand, we deduce from Lemma 3.1 that there exists a weak solution w of (1.2) such that $u(t) \uparrow w$ as $t \to \infty$. Thus $w \le u_{\lambda^*}$. Next, consider any $\lambda < \lambda^*$ and denote by v the corresponding solution of (1.1) with the initial value $v(0) = 0$. It follows from the maximum principle (see Theorem 6.2 (ii)) that $u(t) \ge v(t)$. Since $v(t) \to u_\lambda$ as $t \to \infty$ by Theorem 1.2 (i), we see that $w \ge u_\lambda$. $\lambda < \lambda^*$ being arbitrary and u_{λ^*} being the increasing limit of u_λ as $\lambda \uparrow \lambda^*$, this implies $w \ge u_{\lambda^*}$, thus $w = u_{\lambda^*}$. Since $u_{\lambda^*} \in L^{p+1}(\Omega)$, the result follows from Lemma 3.1. ∎

Proof of Theorem 1.2 (iii). Suppose by contradiction that $T_{\mathrm{m}} = \infty$. It follows from Lemma 3.1 that there exists a positive weak solution of (1.2), which is ruled out by Proposition 2.3. ∎

4 – Further results

We begin with a proposition which extends Property (ii) of Theorem 1.2.

Proposition 4.1. *Assume* $0 < \lambda \le \lambda^*$ *and let* $w \not\equiv 0$ *be any weak solution of* (1.2)*. Let* $u_0 \in L^\infty(\Omega)$, $0 \le u_0 \le w$ *and let* u *be the corresponding maximal, positive solution of* (1.1)*. It follows that* $T_{\mathrm{m}} = \infty$ *and that there exists a constant* C *such that* $\|u(t)\|_{L^\infty} \le Ce^{\frac{\lambda_1}{p-1}t}$ *for all* $t > 0$*. Moreover,* $u(t) \le w$ *for all* $t > 0$*.*

Proof. Suppose first that $w \in L^\infty(\Omega)$. Then it follows from the maximum principle (see Theorem 6.2 (ii)) that $u(t) \le w$ for all $t < T_{\mathrm{m}}$ and the conclusions follow immediately. Thus we assume that

$$w \notin L^\infty(\Omega). \tag{4.1}$$

Note that by Remark 2.2 (iii),

$$w \ge \delta \mathrm{d}_\Omega, \tag{4.2}$$

for some $\delta > 0$. We now proceed in three steps.

Step 1. If there exist $t_0 \in [0, T_{\rm m})$, $\eta > 0$ such that

$$u(t_0) \le w - \eta \mathrm{d}_\Omega, \tag{4.3}$$

then the conclusions of the theorem hold. Indeed, we may argue as in the proof of Theorem 1.2 (ii). More precisely, changing $u(t)$ to $u(t - t_0)$, we may assume that $u_0 \le w - \eta \mathrm{d}_\Omega$. Set $\mu = \eta$, let $T > 0$ and let Z be the solution of (3.10). Let $\varepsilon > 0$ be given by (3.11), so that $Z \ge 0$ on $(0, T) \times \Omega$. We now apply Lemma 2.6 with $M = \|u_0\|_{L^\infty} + \mu \|\mathrm{d}_\Omega\|_{L^\infty}$. Setting $w_\varepsilon = \Phi_\varepsilon(w)$, we see that $u_0 \le w_\varepsilon - \mu \mathrm{d}_\Omega$. Consider the solution v of (3.7) with the initial value $v(0) = w_\varepsilon$, so that by Lemma 3.2 v is globally defined and $v(t) \le w_\varepsilon$ for all $t \ge 0$. On the other hand, arguing as in the proof of Theorem 1.2 (ii), we see that $z(t) = u(t) + Z(t)$ satisfies $z(t) \le v(t)$ for all $t < \min\{T, T_{\rm m}\}$. In particular, $u(t) \le w_\varepsilon$. Applying now (3.11) and Lemma 2.6 (ii), we see that $\|u(t)\|_{L^\infty} \le C e^{\frac{\lambda_1}{p-1} T}$ for all $t < \min\{T, T_{\rm m}\}$. Since $T < \infty$ is arbitrary, the blow up alternative implies that $T_{\rm m} = \infty$; and so $\|u(t)\|_{L^\infty} \le C e^{\frac{\lambda_1}{p-1} t}$ for all $t > 0$. Since $u(t) \le w_\varepsilon$ and $w_\varepsilon \le w$ by Lemma 2.6 (i), the desired conclusions hold.

Step 2. We have

$$u(t) \le w, \tag{4.4}$$

for all $t < T_{\rm m}$. Indeed, let $0 < \alpha < \delta$ with δ given by (4.2) and set $u_0^\alpha = \min\{w - \alpha \mathrm{d}_\Omega, u_0\}$. Let u^α be the corresponding solution of (1.1). Since u^α satisfies (4.3) with $t_0 = 0$ and $\eta = \alpha$, it follows from Step 1 that u^α is globally defined and that $u^\alpha(t) \le w$ for all $t > 0$. On the other hand $u_0^\alpha \uparrow u_0$ as $\alpha \downarrow 0$, so that it follows from the maximum principle (see Theorem 6.2 (ii)) that $u^\alpha \le u$. We easily deduce that $u^\alpha \uparrow u$ in $(0, T) \times \Omega$ for all $t < T_{\rm m}$ and (4.4) follows.

Step 3. Conclusion. Set $v_0 = \min\{w, 1 + u_0\}$. It is clear that $v_0 \in L^\infty(\Omega)$, $v_0 \ge u_0$, and it follows from (4.1) that $v_0 \not\equiv u_0$. Therefore, there exists a function $\delta : [0, \infty) \to [0, \infty)$ with $\delta(t) > 0$ for $t > 0$ such that

$$T(t)(v_0 - u_0) \ge \delta(t) \mathrm{d}_\Omega, \tag{4.5}$$

for all $t > 0$. Let v be the solution of (1.1) with the initial value $v(0) = v_0$ defined on the maximal interval $[0, T_{\mathrm{m}}{}')$. It follows from Step 2 that

$$v(t) \leq w, \tag{4.6}$$

for $t < T_{\mathrm{m}}{}'$. Set $z(t) = u(t) + T(t)(v_0 - u_0) \geq u(t)$ for $t < \min\{T_{\mathrm{m}}, T_{\mathrm{m}}{}'\}$. Since $z(0) = v_0$ and

$$z_t - \Delta z = g(u) \leq g(z),$$

it follows from the maximum principle (see Theorem 6.2 (ii)) that $z(t) \leq v(t)$. In particular, $u(t) \leq v(t) - T(t)(v_0 - u_0)$. Applying (4.5) and (4.6), we deduce that

$$u(t) \leq w - \delta(t)\mathrm{d}_\Omega,$$

for $t > 0$ sufficiently small. Thus (4.3) is satisfied for some $t_0 > 0$ and the result follows from Step 1. $\square$

Proof of Corollary 1.3. Let u_0 and u be as in the statement of Corollary 1.3. We denote by v the unique, positive solution of (1.1) with the initial value $v(0) = 0$, defined on the maximal interval $(0, S_{\mathrm{m}})$. It follows from the maximum principle (see Theorem 6.2 (ii)) that $T_{\mathrm{m}} \leq S_{\mathrm{m}}$ and that $u(t) \geq v(t)$ for all $0 \leq t < T_{\mathrm{m}}$.

Suppose first that $0 < \lambda \leq \lambda^*$ and that $u_0 \leq u_\lambda$. It follows from Proposition 4.1 that $T_{\mathrm{m}} = +\infty$, $u(t) \leq u_\lambda$ for all $t \geq 0$ and that there exists a constant C such that $\|u(t)\|_{L^\infty} \leq Ce^{\frac{\lambda_1}{p-1}t}$ for all $t > 0$. Properties (i) and (ii) now follow from Properties (i) and (ii) of Theorem 1.2.

Finally, if $\lambda > \lambda^*$ then $S_{\mathrm{m}} < \infty$ by Theorem 1.2 (iii), thus $T_{\mathrm{m}} < \infty$. This completes the proof. $\square$

5 – Comparison with the case $q = 1$

Suppose $p < \dfrac{N+2}{N-2}$ and consider the equation (1.1) with now $q = 1$, i.e.

$$\begin{cases} v_t - \Delta v = \lambda v + v^p & (t,x) \in (0,T) \times \Omega, \\ v = 0 & (t,x) \in (0,T) \times \partial\Omega, \\ v(0,x) = \overline{v}(x) & x \in \Omega, \end{cases} \tag{5.1}$$

and the corresponding elliptic problem

$$
\begin{cases} -\Delta v = \lambda v + v^p & x \in \Omega, \\ v = 0 & x \in \partial\Omega. \end{cases} \tag{5.2}
$$

It is well known that for every $0 \le \lambda < \lambda_1$ the equation (5.2) has a unique positive solution v_λ. For $\lambda \ge \lambda_1$, the equation (5.2) has no positive solution, even in the weak sense of Definition 2.1, as follows immediately by using the test function φ_1 and Jensen's inequality. Furthermore, the solution v_λ for $0 \le \lambda < \lambda_1$ is an unstable solution of (5.1) in the sense that $\lambda_1(-\Delta - \lambda - pv_\lambda^{p-1}) < 0$. In fact, the following holds: if $0 \le \overline{v} \le v_\lambda$ and $\overline{v} \not\equiv v_\lambda$, then the solution v of (5.1) is global and converges to 0 as $t \to \infty$; if $\overline{v} \ge v_\lambda$ and $\overline{v} \not\equiv v_\lambda$, then the solution v of (5.1) blows up in finite time (see [4], exercises).

The above instability phenomenon is surprising in view of the stability properties for $q < 1$, i.e. Corollary 1.3 (i) and the fact that $\lambda_1(-\Delta - \lambda q u_\lambda^{q-1} - pu_\lambda^{p-1}) \ge 0$ (see [1]). What happens here is that v_λ is not the limit of $u_\lambda(q)$ as $q \uparrow 1$. More precisely, we have the following result.

THEOREM 5.1. *Let $p > 1$. Given $0 < q < 1$, let $u_\lambda(q)$ be the minimal positive solution of* (1.2)*, defined for all λ in the maximal interval $(0, \lambda^*(q))$. It follows that*

$$
\lambda^*(q) \underset{q\uparrow 1}{\longrightarrow} \lambda_1, \tag{5.3}
$$

and that for all $0 < \mu < \lambda_1$,

$$
u_\lambda(q) \underset{q\uparrow 1}{\longrightarrow} 0, \tag{5.4}
$$

in $L^\infty(\Omega)$, uniformly for $\lambda \in (0, \mu)$.

PROOF. We first show that

$$
\limsup_{q\uparrow 1} \lambda^*(q) \le \lambda_1. \tag{5.5}
$$

Indeed, given $\lambda < \lambda^*(q)$, it follows from (1.2) that

$$\int_\Omega u_\lambda^p \varphi_1 + \lambda \int_\Omega u_\lambda^q \varphi_1 = \lambda_1 \int_\Omega u_\lambda \varphi_1.$$

On the other hand, it follows easily from Young's inequality that

$$\lambda_1 u \le u^p + \frac{p-1}{p-q}\Big(\frac{1-q}{p-q}\Big)^{\frac{1-q}{p-1}} \lambda_1^{\frac{p-q}{p-1}} u^q;$$

and so,

$$\lambda \le \frac{p-1}{p-q}\Big(\frac{1-q}{p-q}\Big)^{\frac{1-q}{p-1}} \lambda_1^{\frac{p-q}{p-1}}.$$

Since $\lambda < \lambda^*(q)$ is arbitrary and the right-hand side of the above inequality converges to λ_1 as $q \uparrow 1$, we deduce (5.5).

Fix now $0 < \lambda < \lambda_1$ and $\varepsilon > 0$. Let ξ be the solution of

$$\begin{cases} -\Delta\xi = \lambda\xi + 1 & \text{in} \quad \Omega, \\ \xi = 0 & \text{in} \quad \partial\Omega. \end{cases}$$

Let $M = \|\xi\|_{L^\infty}$ and let η be small enough so that $\xi \ge \eta\varphi_1$. Given $\delta > 0$, let $w = \delta\xi$. Since $u^q \le u + (1-q)q^{\frac{q}{1-q}}$, we have

$$\lambda w^q + w^p \le \lambda w + \lambda(1-q)q^{\frac{q}{1-q}} + \delta^p M^p. \tag{5.6}$$

We now fix $\delta > 0$ small enough so that $\delta^p M^p \le \delta/2$ and $\delta M \le \varepsilon$. Next, let $q_0 < 1$ be such that $\lambda(1-q)q^{\frac{q}{1-q}} \le \delta/2$ for $q_0 \le q < 1$. Using (5.6), we see that for all $q_0 \le q < 1$,

$$\lambda w^q + w^p \le \lambda w + \delta = -\Delta w.$$

Therefore, w is a supersolution of (1.2), with $w \ge \delta\eta\varphi_1$. Since $\sigma\varphi_1$ is clearly a subsolution of (1.2) for $\sigma > 0$ small enough, it follows that there exists a positive solution u of (1.2), with $u \le \xi$ (see [1]). Thus, $\lambda(q) > \lambda$ for $q_0 \le q < 1$, which, together with (5.5) implies (5.3). Next, since $\delta M \le \varepsilon$, we have $w \le \varepsilon$, thus $u_\lambda(q) \le \varepsilon$ for $q_0 \le q < 1$. Since $\varepsilon > 0$ is arbitrary, and since u_λ is nondecreasing in λ, (5.4) follows. ∎

6 – Appendix

We present existence and uniqueness results for equations of the type (1.1). The proofs are inspired by the papers of Fujita and Watanabe [9] and Escobedo and Herrero [8].

Let $g : [0, \infty) \to [0, \infty)$ be continuous and consider the equation

$$\begin{cases} u_t - \Delta u = g(u) & (t, x) \in (0, \infty) \times \Omega, \\ u = 0 & (t, x) \in (0, \infty) \times \partial\Omega, \\ u(0, x) = u_0(x) & x \in \Omega. \end{cases} \tag{6.1}$$

We assume that the initial value u_0 is nonnegative, and we consider nonnegative solutions u.

Given $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, we call a solution of (6.1) a function $u \in L^\infty((0, T) \times \Omega)$ for some $T > 0$, $u \geq 0$ which satisfies

$$u(t) = T(t)u_0 + \int_0^t T(t - s)g(u(s))\, ds, \tag{6.2}$$

for all $t \in [0, T]$. Note that the above definition makes sense. Indeed, if $u \in L^\infty((0, T) \times \Omega)$, then $g(u) \in L^\infty((0, T) \times \Omega)$, so that the right-hand side of (6.2) is well-defined.

Remark 6.1. We collect below some immediate properties of the solutions of (6.1).

(i) Since $g(u) \in L^\infty((0, T) \times \Omega)$, standard regularity results imply that $u \in C([0, T], L^r(\Omega))$, that $u - T(t)u_0 \in L^r((0, T), W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega)) \cap W^{1,r}((0, T), L^r(\Omega))$ for every $r < \infty$ and that u satisfies the equation (6.1) for a.a. $t \in (0, T)$.

(ii) The property $g(u) \in L^\infty((0, T) \times \Omega)$ also implies that $u - T(t)u_0 \in C([0, T], C_0(\Omega))$. In particular if $u_0 = 0$ (or more generally if $u_0 \in C_0(\Omega)$) then $u \in C([0, T], C_0(\Omega))$.

(iii) If g is locally Lipschitz $(0, \infty) \to [0, \infty)$ and if $u > 0$ in $(0, T) \times \Omega$, then it also follows from parabolic regularity that u is smooth in $(0, T) \times \Omega$, i.e. u is C^1 in $t \in (0, T)$ and C^2 in $x \in \Omega$.

(iv) It follows from (6.2) that

$$u(t) = T(t-s)u(s) + \int_0^{t-s} T(t-s-\sigma)g(u(s+\sigma))\,d\sigma, \tag{6.3}$$

for all $0 \le s \le t < T$.

(v) It follows in particular from (6.3) that

$$u(t) \ge T(t-s)u(s), \tag{6.4}$$

for all $0 \le s \le t < T$. Therefore, if $u(s) \ne 0$ for some $s \in [0,T)$, then by the strong maximum principle $u(t+s) \ge \delta(t)\mathrm{d}_\Omega$ for $0 < t < T-s$, with $\delta \in C([0,\infty))$ and $\delta(t) > 0$ for $t > 0$. In particular, there exists $t_0 \in [0,T]$ such that $u(t) = 0$ for all $t \in (0,t_0)$ and $u(t) \ge \delta(t-t_0)\mathrm{d}_\Omega$ for all $t \in (t_0,T)$ with $\delta \in C([0,\infty))$ and $\delta(t) > 0$ for $t > 0$.

(vi) Consider any solution u of (6.1) on $(0,T)$. If $u_0 \ge \delta\mathrm{d}_\Omega$ for some $\delta > 0$, then it follows from (6.4) above (applied with $s = 0$) that $u(t) \ge \delta(t)\mathrm{d}_\Omega$ for all $t \in [0,T]$ with $\delta \in C([0,\infty))$ and $\delta(t) > 0$ for all $t \ge 0$. If $u_0 \not\equiv 0$, then $u(t) \ge \delta(t)\mathrm{d}_\Omega$ for all $t \in [0,T]$ with $\delta \in C([0,\infty))$ and $\delta(t) > 0$ for all $t > 0$. If $u_0 = 0$ and $g(0) > 0$, then it follows from (6.3) that $u(t) \ne 0$ for all $t \in (0,T)$. Therefore, by (v) above, we also obtain that $u(t) \ge \delta(t)\mathrm{d}_\Omega$ for all $t \in [0,T]$ with $\delta \in C([0,\infty))$ and $\delta(t) > 0$ for all $t > 0$.

We are interested in the case where g is possibly not locally Lipschitz at 0, so that we may not apply the standard theory. We assume that

$$g : [0,\infty) \to [0,\infty) \quad \text{is continuous.} \tag{6.5}$$

In addition, depending on the results, we will assume that g satisfies some of the following hypotheses.

$$\begin{gathered} \forall\, M > 0, \quad \exists\, L_M < \infty \quad \text{such that} \\ g(u) - g(v) \le \frac{L_M}{v}(u-v) \quad \forall\, 0 < v \le u \le M, \end{gathered} \tag{6.6}$$

$$\exists\, a > 0 \text{ such that} \quad \int_0^a \frac{ds}{g(s)} < \infty. \tag{6.7}$$

(6.8) $\exists\, a > 0$ such that g is concave and nondecreasing on $(0, a)$.

Note that (6.6) is a one-sided condition, which means when g is C^1 that $g'(u) \le L_M/u$ for all $u \in (0, M)$. (6.7) and (6.6) imply that $g(u) > 0$ for $0 < u < a$. Our main result of this section is the following.

THEOREM 6.2. *Suppose g satisfies* (6.5), (6.6), (6.7) *and* (6.8). *For all $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$, there exists a unique, positive solution u of* (6.1) *defined on a maximal time interval* $[0, T_\mathrm{m})$, *$u \in L^\infty((0,T) \times \Omega)$ for all $T < T_\mathrm{m}$. Moreover, the following properties hold.*

(i) *There is the blow up alternative: either $T_\mathrm{m} = +\infty$ or else $T_\mathrm{m} < \infty$ and $\|u(t)\|_{L^\infty} \underset{t \uparrow T_\mathrm{m}}{\longrightarrow} \infty$.*

(ii) *There is a maximum principle: suppose u is a subsolution of (6.1) and v is a positive supersolution of (6.1) on some interval $[0,T]$. If u and v are smooth enough (i.e. $u, v \in L^\infty((0,T) \times \Omega) \cap C([0,T], L^2(\Omega))$ and $u, v \in L^2_{\mathrm{loc}}((0,T), H^1(\Omega)) \cap W^{1,2}_{\mathrm{loc}}((0,T), H^{-1}(\Omega))$), then $u(t) \le v(t)$ for all $0 \le t \le T$.*

REMARK 6.3. Here are some comments on the assumptions of Theorem 6.2. The continuity assumption (6.5) is natural. The assumption (6.7) is essential for the existence of a positive solution when $u_0 = 0$, see Remark 6.8. (6.8) is essential for our proof of uniqueness, but we do not know if it is necessary. On the other hand, the conclusions of Theorem 6.2 hold in the case where $g(u) + Cu$ is nondecreasing for some C, if one replaces the assumption (6.6) by the weaker one

$$\begin{aligned}&\text{for every } 0 < \varepsilon < M < \infty,\ \exists\, L \text{ such that } g(u) - g(v) \le M(v - u)\\ &\text{for all } \varepsilon \le v \le u \le M.\end{aligned}$$

See [7].

REMARK 6.4. Theorem 6.2 states in particular the uniqueness of positive solutions of (6.1). Note that if $u_0 \not\equiv 0$ or if $u_0 \equiv 0$ and $g(0) > 0$, then any solution of (6.1) is positive by Remark 6.1 (vi). Thus positivity is not an actual limitation to the uniqueness property in that case. On the other hand, if $u_0 \equiv 0$ and $g(0) = 0$, then there is the solution $u \equiv 0$. Applying Theorem 6.2 and Remark 6.1 (v), we can describe all the solutions of (6.1) with the initial value $u_0 = 0$: denoting by u the (unique) positive solution, any solution v has the form

$$v(t) = \begin{cases} 0 & 0 \le t \le t_0, \\ u(t - t_0) & t_0 \le t < t_0 + T_{\mathrm{m}}, \end{cases}$$

for some $t_0 \ge 0$.

Before proceeding to the proof of Theorem 6.2, we establish some preliminary results. We first show a general existence result of a (unique) larger solution under the assumption (6.5).

Proposition 6.5. *Assume* (6.5). *Suppose further that*

(6.9) *there exists $M > 0$ such that $g(s) = g(M)$ for all $s \ge M$.*

Given any $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$, there exists a unique larger solution $u \ge 0$ of (6.1) *defined for all $t \ge 0$, $u \in L^\infty((0, \infty) \times \Omega)$. u is the larger solution in the sense that if $v \ge 0$ is any subsolution of* (6.1) *on some interval* $[0, T]$ *and if v is smooth enough (i.e. $v \in L^\infty((0, \infty) \times \Omega) \cap C([0, T], L^2(\Omega))$ and $v \in L^2_{\mathrm{loc}}((0, T), H^1(\Omega)) \cap W^{1,2}_{\mathrm{loc}}((0, T), H^{-1}(\Omega))$), then $v(t) \le u(t)$ for all $t \le T$.*

Proof. We proceed in four steps.

Step 1. Approximation of g by smooth functions. Consider a regularizing sequence $(\rho_\varepsilon)_{\varepsilon > 0}$ and extend g to $\mathbb{R}$ by setting $g(s) = g(0)$ for $s \le 0$. In particular, $g \in C(\mathbb{R})$ and $g(s)$ is constant for both $s \le 0$ and $s \ge M$. It follows that $\rho_\varepsilon \star g \to g$ in $L^\infty(\mathbb{R})$. For convenience, we define

$$\alpha(\varepsilon) = \|g - \rho_\varepsilon \star g\|_{L^\infty} \underset{\varepsilon \downarrow 0}{\longrightarrow} 0.$$

We define the sequence $(g_n)_{n \ge 1}$ as follows. Given $n \ge 1$, let ε_n be small enough so that $\alpha(\varepsilon_n) \le 2^{-n}/3$, and set $g_n = 2^{-n} + \rho_{\varepsilon_n} \star g$. It follows that

$$g + \frac{2}{3} 2^{-n} \le g_n \le g + \frac{4}{3} 2^{-n}.$$

In particular, we see that $g_{n+1} \le g_n$, so that $(g_n)_{n \ge 1}$ is nonincreasing in n, and that g_n converges to g uniformly as $n \to \infty$.

Step 2. Construction of approximate solutions. Given $n \geq 1$, we consider the solution u_n of the problem

$$
\begin{cases} \frac{\partial u_n}{\partial t} - \Delta u_n = g_n(u_n) & t \geq 0, x \in \Omega, \\ u_n = 0 & x \in \partial\Omega, \\ u_n(0,x) = u_0(x) & x \in \Omega. \end{cases} \tag{6.10}
$$

Since g_n is smooth and $g_n > 0$, u_n is well-defined and positive. In addition, g_n is bounded, so that u_n exists globally in time and $\sup\limits_{t\geq 0} \|u_n(t)\|_{L^\infty} < \infty$. Finally, g_n being nonincreasing in n, it follows from the maximum principle that u_n is also nonincreasing in n.

Step 3. Passage to the limit. By Step 2, u_n is nonnegative, bounded in t and nonincreasing in n. Thus u_n has a limit u as $n \to \infty$, which clearly satisfies (6.1). We deduce easily from (6.1) that

$$
\|u(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + t\|g\|_{L^\infty}, \tag{6.11}
$$

for all $t \geq 0$.

Step 4. Conclusion. Consider the solution u constructed in Step 3. Next, consider a subsolution v as in the statement. Since $g_n \geq g$, it follows that v is a subsolution of (6.10); and so, $v(t) \leq u_n(t)$ for all $t \leq T$ and all $n \geq 1$. By letting $n \to \infty$ we deduce that $v(t) \leq u(t)$. ∎

Remark 6.6. In the proof of Proposition 6.5, we approximate g by a nonincreasing sequence of smooth, positive nonlinearities $(g_n)_{n\geq 1}$. We may as well approximate g by a nondecreasing sequence of smooth, nonnegative nonlinearities $(\widetilde{g}_n)_{n\geq 1}$. One can take for example $\widetilde{g}_n$ of the form $\widetilde{g}_n = (-2^{-n} + \rho_{\varepsilon_n} \star g)^+$. Arguing as in the proof of Proposition 6.5, we obtain the existence of a smaller solution $\underline{u}$ of (6.1). $\underline{u}$ is the smaller solution in the sense that if $v \geq 0$ is any supersolution of (6.1) on some interval $[0, T]$, then $v(t) \geq \underline{u}(t)$ for all $t \leq T$.

If $u_0 \not\equiv 0$, then it follows from Remark 6.1 (vi) that the corresponding larger solution u is positive. Under the additional assumptions (6.7) and (6.8), we show that the same property holds for $u_0 = 0$.

Proposition 6.7. *Assume* (6.5), (6.9), (6.7) *and* (6.8). *Set* $u_0 \equiv 0$ *and let* u *be the corresponding larger solution of* (6.1). *It follows that* $u(t) > 0$ *for all* $t > 0$.

Proof. We show that u is positive by constructing appropriate subsolutions. Note that we need only consider small times by Remark 6.1 (v). We claim that there exists $c > 0$ such that

$$u(t) \geq c\alpha(t)\mathrm{d}_\Omega. \tag{6.12}$$

for t sufficiently small, where α is defined below by (6.13). Set

$$h(s) = \int_0^s \frac{d\sigma}{g(\sigma)}.$$

It follows from (6.7) that $h \in C([0, a])$, $h(0) = 0$ and h is increasing on $[0, a]$. Set now

$$\alpha(t) = h^{-1}(t), \tag{6.13}$$

and let $w(t, x) = \alpha(t)T(t)1_\Omega$ for $0 \leq t \leq h(a)$. It follows that

$$w_t - \Delta w = \alpha'(t)T(t)1_\Omega = g(\alpha(t))T(t)1_\Omega.$$

Since $T(t)1_\Omega \leq 1$ and $\alpha(t) \leq a$, it follows from (6.8) that $g(\alpha(t))T(t)1_\Omega \leq g(\alpha(t)T(t)1_\Omega) = g(w)$; and so $w_t - \Delta w \leq g(w)$. We deduce that w is a subsolution of (6.1) on $[0, h(a)]$, so that $u \geq \alpha(t)T(t)1_\Omega$ for all $0 \leq t \leq h(a)$ by Proposition 6.5. Finally, since $1_\Omega \geq \gamma\varphi_1$ for some $\gamma > 0$, it follows that $T(t)1_\Omega \geq \gamma e^{-\lambda_1 t}\varphi_1$, and we obtain (6.12). ∎

Remark 6.8. If $g(0) = 0$, then the assumption (6.7) is necessary for the existence of a positive solution with the initial value $u_0 = 0$. Indeed, suppose that (6.7) fails, i.e.

$$\int_0^a \frac{ds}{g(s)} = +\infty,$$

for all $a > 0$ and let u be a solution of (6.1) with $u_0 = 0$. We claim that $u \equiv 0$. Given $0 < \varepsilon \leq 1$, consider the solution z_ε of the equation $z_\varepsilon' = \varepsilon + g(z_\varepsilon)$ with the initial condition $z_\varepsilon(0) = \varepsilon$. It is clear that z_ε is an increasing function of ε. Since z_ε is a supersolution of (6.1), we have $u \leq z_\varepsilon$ by Lemma 6.9. On the other hand,

$$\int_\varepsilon^{z_\varepsilon(t)} \frac{ds}{g(s) + \varepsilon} = t,$$

so that $z_\varepsilon(t) \downarrow 0$ as $\varepsilon \downarrow 0$. Thus $u \equiv 0$ on some time interval. The conclusion follows by iteration.

For the uniqueness property, we will make use of the following comparison principle.

LEMMA 6.9. *Assume* (6.5) *and* (6.6) *and let* $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$. *Suppose* u *is a supersolution of* (6.1) *and* v *is a subsolution of* (6.1) *on some interval* $[0,T]$. *If* u *and* v *are sufficiently smooth, i.e.* $u, v \in L^\infty((0,T) \times \Omega) \cap C([0,T], L^2(\Omega))$ *and* $u, v \in L^2_{\mathrm{loc}}((0,T), H^1(\Omega)) \cap W^{1,2}_{\mathrm{loc}}((0,T), H^{-1}(\Omega))$ *and if* $u(0) \geq \delta \mathrm{d}_\Omega$ *for some* $\delta > 0$, *then* $u(t) \geq v(t)$ *for all* $t \in [0,T]$.

PROOF. The proof is based on Hardy's inequality

$$\int_\Omega \frac{\varphi^2}{\mathrm{d}_\Omega^2} \leq C \int_\Omega |\nabla \varphi|^2, \tag{6.14}$$

for all $\varphi \in H^1_0(\Omega)$. Note that $u_t - \Delta u \geq 0$, so that $u(t) \geq T(t)u(0)$. In particular, there exists $\gamma > 0$ such that $u(t) \geq \gamma \mathrm{d}_\Omega$ for all $t \in [0,T]$. Multiplying by $(v-u)^+$ the difference of the inequalities satisfied by v and u, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega (v-u)^{+2} + \int_\Omega |\nabla (v-u)^+|^2 &\leq \int_{\{v>u\}} (g(v)-g(u))(v-u)^+ \leq \\ &\leq L_M \int_{\{v>u\}} \frac{(v-u)^{+2}}{u},\end{aligned}$$

by (6.6), with $M = \max\{\|u\|_{L^\infty((0,T)\times\Omega)}, \|v\|_{L^\infty((0,T)\times\Omega)}\}$. Since $u^{-1} \leq (\gamma \mathrm{d}_\Omega)^{-1} \leq \varepsilon \mathrm{d}_\Omega^{-2} + C(\varepsilon)$ for all $\varepsilon > 0$, we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (v-u)^{+2} + \int_\Omega |\nabla (v-u)^+|^2 \leq \varepsilon \int_\Omega \frac{(v-u)^{+2}}{\mathrm{d}_\Omega^2} + C(\varepsilon)\int_\Omega (v-u)^{+2}.$$

Applying (6.14) and choosing $\varepsilon > 0$ sufficiently small, we then obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (v-u)^{+2} \leq C \int_\Omega (v-u)^{+2},$$

from which the result follows. ∎

Corollary 6.10. *Assume* (6.5) *and* (6.6). *Let* $u_0 \equiv 0$ *and* $T > 0$. *Let* $u, v \in L^\infty((0,T)\times\Omega) \cap C([0,T], L^2(\Omega))$ *with* $u, v \in L^2_{\mathrm{loc}}((0,T), H^1(\Omega)) \cap W^{1,2}_{\mathrm{loc}}((0,T), H^{-1}(\Omega))$. *If* u *is a positive supersolution of* (6.1) *and if* v *is a subsolution, then* $u(t) \geq v(t)$ *for all* $0 \leq t \leq T$.

Proof. Since $u_t - \Delta u \geq 0$, we see that $u(t) \geq T(t-s)u(s)$ for all $0 \leq s \leq t \leq T$. Since u is positive, it follows from the strong maximum principle that $u(t) \geq \delta(t)\mathrm{d}_\Omega$ for all $t \in (0,T)$ with $\delta(t) > 0$. Applying Lemma 6.9, we conclude that $u(t+s) \geq v(t)$ for all $0 \leq t < t+s \leq T$. The result follows by fixing $0 \leq t \leq T$ and letting $s \downarrow 0$. □

The last ingredient we need for proving Theorem 6.2 is the following elementary lemma.

Lemma 6.11. *Let* $g \in C([0,\infty), \mathbb{R})$ *with* $g(0) \geq 0$. *If* g *is concave on* $(0,a)$ *for some* $a > 0$, *then the following properties hold.*

(i) *If* $g(u) - g(v) = g(u-v)$ *for some* $0 < v < u \leq a$, *then* $g(0) = 0$ *and* g *is linear on* $[0,u]$.

(ii) *If in addition* $\dfrac{d^+g}{dx}(0) = +\infty$, *then for every* $M > 0$, *there exists* $0 < \varepsilon \leq a$ *such that*

$$g(u) - g(v) \leq g(u-v), \tag{6.15}$$

for all $0 \leq v \leq u \leq M$ *with* $u - v \leq \varepsilon$, *and*

$$g(w) - g(u) + g(v) \geq g(w-u+v) \tag{6.16}$$

for all $0 \leq v \leq u \leq M$, $0 \leq w \leq u$ *with* $u - v \leq w \leq \varepsilon$.

Proof. We first prove (i). By concavity, $g(u)-g(v) \leq g(u-v)-g(0)$. Thus, $g(0) = 0$. Next, if g is not linear on $[0,u]$, then there exists $z \in (0,u)$ such that

$$g(z) > \frac{z}{u} g(u). \tag{6.17}$$

By concavity, we deduce that (6.17) holds for all $z \in (0,u)$. Applying (6.17) with successively $z = v$ and $z = u - v$, we obtain $g(u-v) + g(v) > g(u)$, which is absurd.

We now show (ii). Since $\frac{d^+g}{dx}(0) > 0$, g is nondecreasing near 0. Thus (6.8) holds and

$$g(x+\alpha) - g(y+\alpha) \le g(x) - g(y),$$

for all $0 \le y \le x \le x+\alpha \le a$. Letting $x = v$, $\alpha = u-v$ and $y = w-u+v$, we deduce that (6.16) holds for all $0 \le v \le u \le a$, $0 \le w \le a$ such that $u-v \le w \le u$. Fix now $0 < \varepsilon \le a$ and let $0 \le w \le \varepsilon$ and $0 \le v \le u \le M$ such that $u-v \le w \le u$. We then know that (6.16) holds when in addition $u \le a$. Since the case $u = v$ is trivial, we now suppose $a \le u \le M$ and $u > v$. Note that by concavity,

$$\begin{aligned}\frac{g(w) - g(w-u+v)}{u-v} &\ge \frac{g(\varepsilon+u-v) - g(\varepsilon)}{u-v} \ge \frac{g(2\varepsilon) - g(\varepsilon)}{\varepsilon} \ge \\ &\ge \frac{d^+g}{dx}(\varepsilon).\end{aligned} \tag{6.18}$$

Set

$$K(a,M) = \sup_{\substack{a \le x \le M \\ 0 \le y \le x}} \frac{g(x)-g(y)}{x-y} \le \max\left\{\frac{ML_M}{a/2}, \frac{\|g\|_{L^\infty(0,M)}}{a/2}\right\} < \infty, \tag{6.19}$$

and choose $0 < \varepsilon \le a$ small enough so that $\frac{d^+g}{dx}(\varepsilon) \ge K(a,M)$. We deduce from (6.18) and (6.19) that

$$\frac{g(w) - g(w-u+v)}{u-v} \ge K(a,M) \ge \frac{g(u)-g(v)}{u-v}.$$

Hence (6.16). (6.15) follows from (6.16) by letting $w = u - v$. □

Proof of Theorem 6.2. We proceed in two steps.

Step 1. The case where g satisfies (6.9). The existence of a (global in time) larger solution follows from Proposition 6.5. The positivity of the solution follows from Proposition 6.7 in the case $u_0 \equiv 0$ and from Remark 6.1 (vi) in the case $u_0 \not\equiv 0$.

We now show uniqueness in the case $u_0 \equiv 0$. Let u be the larger solution and let v be another positive solution on some time interval

$[0, T]$. We have in particular $v(t) \leq u(t)$ for all $0 \leq t \leq T$. Since v is positive, we may apply Corollary 6.10 and we deduce that $v(t) \geq u(t)$ for $0 \leq t \leq T$. Thus $u = v$.

We next show uniqueness in the general case. Since this is a local property, we need only show uniqueness for a small time interval. Note that if $g'_+(0) < \infty$, then we deduce from (6.6) that for every $M > 0$ there exists K_M such that $g(u) - g(v) \leq K_M(u - v)$ for all $0 < v \leq u \leq M$, so that uniqueness is immediate by the standard technique. (Multiply the difference of the equations satisfied by u and v by $(u - v)^+$.) We thus assume $g'_+(0) = \infty$, and we may apply Lemma 6.11 (ii). Let u be the larger solution and let v be another solution of (6.1) on some interval $[0, T]$. It follows that

$$u(t) \geq v(t), \tag{6.20}$$

for all $0 \leq t \leq T$. Set $M = \max\{\|u\|_{L^\infty((0,T)\times\Omega)}, \|v\|_{L^\infty((0,T)\times\Omega)}\}$ and let $\varepsilon > 0$ be given by Lemma 6.11 (ii). We have

$$(u - v)_t - \Delta(u - v) = g(u) - g(v), \tag{6.21}$$

with $(u - v)(0) = 0$. By choosing T possibly smaller, we may assume that $u(t) - v(t) \leq \varepsilon$ (see Remark 6.1 (ii)), so that $g(u) - g(v) \leq g(u - v)$ by (6.15). We now may apply Corollary 6.10 and we see that

$$u(t) - v(t) \leq w(t), \tag{6.22}$$

for all $0 \leq t \leq T$, where w denotes the positive solution of (6.1) with the initial value $w(0) = 0$. Note that by Remark 6.1 (ii) we have, by possibly choosing T smaller,

$$w(t) \leq \varepsilon, \tag{6.23}$$

for all $0 \leq t \leq T$; and by Corollary 6.10,

$$w(t) \leq u(t), \tag{6.24}$$

for all $0 \leq t \leq T$. Finally, set $z(t) = w(t) - u(t) + v(t)$. It follows from (6.16) that

$$z_t - \Delta z = g(w) - g(u) + g(v) \geq g(z) \geq 0. \tag{6.25}$$

(Note that the assumptions of (6.16) are satisfied by (6.20), (6.22), (6.23) and (6.24).) We deduce in particular from (6.25) that $z(t) \geq T(t-s)z(s)$ for all $0 \leq s \leq t \leq T$. By the strong maximum principle, it follows that either $z(t) > 0$ for all $0 < t < T$, or else there exists $t_0 > 0$ such that $z(t) = 0$ for $0 \leq t \leq t_0$. In the first case, it follows from (6.25) and Corollary 6.10 that $z(t) \geq w(t)$ for all $0 < t < t+s \leq T$. Therefore, $u(t) \leq v(t)$, thus $u \equiv v$ by (6.20). We finally show that the second case is impossible. Indeed, we would have $w(t) \equiv u(t) - v(t)$ on $(0, t_0)$. In particular,

$$(u-v)_t - \Delta(u-v) = g(u-v), \tag{6.26}$$

by definition of w. Note that $g \geq 0$ and that $g(s) > 0$ for $s > 0$ and small. Thus if $u \not\equiv v$, then $g(u-v) \not\equiv 0$. It follows from the maximum principle that $u(t_0) > v(t_0)$ in Ω. Comparing (6.26) with (6.21), we obtain that $g(u) - g(v) \equiv g(u-v)$ on $(0, t_0) \times \Omega$. Note that $v > 0$ on $(0, t_0) \times \Omega$ by Remark 6.1 (vi). Therefore, we may apply Lemma 6.11 (i) and we deduce that $u \equiv v$ on the set $\{(t,x) \in (0,t_0) \times \Omega;\, u(t,x) \leq a\}$. This is absurd.

It remains to prove the maximum principle, i.e. statement (ii). Let u and v be as in statement (ii) and denote by $\overline{u}$ the positive solution of (6.1). In the case $u_0 \equiv 0$, the result follows from Corollary 6.10. We now assume $u_0 \not\equiv 0$. Since the smaller solution $\underline{u}$ of (6.1) is positive by Remark 6.1 (vi), we have that $\underline{u} = \overline{u}$. It follows from Proposition 6.5 that $u \leq \overline{u}$ and from Remark 6.6 that $v \geq \overline{u}$. Hence the result.

Step 2. The general case. Given $M > 0$, we set

$$g^M(s) = \begin{cases} g(s) & \text{for} \quad 0 \leq s \leq M, \\ g(M) & \text{for} \quad s \geq M, \end{cases}$$

so that g^M satisfies (6.9). Let $u_0 \in L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$, and consider the equation

$$\begin{cases} u^M_t - \Delta u^M = g^M(u^M) & (t,x) \in (0,\infty) \times \Omega, \\ u^M = 0 & (t,x) \in (0,\infty) \times \partial\Omega, \\ u^M(0,x) = u_0(x) & x \in \Omega. \end{cases} \tag{6.27}$$

We first show uniqueness. Let u, v be two solutions of (6.1) on some interval $[0,T]$ and set $M = \max\{\|u\|_{L^\infty((0,T)\times\Omega)}, \|u\|_{L^\infty((0,T)\times\Omega)}\}$. It follows that both u and v are solutions of (6.27) on $[0,T]$. Thus $u=v$ by Step 1.

We now prove existence. Set $M=\|u_0\|_{L^\infty}+1$ and consider the positive solution u^M of (6.27). It follows from (6.11) that $\|u^M(t)\|_{L^\infty}\le \|u_0\|_{L^\infty}+t\|g\|_{L^\infty(0,M)}$ for all $t\ge 0$. In particular, if we set

$$T=T(\|u_0\|_{L^\infty})=\frac{1}{\|g\|_{L^\infty(0,M)}},$$

then $\|u^M(t)\|_{L^\infty}\le M$ for all $t\le T$. It follows that u^M is indeed a solution of (6.1) on $[0,T]$, which we denote by u. By uniqueness, we may extend the solution u to the maximal time interval $[0,T_{\mathrm{m}})$. Since T only depends on $\|u_0\|_{L^\infty}$, the blow up alternative (i) follows from a standard argument.

Finally, we show the maximum principle (ii). Let u and v be as in statement (ii) and set $M=\max\{\|u\|_{L^\infty((0,T)\times\Omega)},\|u\|_{L^\infty((0,T)\times\Omega)}\}$. It follows that u is a subsolution of (6.27) and that v is a positive supersolution of (6.27) on $[0,T]$. Thus $u(t)\le v(t)$ for all $0\le t\le T$ by Step 1. ∎

Proposition 6.12. *Assume* (6.5)*,* (6.6)*,* (6.7) *and* (6.8)*. If u is the positive solution of* (6.1) *with the initial value $u(0)=0$, then $u_t\ge 0$ a.e. in $(0,T_{\mathrm{m}})\times\Omega$. If g is nondecreasing, then $u_t>0$ in $(0,T_{\mathrm{m}})\times\Omega$.*

Proof. Fix $T<T_{\mathrm{m}}$. Arguing as in Step 2 of the proof of Theorem 6.2, we may assume that g satisfies (6.9).

Note that $u_t\in L^2((0,T)\times\Omega)$ by Remark 6.1 (i). Since $u(s)>\delta(s)\mathrm{d}_\Omega$ with $\delta(s)>0$ for all $s\in(0,T)$, it follows from Lemma 6.9 that $u(t+s)\ge u(t)$ for $0\le t<t+s<T$, which implies that $u_t\ge 0$ a.e. in $(0,T)\times\Omega$. Since $T<T_{\mathrm{m}}$ is arbitrary, this proves the first part of the result.

We now assume that g is nondecreasing. It then follows from (6.6) that $g:(0,\infty)\to[0,\infty)$ is locally Lipschitz. Since $u>0$, we deduce from Remark 6.1 (iii) that $u_t\in C((0,T)\times\Omega)$. Suppose $s\in(0,T)$ is such that $\Delta u(s)+g(u(s))\in L^2(\Omega)$, set $\widetilde{u}_0=u(s)$ and $\widetilde{u}(t)=u(t+s)$ for $0\le t<T-s$ and let $\widetilde{u}_n$ be the corresponding solution of (6.10). We have $\Delta\widetilde{u}_0+g_n(\widetilde{u}_0)\ge\Delta\widetilde{u}_0+g(\widetilde{u}_0)\ge 0$. Since g_n is locally Lipschitz, it follows from the maximum principle that $(\widetilde{u}_n)_t\ge 0$. Setting $\widetilde{v}_n=(\widetilde{u}_n)_t$, we see that $(\widetilde{v}_n)_t-\Delta\widetilde{v}_n=g_n'(\widetilde{u}_n)\widetilde{v}_n\ge 0$. It follows that $\widetilde{v}_n(t)\ge T(t)\widetilde{v}_n(0)$, thus $\widetilde{v}_n(t)\ge T(t)(\Delta u(s)+g(u(s)))$. Letting $n\to\infty$, we deduce easily that

$$u_t(t+s)\ge T(t)(\Delta u(s)+g(u(s))). \tag{6.28}$$

Since $u \in L^2((0,T), H^2(\Omega) \cap H^1_0(\Omega))$, $u(t) > 0$ for $t \in (0,T)$ and $u(0) = 0$, it follows that there exists $s_n \downarrow 0$ such that $\Delta u(s_n) + g(u(s_n)) \in L^2(\Omega)$, $\Delta u(s_n) + g(u(s_n)) \geq 0$ and $\Delta u(s_n) + g(u(s_n)) \neq 0$. By (6.28) and the strong maximum principle, we deduce that $u_t > 0$ for all $t \in (s_n, T)$. Letting $n \to \infty$, we obtain that $u_t > 0$ for all $t \in (0,T)$. Since $T < T_{\mathrm{m}}$ is arbitrary, the result follows. ∎

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