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On torsion free sheaves on singular projective curves

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Riassunto: *Sia X una curva proiettiva ridotta e irriducibile. Qui studiamo i fasci coerenti senza torsione su X: esistenza di fasci stabili o generati dalle sezioni globali e con grado assegnato e con fissata classe di isomorfismo ad ogni punto di* $\mathrm{Sing}(X)$ (*teoremi di tipo Clifford, esistenza di fibrati stabili e classificazione di fasci di grado basso generati da sezioni globali*).

Abstract: *Let X be a reduced and irreducible projective curve. Here we study torsion free sheaves on X: existence of stable or spanned sheaves with given degree, number of sections and fixed formal completion at each point of* $\mathrm{Sing}(X)$ (*Clifford's type theorems, non-emptyness of moduli spaces, and classification of low degree spanned torsion free sheaves*).

1 – Introduction

Let X be a reduced and irreducible projective curve. Here we study torsion free sheaves on X from the point of view of stability and the spanned ones with many sections with respect to their degree (Clifford's type theorems). At the beginning of Section 2 we fix the notation and recall a very powerful tool for this topic: elementary transformations. Then we give a Clifford's type theorem for rank r torsion free sheaves on a singular projective curve (see Theorem 2.2). Its proof follows easily

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from the case $r = 1$ proved in [6] and from the proof of the case of vector bundles on smooth curves given in [3], Th. 2.1. Then we study the cases in which Clifford's inequality is an equality (see 2.4, 2.5 and 2.6). In the third section we study spanned torsion free sheaves with low degree. In particular we classify the rank 1 spanned torsion free sheaves with degree 2 and 3 (see Theorems 3.1 and 3.2). For low degree spanned torsion free sheaves of rank at least two, see Corollary 2.6, Remark 3.12, Theorem 1.3 below and Corollary 5.4.

In the second part of this paper (Sections 4 and 5) we will use the elementary transformations to prove the following existence Theorems 1.1, 1.2 and 1.3 for stable and for spanned torsion free sheaves.

THEOREM 1.1. *Let X be an integral projective curve with $g := p_a(X) \geq 2$. Let F be a rank r torsion free sheaf on X. Then there exists an integer $t(F)$ such that for all integers $t \geq t(F)$ and for all sequences $\{Q_i\}_{1 \leq i \leq t}$ of generic points of X_{reg} the general torsion free sheaf obtained from F making t general positive elementary transformations supported at $Q_1, \dots, Q_t$ is stable.*

THEOREM 1.2. *Let X be an integral projective curve with $g := p_a(X) \geq 2$. Fix an integer $r \geq 1$ and an integer d. For every $P \in \mathrm{Sing}(X)$ fix a formal isomorphism type $F_P^\wedge$ for* rank r *torsion free $O_{X,P}^\wedge$-modules. Then there exists a stable* rank r *torsion free sheaf F on X with $\{F_P^\wedge\}_{P \in \mathrm{Sing}(X)}$ as formal isomorphism type at* $\mathrm{Sing}(X)$ *and* $\deg(F) = d$.

Theorem 1.1 gives the existence, for each formal isomorphism type along $\mathrm{Sing}(X)$, of sheaves with a very high degree of stability. Theorem 1.2 will be an immediate corollary of it. Theorem 1.2 should be well-known and hence we cannot claim any priority, but we stress the proof given here which uses elementary transformations. In some sense, Theorem 1.2 gives the non-emptyness part for the study of the moduli schemes of torsion free sheaves with fixed formal isomorphism type along $\mathrm{Sing}(X)$, i.e. of the set of all stable torsion free sheaves, F, on X such that for every $P \in X$ the formal completion $F_P^\wedge$ of the germ of F at P is a fixed module of the complete local ring $\mathbf{O}_{X,P}^\wedge$. In [2] we gave an upper bound for the number of irreducible components of these moduli schemes. In Section 5 we prove the following result (in which $\mathrm{emb}(F)$

is the minimal integer such that each germ F_P is generated by $\mathrm{emb}(F)$ elements).

THEOREM 1.3. *Let G be a* rank r *torsion free sheaf on X. Fix an integer $\nu \geq \max\{r+1, \mathrm{emb}(G)\}$ and set $d := \nu + r(g-1)$. Then there exists a* rank r *torsion free sheaf A on X with* $\deg(A) = d$, $h^1(X,A) = 0$, *A spanned by its global sections and obtained from G making some positive and some negative elementary transformation supported at smooth points of X. Furthermore, for every integer w with $r+1 \leq w \leq \mathrm{emb}(G)$ there exists a* rank r *torsion free sheaf B on X with* $\deg(B) = w + r(g-1)$, $h^1(X,B) = 0$, *B spanned by its global sections at each point P of X with* $\mathrm{emb}_P(F) \leq w$ *and obtained from G making some positive and some negative elementary transformation supported at smooth points of X.*

We work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$.

2 – Clifford's theorem

We fix some notation which we will use in this paper. Let X be an integral projective curve of genus g and $\pi : Y \to X$ the normalization. Let $g'' := p_a(Y)$ be the geometric genus of X. For any $P \in X$ (or Y) let $\mathbf{K}_P$ be the skyscraper sheaf on X (resp. Y) with $\{P\}$ as support and with $h^0(X, \mathbf{K}_P) = 1$ (resp. $h^0(Y, \mathbf{K}_P) = 1$). As usual in the literature the degree $\deg(M)$ of a coherent sheaf, M, on X whose generic rank is r is defined by the Riemann-Roch type formula $\deg(M) = \chi(M) - r_\chi(\mathbf{O}_X)$. If A is an torsion free sheaf on an integral projective curve, let $\mu(A) := \deg(A)/\mathrm{rank}(A)$ be its slope. For any coherent sheaf F on X call $G(F)$ the subsheaf of F spanned by $H^0(X,F)$. If $\mathrm{rank}(G(F)) = \mathrm{rank}(F)$, we have $\deg(G(F)) \leq \deg(F)$ and $\deg(G(F)) = \deg(F)$ if and only if $F = G(F)$, i.e. F is spanned. For every coherent sheaf G on Y, $\mathrm{Tors}(G)$ will denote its torsion part; we have $G \cong (G/\mathrm{Tors}(G)) \oplus \mathrm{Tors}(G)$ and $G/\mathrm{Tors}(G)$ is locally free (or zero). For every torsion free sheaf F on X set $F' := \pi^*(F)/\mathrm{Tors}(\pi^*(F))$. Assume F spanned. We have $\pi^*(F) \cong F' \oplus \mathrm{Tors}(\pi^*(F))$. The sheaf $\pi^*(F)$ (resp. F') is spanned by the image of $\pi^*(H^0(X,F))$ into $H^0(Y, \pi^*(F))$ (resp. $H^0(Y, F')$). If $P \in X$ and F is a coherent sheaf on X, $\mathbf{m}_P$ will denote the maximal ideal of $\mathbf{O}_{X,P}$ and F_P the $\mathbf{O}_{X,P}$-module induced by F; set $\mathrm{emb}_P(F) = \dim_{\mathbf{K}}(F_P/\mathbf{m}_P F_P)$; hence

$\mathrm{emb}_P(F)$ is the minimal integer t such that F_P is the quotient of a free rank $t\mathbf{O}_{X,P}$-module; set $\mathrm{emb}(F) = \max_{P\in X}\{\mathrm{emb}_P(F)\}$; hence $\mathrm{emb}(F)$ is the minimal integer t such that F is the quotient of a rank t vector bundle on X. For any torsion free sheaf F on X, set $\mathrm{Sing}(F) := \{P \in X\colon F$ is not locally free at $P\}$. Notice that $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(X)$ for every torsion free sheaf F on X. Fix $P \in \mathrm{Sing}(X)$ and let M be a rank r torsion free $\mathbf{O}_{X,P}$-module; as in [5], def. 2.2.3, let $l(M)$ be the minimal integer t such that M contains a free $\mathbf{O}_{X,P}$-module N of rank r with $\dim_{\mathbf{K}}(M/N) = t$. Hence $l(M) = 0$ if and only if M is free. For every rank r torsion free sheaf F on X it is important to consider the integer $l(F) := \sum_{P\in\mathrm{Sing}(X)} l(F_P)$, where F_P is the torsion free $\mathbf{O}_{X,P}$-module of rank r induced by F (see e.g. [5], Ch. III, for its use). Hence $l(F) = 0$ if and only if F is locally free. If $P \in X$ (resp. Y) and F is a torsion free rank r sheaf on X (resp. Y), $F|\{P\}$ will denote the fiber of F at P. If $P \notin \mathrm{Sing}(F)$ we have $F|\{P\} \cong K^{\oplus r}$. Now assume $P \in X_{\mathrm{reg}}$. There is a natural bijection between the surjections $u\colon F \to \mathbf{K}_P$ and the linear surjective maps $F|\{P\} \to \mathbf{K}$. Fix any such surjection u. Then $\ker(u)$ is a torsion free subsheaf of F with $\deg(\ker(u)) = \deg(F) - 1$ and $\mathrm{Sing}(\ker(u)) = \mathrm{Sing}(F)$. As usual in the case of a smooth curve we will say that $\ker(u)$ is obtained from F making a negative elementary transformation supported at P and that F is obtained from $\ker(u)$ making a positive elementary transformation supported at P. The set of all sheaves obtained from a rank r torsion free sheaf G making a positive (or negative) elementary transformation supported at P is parametrized by $\mathbf{P}^{r-1}$. Hence for any pair of non-negative integers (α, β) and any fixed torsion free sheaf G the set of all sheaves obtained from G making α positive elementary transformations and β negative elementary transformations supported at arbitrary points of X_{reg} is parametrized by an irreducible variety; any such sheaf has the formal isomorphism type of G at every point of $\mathrm{Sing}(X)$.

Proposition 2.1. *Let F be a* rank r *semistable torsion free sheaf on X with $0 \le \mu(F) \le 2g-2$. Then $h^0(X,F) \le \deg(F)/2 + r$.*

Proof. If $r = 1$, the result was proved in [6]. Assume $r \ge 2$. Let G be a subsheaf of F with $0 < \mathrm{rank}(G) < r$ and with $\mu(G)$ maximal. We may assume $h^0(X,F) > 0$ and hence $\deg(G) \ge 0$. By the maximality of $\mu(G)$ the sheaf G is saturated in F, i.e. F/G is torsion free. By the maximality of $\mu(G)$ the sheaf F/G is semistable. By Riemann-Roch and Serre duality we may assume $H^0(X, \mathrm{Hom}(F, \omega_Y)) = h^1(X, F) > 0$. Hence there is a rank 1 subsheaf A of ω_X which is a quotient of F. Since $\deg(A) \leq 2g-2$, we have $\mu(F/G) \leq 2g-2$. By the semistability of F we have $\mu(G) \leq 2g-2$ and $\mu(F/G) \geq 0$. Hence we conclude by induction on r. □

Call $T(g)$ one of the integral curves of arithmetic genus g, rational normalization and with a unique singular point with maximal ideal as conductor introduced in [6], Th. A, part (c). Call $L(g,d)$, $0 \leq d \leq g-1$, one of the degree d rank 1 torsion free sheaves on $T(g)$ described in the same statement. For a detailed description of the curves $T(g)$ (see [6], p. 533). We just observe that for $g \geq 2\, T(g)$ is not Gorenstein because the conductor of its only singular point, P, is the maximal ideal of $\mathbf{O}_{T(g),P}$.

The proof of Proposition 2.1, induction on r and [6], Th. A, give immediately the following result.

Theorem 2.2. *Let X be an integral curve of arithmetic genus g and F a* rank *r semistable torsion free sheaf on X with $0 \leq \mu(F) \leq 2g-2$ such that $h^0(X,F) = \deg(F)/2 + r$. Then:*

(1) *F has an increasing filtration $\{F_i\}_{0 \leq i \leq r}$ with $F_0 = \{0\}$, $F_r = F$, F_{i+1}/F_i* rank 1 *torsion free sheaf on X, $0 \leq i < r$, and such that for every integer i with $0 \leq i < r$, F_{i+1}/F_i is one of the following sheaves:*
   - (a) *$F_{i+1}/F_i \cong \mathbf{O}_X$;*
   - (b) *$F_{i+1}/F_i \cong \omega_X$;*
   - (c) *F_{i+1}/F_i is a tensor power $L^{\oplus m}$, $1 \leq m \leq g-2$, of a degree 2 spanned line bundle L inducing a degree 2* map *$X \to \mathbf{P}^1$; hence if this case occurs X is hyperelliptic* (*but perhaps singular*)*;*
   - (d) *$F_{i+1}/F_i \cong L(g,d)$ for some integer d with $0 \leq d \leq g-1$; hence if this case occurs, then $X \cong T(g)$.*

(2) *For every integer i with $0 \leq i \leq r-2$ the* rank *1 sheaf F_{i+1}/F_i is a* rank *1 subsheaf of maximal degree of the* rank *2 torsion free sheaf F_{i+2}/F_i.*

(3) *For all integers i, j with $0 \leq i \leq j-2 \leq r-2$ we have $h^0(X, F_j/F_i) = \sum_{i+1 \leq k \leq j} h^0(X, F_k/F_{k-1})$.*

Now we will show how to use the last two assertions of Theorem 2.2 to obtain more precise results. First, we discuss the use of part (2) and part (3) for $j = i + 2$. Hence we are in the following situation.

(2.3) E is a rank 2 semistable torsion free sheaf on X which fits in an exact sequence

$$0 \to A \to E \to B \to 0 \tag{1}$$

with A and B rank 1 torsion free sheaves on X, $0 \leq \mu(A) \leq 2g - 2$, $0 \leq \mu(E) \leq 2g - 2$, $0 \leq \mu(B) \leq 2g - 2$, $h^0(X, A) = \deg(A)/2 + 1$, $h^0(X, E) = \deg(E)/2 + 2$, $h^0(X, B) = \deg(B)/2 + 1$ and A is a rank 1 subsheaf with maximal slope of E. Furthermore, both A and B are one of the sheaves described in [6], Th. A, i.e. one of the rank 1 torsion free sheaves F_i/F_{i-1} appearing in the statement of 2.2. By assumption $h^0(X, E) = h^0(X, A) + h^0(X, B)$, i.e. the natural map $H^0(X, E) \to H^0(X, B)$ is surjective and the cobundary map $H^0(X, B) \to H^1(X, A)$ is zero. If $B \cong \mathbf{O}_X$ this condition is equivalent to the splitting of the exact sequence (1) because if $B \cong \mathbf{O}_X$ the image of the constant function 1 by the coboundary map gives, up to a sign, the extension class of the extension (1).

(2.3.1) Here we assume $B \cong \mathbf{O}_X$. Hence the exact sequence (1) splits. Since E is semistable we have $E \cong \mathbf{O}_X^{\oplus 2}$. Hence in the original rank r situation there is an integer i with $1 \leq i \leq r - 1$ and such that F_{k+1}/F_k is trivial if and only if $k \leq i$.

(2.3.2) Here we assume $A \cong \mathbf{O}_X$ and $B \cong \omega_X$. Furthermore, we assume $g \geq 2$, i.e. we assume B not trivial. We fix $P \in X_{\text{reg}}$. Since $h^0(X, E) = g + 1 \geq 3$, there is $u \in H^0(X, E)$, $u \neq 0$, with $u(P) = 0$, i.e. $\mathbf{O}_X(P)$ is a subsheaf of E. Hence A is not a maximal degree rank 1 subsheaf of E, contradiction.

(2.3.3) Here we assume $A \cong \omega_X$ and $B \cong \omega_X$. Furthermore, we assume $g \geq 2$, i.e. we assume B not trivial. By Riemann-Roch we have $h^1(X, E) = 2$. Hence by duality on the locally Cohen-Macaulay curve X we have $h^0(X, \mathrm{Hom}(E, \omega_X)) = 2$. We have $h^0(X, \mathrm{Hom}(\omega_X, \omega_X)) = 1$ because a rank 1 torsion free sheaf is stable (see [5], part 1 of Lemma 3.5.1 or statement of Cor. 3.1.9). Hence there is a morphism $E \to \omega_X$ whose composition with the inclusion in the exact sequence (1) induces a splitting of (1). Hence $E \cong \omega_X^{\oplus 2}$.

(2.3.4) Here we assume that X is hyperelliptic and that both A and B are locally free. Since X is Gorenstein (see e.g. [6], part (b) of Th. A) we may apply the proof of [8], Prop. 2. Hence we obtain $E \cong A \oplus B$. Notice that the g^1_2 on X is locally free (see e.g. [6], part (*b*) of Th. A).

(2.3.5) Notice that if $A \cong \mathbf{O}_X$ and $h^0(X, B) \geq 2$ we obtain a contradiction as in subcase 2.3.2.

In summary, from the subcases 2.3.1, 2.3.2, 2.3.3, 2.3.4 and 2.3.5 we obtain the following results 2.4 and 2.5.

THEOREM 2.4. *Assume X hyperelliptic. Let $f\colon X \to \mathbf{P}^1$ be the associated degree 2 morphism. Let F be a* rank r *semistable torsion free sheaf on X with $0 \leq \mu(F) \leq 2g-2$ and $h^0(X, F) = \deg(F)/2+r$. Then F is locally free and there is a* rank r *semistable vector bundle U on $\mathbf{P}^1$ with $F \cong f*(U)$ and $h^0(X, F) = h^0(\mathbf{P}^1, U)$.*

PROOF. Since $T(g)$ is not Gorenstein, $T(g)$ is not hyperelliptic (see e.g. [6], part (*b*) of Th. A). Hence X is not isomorphic to $T(g)$. X is Gorenstein (see e.g. [6], part (*b*) of Th. A). Let $\{F_i\}_{0\leq i\leq r}$ be a filtration of F satisfying the thesis of Theorem 2.2. By assumption and part (2) of Theorem 2.2 all the torsion free sheaves F_{i+1}/F_i are locally free and are pull-backs through f of line bundles on $\mathbf{P}^1$. Hence F is locally free. Furthermore, all the maps $H^0(X, F_{i+1})) \to H^0(X, F_{i+1}/F_i)$ are surjective. Hence F is spanned. We claim that part (3) of Theorem 2.2 implies that the r extensions induced by the filtration $\{F_i\}_{0\leq i\leq r}$ split. A moment's thought shows that the claim implies all the assertions of 2.4. To check the claim we consider the morphism u: $X \to Gr(r, v)$, $v = h^0(X, F)$, induced by the pair $(F, H^0(X, F))$. Since all the spanned line bundles F_{i+1}/F_i induce the hyperelliptic double covering f, we see that u factors through f, proving the claim. ∎

THEOREM 2.5. *Assume X neither hyperelliptic nor isomorphic to the curve $T(g)$. Let F be a* rank r *semistable torsion free sheaf with $0 \leq \mu(F) \leq 2g-2$ and $h^0(X, F) = \deg(F)/2 + r$. Then either $F \cong \mathbf{O}_X^{\oplus r}$ or $F \cong \omega_X^{\oplus r}$.*

PROOF. Just use 2.3.1, 2.3.2 and 2.3.3. ∎

Theorems 2.4 and 2.5 give the following result.

COROLLARY 2.6. *Assume $g \geq 3$. Let X be an integral projective curve with $g := p_a(X) \geq 3$ and not isomorphic to $T(g)$. Let F be a* rank r

spanned torsion free sheaf with $0 < \deg(F) \leq 2r$ *and with no trivial factor. Then* X *is hyperelliptic and* $F \cong L^{\oplus r}$*, where* L *is the degree 2 line bundle on* X *inducing the double covering* $X \to \mathbf{P}^1$*.*

3 – Spanned torsion free rank 1 sheaves of degree 2 and 3

In this section we will study low degree rank one torsion free sheaves, F, on an integral projective curve X with $g := p_a(X) \geq 2$ and such that $h^0(X, F) \geq 2$. We prove the following results; for an explanation of the notations $l(F)$ and $l(F_P)$, see Section 2.

THEOREM 3.1. *Let* X *be an integral projective curve with* $g := p_a(X) \geq 1$ *and* F *a* rank *one torsion free sheaf on* X *with* $\deg(F) = 2$ *and* $h^0(X, F) \geq 2$*. Then one of the following cases occurs:*

(i) $F \in \mathrm{Pic}(X)$*,* $h^0(X, F) = 2$*,* F *is spanned and* $h^0(X, F)$ *induces a degree 2 morphism* $f\colon X \to \mathbf{P}^1$*, i.e.* X *is hyperelliptic; in particular* X *is Gorenstein;*

(ii) $g \geq 2$ *and* X *is one of the genus* g *curves* (*call any of them* $T(g)$) *described in* [6]*, part* (*c*) *of Th. A and Remark at p. 533; hence the normalization of* X *is* $\mathbf{P}^1$*,* X *has a unique singular point with the maximal ideal as conductor and* X *can be embedded into* $\mathbf{P}^{g+1}$ *as an arithmetically Cohen-Macaulay curve of degree* $2g+1$*;* F *is the sheaf described in the statement of* [6]*, part* (*c*) *of Th. A, for the integer* $d = 1$*; in particular we have* $h^0(X, F) = 2$*;*

(iii) $g = 1$*; for a description of all such* F*, see Example* 3.8.

THEOREM 3.2. *Let* X *be an integral projective curve with* $g := p_a(X) \geq 3$ *and* F *a* rank *one torsion free sheaf on* X *with* $\deg(F) = 3$ *and* $h^0(X, F) \geq 2$*. Then one of the following cases occurs:*

(i) $F \in \mathrm{Pic}(X)$*,* $h^0(X, F) = 2$*,* F *is spanned and* $h^0(X, F)$ *induces a degree 3 morphism* $f\colon X \to \mathbf{P}^1$*, i.e.* X *is "trigonal";*

(ii) F *is not spanned; call* $G(F)$ *the subsheaf of* F *spanned by* $H^0(X, F)$*; we have* $\deg(G(F)) = 2$ *and the pair* $(X, G(F))$ *is one of the two pairs listed in parts* (i) *and* (ii) *of the statement of Theorem* 3.1*;*

(iii) F *is not locally free and* $l(F) = 2$*;* $\mathrm{card}(\mathrm{Sing}(F)) = \mathrm{card}(\mathrm{Sing}(X)) = 2$*; the normalization of* X *is isomorphic* $\mathbf{P}^1$*;*

(iv) *F is not locally free and $l(F) = 1$; let $\pi\colon Y \to X$ be the normalization* map*; Y is hyperelliptic or elliptic or rational and there is a two to one* map $Y \to \mathbf{P}^1$ *which factors through π and which is induced by two sections of the degree 2 line bundle $F' := \pi^*(F)/\mathrm{Tors}(\pi^*(F))$.*

For a complete classification of the pairs (X, F) arising in case (iii) of Theorem 3.2, see the last part of its proof. For a partial classification of case (iv) of Theorem 3.2, see Remark 3.10;

Remark 3.3. Let F be a rank r torsion free sheaf on X spanned by its global sections. We have $\deg(F) \geq 0$ and $\deg(F) = 0$ if and only if $F \cong \mathbf{O}_X^{\oplus r}$. If $g > 0$ by Riemann-Roch the sheaf $\mathbf{O}_X$ is the only rank 1 spanned torsion free sheaf with degree at most 1.

Remark 3.4. Let F be a rank r torsion free sheaf on X spanned by its global sections. We have $h^0(X, F) \geq r$. It is easy to check that $h^0(X, F) = r$ if and only if $F \cong \mathbf{O}_X^{\oplus r}$.

Remark 3.5. Let F be a rank 1 torsion free sheaf on X and set $F' := \pi^*(F)/\mathrm{Tors}(\pi^*(F))$. By [6], Lemma 1 at p. 534, we have $\deg(F') = \deg(F) - l(F) \leq \deg(F)$ and $\deg(F') = \deg(F)$ if and only if $F \in \mathrm{Pic}(X)$. The integer d appearing in the classification of the sheaves F on the rational singular curve considered in [6], part (c) of Th. A, is the integer $\deg(F')$ and we have $h^0(X, F') = \deg(F') + 1$. For the corresponding result for a torsion free sheaf with arbitrary rank, see [5], part 2) of Prop. 3.2.4; more precisely we have $\deg(F') = \deg(F) - l(F)$.

From now on in this section we fix a rank 1 torsion free sheaf F on X and set $d := \deg(F)$ and $v := h^0(X, F)$.

Remark 3.6. Here we will check that $v \leq d+1$. Since $d = \deg(F') + l(F)$, the case $d \leq 0$ is obvious. Hence we may assume $v \geq 2$. Fix a general $P \in X_{\mathrm{reg}}$. We have $\deg(F(-P)) = d - 1$. Since F is torsion free and P is general we have $h^0(X, F(-P)) = v - 1$. Hence we conclude by induction on d. Now we assume $g \geq 1$. We want to check that $v \leq d$ if $v > 0$. We may use the same trick and reduce our task to the proof that $v \leq 1$ if $d = 1$. Since C is not $\mathbf{P}^1$, for general P and Q in X_{reg} the line bundles $\mathbf{O}_X(P)$ and $\mathbf{O}_X(Q)$ are not isomorphic. Using again $F(-P)$ with P general we reduce to the obvious assertion that if $d = 0$ and $v > 0$, then $F \cong \mathbf{O}_X$ (Remark 3.3). Now assume $g \geq 1$ and $v = d > 0$. Since *a* proper subsheaf of F has degree $< d$, the result just proven implies that F is spanned. By Riemann-Roch and Clifford's theorem for rank 1 torsion free sheaves we obtain $h^1(X, F) = 0$ and $g = 1$.

REMARK 3.7. Let G be a subsheaf of F with length$(F/G) = 1$. Hence F/G is a skyscraper sheaf supported at one point, P, of X. If $P \in X_{\mathrm{reg}}$, then $F \cong G(P)$. However, if $P \in \mathrm{Sing}(X)$ the situation is much more complicated and depends heavily on the formal isomorphism type of the singularity of X at P. F is just given (and uniquely determined by) a non-trivial extension

$$0 \to G \to F \to \mathbf{K}_P \to 0 \tag{2}$$

where $\mathbf{K}_P$ is the skyscraper sheaf of length 1 with P as support. By the local-to-global spectral sequence of the Ext-functors, $\mathrm{Ext}^1(X; \mathbf{K}_P, G) \cong H^0(X, \mathrm{Ext}^1(\mathbf{K}_P, G))$, where Ext^1 is computed either over the local ring $\mathbf{O}_{X,P}$ or over its completion. For such extension (2) the sheaf F is torsion free if and only if (2) does not split and it is spanned if and only if $h^0(X, F) > h^0(X, G)$, while if $h^0(X, F) = h^0(X, G)G$ is the subsheaf of F spanned by $H^0(X, F)$. In this sense case (ii) of Theorem 3.2 is not a full classification of all possible types. For some kind of singularities we can give some informations using the classification of all torsion free modules on certain complete one-dimensional local rings. For instance if X has an ordinary node or an ordinary cusp at P the sheaf G is locally free at P if and only if F is not locally free at P, i.e. exactly one of the sheaves G and F are locally free at P while the other is formally isomorphic at P to the maximal ideal of $\mathbf{O}_{X,P}$ (see e.g. [5], pp. 24 and 25, and Example 3.8 below).

EXAMPLE 3.8. Here we assume $g = 1$. If X is smooth, everything is known. Hence we assume that X is singular, i.e. that it is rational and with a unique singular point, P, which is either an ordinary node or an ordinary cusp. There is a complete classification of all torsion free modules over the completion $\mathbf{O}^{\wedge}_{X,P}$ of $\mathbf{O}_{X,P}$ (see e.g. [5], pp. 24 and 25): calling $\mathbf{m}^{\wedge}_P$ the maximal ideal of $\mathbf{O}^{\wedge}_{X,P}$ any such module is isomorphic to $\mathbf{m}_P^{\wedge \oplus a} \oplus \mathbf{O}_{X,P}^{\wedge \oplus b}$ for some integers $a \geq 0, b \geq 0$ uniquely determined. All torsion free rank 1 sheaves A of degree ≥ 2 on X are not special and have $h^0(X, A) = \deg(A)$. Fix $M \in \mathrm{Pic}^t(X)$ with $t > 0$. By the local-to-global spectral sequence for the Ext-functors up to a multiplicative constant there is a unique non-split extension

$$0 \to M \to F \to \mathbf{K}_P \to 0 \tag{3}$$

The sheaf F is torsion free of rank 1 and with $\deg(F) = \deg(M)+1$. Since $h^1(X, M) = 0$, we have $h^0(X, F) = \deg(F)$. Vice versa, take any such F. The kernel of any surjection $F \to \mathbf{K}_P$ is locally free by the classification of modules over ordinary nodes or ordinary cusps (see e.g. [5], pp. 24 and 25). Hence any torsion free non locally free F fits in an exact sequence (3) with $\deg(M) = \deg(F) - 1$.

Proof of 3.1. If $g \geq 2$ we have $h^1(X, F) > 0$ (Riemann-Roch). Hence if $g \geq 2$ we conclude by [6], Th. A. Now assume $g = 1$. We have $v = 2$ and F spanned by Remark 3.6. Vice versa, any pair (X, F) with $p_a(X) = 1$ and F torsion free has $v \geq 2$ by Riemann-Roch. Any such sheaf F is described in Example 3.8 taking $\deg(M) = 1$. □

Proof of 3.2. Let Y be the normalization of X. First assume $h^0(X, F) \geq 3$. By [6], Th. A, we have $h^1(X, F) = 0$. Hence by Riemann-Roch we have $g = 1$, contradiction. Hence we may assume $h^0(X, F) = 2$. By case (ii) we may assume F spanned. If $F \in \mathrm{Pic}(X)$, then $h^0(X, F) = 2$ and X is "trigonal". Hence we may assume that F is not locally free. Hence we have $0 \leq \deg(F') = 3 - l(F) \leq 2$. Notice that the pair $(F, H^0(X, F))$ induces *a* non-constant morphism u: $U \to \mathbf{P}^1$, where U is the open subset of X on which F is locally free and that u induces *a* non-constant morphism u': $Y \to \mathbf{P}^1$ with F' as associated line bundle. Hence $\deg(F') > 0$ and if $g'' > 0$, then $\deg(F') = 2$. If $\deg(u) = 2$, the curve Y is hyperelliptic (or elliptic or rational) and we are in case (iv). From now on we assume $\deg(F') = 1$ and hence $Y \cong \mathbf{P}^1$ and $l(F) = 2$. Fix $P \in \mathrm{Sing}(F)$. Since $v = 2$, F is a quotient of $\mathbf{O}_X^{\oplus 2}$. Hence $l(F_P) \leq 1$, i.e. $l(F_P) = 1$. Thus $\mathrm{card}(\mathrm{Sing}(F)) = 2$. Since the pair $(F', H^0(Y, F'))$ induces an isomorphism from Y to $\mathbf{P}^1$, we must have $\mathrm{Sing}(F) = \mathrm{Sing}(X)$. Set $\{P_1, P_2\} := \mathrm{Sing}(X)$ and let f_i: $Y_i \to X$ be the partial normalization of X at P_i. Set $F_i := f_i^*(F)/\mathrm{Tors}(f_i^*(F))$. Since $l(F_{P_i}) = 1$, we have $\deg(F_i) = 2$ for every i. By construction F_i is a degree 2 spanned torsion free sheaf on Y_i. Hence by Theorem 3.1 and the assumption $\deg(F') = 1$ there is an integer $g_i \geq 2$ such that $Y_i \cong T(g_i)$ and F_i is one of the sheaves described in [6], part (c) of Th. A, for the integer $d = 1$. We have $g_1 + g_2 = g$. Vice versa, given any pair of points, Q_1 and Q_2, on $\mathbf{P}^1$ with $Q_1 \neq Q_2$ and any pair of integers g_1, g_2 with $g_1 \geq 2$ and $g_2 \geq 2$, set $g := g_1 + g_2$ and define a singular rational curve $T(g_1, g_2)$ of arithmetic genus g and with 2 unibranch singularities, say P_1 and P_2, with P_i formally isomorphic to the singularity of a curve $T(g_i)$ and with Q_i as counterimage of the singular point. Since the two linearly independent sections of $\mathbf{O}_{\mathbf{P}^1}(1)$ descend to two spanning sections of F_i on $T(g_i)$ by [6], we obtain the existence on $T(g_1, g_2)$ of a unique torsion free degree 3 sheaf, F, with $h^0(T(g_1, g_2), F) = 2$. We claim that F is spanned and hence that it is a solution of our classification Theorem 3.2, solving completely part (iii) of 3.2. Call $G(F)$ the subsheaf of F spanned by $H^0(T(g_1, g_2), F)$. Since $h^0(T(g_1, g_2), F) = 2$, we have $\deg(G(F)) > 0$. Since $p_a(T(g_1, g_2)) \geq 4 > 0$, we have $\deg(G(F)) \geq 2$. Since the spanned line bundle F' obtained on the normalization $\mathbf{P}^1$ has degree 1, and $T(g_1, g_2)$ is not isomorphic to $T(g)$, we obtain $\deg(G(F)) \geq 3$ using the statement of 3.1. Hence $G(F) = F$, i.e. F is spanned. □

Remark 3.9. Motivated by Remark 3.7, we want to discuss in detail one of the pairs (X, F) covered by subcase (ii) of 3.2. The curve X is one of the genus g curves (call any of them $T(g)$) described in [6], part (c) of Th. A and Remark at p. 533; hence the normalization of X is $\mathbf{P}^1$, X has a unique singular point with the maximal ideal as conductor and X can be embedded into $\mathbf{P}^{g+1}$ as an arithmetically Cohen-Macaulay curve of degree $2g + 1$. Call $L(g, d)$, $d \geq 0$, the sheaf described in the statement of [6], part (c) of Th. A, for the integer d. In particular we have $h^0(X, L(g, d)) = d + 1$, $\deg(L(g, d)) = 2d$ and the fiber of $L(g, d)$ at P is a vector space of dimension $d + 1$. There is a unique point $Q \in X_{\mathrm{reg}}$ such that $F \cong L(g, 2)(-Q)$; we have $h^0(X, L(g, 2)(-Q)) = 2$, and hence $L(g, 2)(-Q)$ is not spanned by $H^0(X, L(g, 2)(-Q))$ at P. It is easy to check (and it follows from 2.1) that the subsheaf of $L(g, 2)(-Q)$ spanned by $H^0(X, L(g, 2)(-Q))$ is isomorphic to $L(g, 1)$. $L(g, 2)(-Q)$ and $L(g, 2)(-Q')$, $Q, Q' \in X_{\mathrm{reg}}$, are not isomorphic if $Q \neq Q'$ beacause $\mathbf{O}_X(Q)$ and $\mathbf{O}_X(Q')$ are not isomorphic.

Remark 3.10. Let Y be a smooth curve of genus $g'' \geq 0$. If $g'' \geq 3$ we assume Y hyperelliptic. If $g'' \geq 2$ we call F' the g^1_2 on Y. If $g'' = 1$ we fix $F' \in \mathrm{Pic}^2(Y)$. If $g'' = 0$ we take as F' the degree 2 line bundle on Y but later we will fix a vector space $W \subset H^0(Y, F')$ with $\dim(W) = 2$ and

W spanning F'. Let X be an integral curve of arithmetic genus $g > g''$ with π: $Y \to X$ as normalization. Take an effective degree 2 divisor $Q + \sigma(Q)$ of F' (or of the projective space associated to W if $g'' = 0$) such that $\pi(Q \cup \sigma(Q)) \subset X_{\mathrm{reg}}$. Hence $\pi(Q \cup \sigma(Q))$ is a Cartier divisor of X and defines $L \in \mathrm{Pic}^2(X)$ with $h^0(X, L) > 0$. For any such L we have $h^0(X, L) \leq 2$ if $g'' > 0$ and $h^0(X, L) \leq 3$ if $g'' = 0$. We assume $h^0(X, L) = 1$; this is true for every such L if the degree 2 morphism $Y \to \mathbf{P}^1$ associated to F' does not factor through π. By the projection formula we have an exact sequence on X:

$$0 \to L \to \pi_*(F') \to \pi_*(\mathbf{O}_Y)/\mathbf{O}_X \to 0 \tag{4}$$

where $\mathrm{length}(\pi_*(\mathbf{O}_Y)/\mathbf{O}_X) = g - g''$ and hence $\deg(\pi_*(F')) = 2 + g - g''$. Notice that $h^0(X, \pi_*(F')) \geq h^0(Y, F') \geq 2$ and $\pi_*(F')$ is torsion free. Since π is finite the functor $R^1\pi_*$ is zero. Hence $\pi_*(F')$ is spanned. If $g'' \leq 1$ for every coherent sheaf F with $L \subset F \subseteq \pi_*(F')$ we have $h^1(Y, F) = 0$. If $g = g'' + 1$ (i.e. if X has a unique singular point which is an ordinary node or an ordinary cusp) we take $F{:=}\ \pi_*(F')$ and obtain an example. In the general case if $g'' = 0$ we fix a spanning subspace W of $H^0(Y, F')$ containing $\pi^*(H^0(Y, L))$ and take as F the subsheaf of $\pi_*(F')$ generated by $\pi_*(W)$. The triple $(\mathbf{P}^1, F', X)$ gives an example if and only if $\deg(F) = 3$. We see how to obtain examples of degree ≥ 3 if $g - g'' = \mathrm{length}(\pi_*(\mathbf{O}_Y)/\mathbf{O}_X) \geq 2$. Since we may have $h^0(X, \pi_*(F')) > h^0(X, L)$, the condition "$g = g'' + 1$" is not a necessary condition for the existence of a spanned F with $\deg(F) = 3$. To refine this sufficient condition we observe that, since F is not locally free, the hyperelliptic pencil u: $Y \to \mathbf{P}^1$ induced by F' does not factor through the normalization map π. However, u may factor through a partial normalization map, i.e. there are birational morphism π': $Y \to Y'$ and π'': $Y' \to X$ with $\pi = \pi'' \circ \pi'$ and such that u factors through π'. We allow the case $Y' = Y$, $\pi' = Id_Y$ and we take (Y', π') with $\mathrm{length}(\pi'_*(\mathbf{O}_Y)/\mathbf{O}_{Y'})$ maximal. By construction there is a hyperelliptic pencil u': $Y' \to \mathbf{P}^1$ and in particular Y' is Gorenstein. By construction there is $G' \in \mathrm{Pic}^2(Y')$ inducing the hyperelliptic pencil and with $\pi''^*(F)/\mathrm{Tors}(\pi''^*(F)) \cong G'$. By the maximality of $\mathrm{length}(\pi'_*(\mathbf{O}_Y)/\mathbf{O}_{Y'})$ we obtain $h^0(X, \pi''_*(G')) = 2$. Hence we have $F = \pi''_*(G')$ and hence $\mathrm{length}(\pi''_*(\mathbf{O}_{Y'})/\mathbf{O}_X) = 1$. Vice versa, we start with (Y', G') and take any pair $(X, \pi''_*(G'))$ with $\mathrm{length}(\pi''_*(\mathbf{O}_{Y'})/\mathbf{O}_X) = 1$ as an example. In particular we may add to Y' a new ordinary node to Y gluing together two points of Y_{reg} not on the same fiber of u' or taking a cusp at a point of Y' at which u is étale. However, at singular points of Y' we may do more complicated operations to obtain X starting from Y'.

REMARK 3.11. Let X be a trigonal curve with $g := p_a(X) \geq 3$. Let π: $Y \to X$ be its normalization and f: $X \to \mathbf{P}^1$ the associated triple covering. Set $h := f \circ \pi$: $Y \to \mathbf{P}^1$. The line bundle $L := \pi^*(\omega_X)/\mathrm{Tors}(\pi^*(\omega_X))$ define a morphism u: $Y \to \mathbf{P}(H^0(X, \omega_X)) \cong \mathbf{P}^{g-1}$ and $u(Y)$ is contained in a ruled rational surface, S, because every fiber of f contained in $X_{\mathrm{reg}} \cong \pi^{-1}(X_{\mathrm{reg}})$ is contained in a line. The morphism u factors through π if X is Gorenstein. However, in general S is just the projection of a linearly normal minimal degree ruled surface $W \cong F_e$ (for some integer $e \geq 0$) contained in the projective space $\mathbf{P}(H^0(Y, L))$ or of a cone W over a rational normal curve contained in a hyperplane of $\mathbf{P}(H^0(Y, L))$ (see e.g. [9], §2). Set $N := h^0(Y, L) - 1$. Hence $\deg(W) = N$ and there are non-negative integers e_1 and e_2 with $e_1 + e_2 = N$, $e_1 \geq e_2$ ([9], §2). It seems natural to call the integer $e_1 - e_2$ the Maroni invariant of X, even when X is not Gorenstein. Notice that for every smooth trigonal curve h: $Y \to \mathbf{P}^1$ any factorization of f through a singular curve gives a singular trigonal curve with Y as a normalization. If $X \neq Y$, then the associated ruled surfaces of X and Y seem to be different becuse $p_a(X) > p_a(Y)$. For a detailed study of the canonical map for a Gorenstein curve X, see [4], §3.

REMARK 3.12. For low degree spanned rank r torsion free sheaves, F, with $h^1(X, F) = 0$ Theorem 1.3 gives the best possible answer (see also the discussion in the first part of Section 5 and Corollary 5.4). For rank r torsion free sheaves with degree at most $2r$ and $g \geq 3$ (hence in the special range) see Corollary 2.6.

4 – Proof of 1.1 and 1.2

In this section we prove 1.1 and 1.2. Notice that if $Q \in X_{\mathrm{reg}}$ and H is a torsion free sheaf obtained from the torsion free sheaf F making a positive or negative elementary transformation supported at Q, then F and H have the same formal isomorphism type at each point of $\mathrm{Sing}(X)$.

Remark 4.1. Let F be a torsion free sheaf on X and $Q \in X_{\mathrm{reg}}$. F is stable (resp. semistable) if and only if $F(-Q)$ is stable (resp. semistable). Hence if G is a rank r stable torsion free sheaf the general sheaf obtained from G making r negative elementary transformations is stable.

Proof of 1.1. (a) Since the result is trivial if $r = 1$, we may assume $r > 1$ and use induction on r. Fix $Q \in X_{\mathrm{reg}}$. Let G be a rank r torsion free sheaf on X and A a saturated subsheaf of G with $1 \leq \mathrm{rank}(A) < r$, i.e. assume the existence of an exact sequence

$$0 \to A \to G \to B \to 0 \tag{5}$$

with B torsion free. The general sheaf, G', obtained from G making a positive elementary transformation supported at Q fits in an exact sequence

$$0 \to A \to G' \to B' \to 0 \tag{6}$$

in which B' is obtained from B making a general elementary transformation supported at Q. Let A' any sheaf obtained from A making a positive elementary transformation supported at Q. There exists a positive elementary transformation of G which induces A', i.e. such that the corresponding sheaf G'' fits in an exact sequence

$$0 \to A' \to G'' \to B \to 0 \tag{7}$$

By the openness of stability to show that the general sheaf obtained from G making a positive elementary transformation supported at Q is stable it is sufficient to prove that G'' is stable.

(b) Let M be a saturated subsheaf of F with $\mathrm{rank}(M) = r-1$. Hence we have an exact sequence

$$0 \to M \to F \to L \to 0 \tag{8}$$

with L torsion free of rank 1. Fix an integer $k \geq \max\{t(M), (r-1)\deg(L) - \deg(M)\}$. We apply to M k general elementary transformations supported at $Q_1, \ldots, Q_k$ and make the corresponding elementary transformations of F. Then we apply to the sheaf obtained from F in this way some (say, x) general positive elementary transformations supported at $Q_{k+1}, \dots, Q_{k+x}$. Here we take $x = [(\deg(M)+k+1)/(r-1)] - \deg(L)$. Set $L' := L(\sum_{k<j\le k+x} Q_j)$. In this way we obtain an exact sequence

$$0 \to M' \to F' \to L' \to 0 \tag{9}$$

with $\mu(M') < \mu(L') \le \mu(M') + 1/(r-1)$. M' is stable because $k \ge t(M)$. Since $g > 0$ it is easy to check that for general elementary transformations the extension (9) does not split. By Remark 4.1 to prove the theorem it is sufficient to prove that F' is stable. In order to obtain a contradiction we assume that F' is not stable. Let U be a saturated subsheaf of F' with $\mu(U) \ge \mu(F')$ and $\operatorname{rank}(U) < r$. Taking $\mu(U)$ maximal and $\operatorname{rank}(U)$ minimal we may assume that U is stable. Set $u := \operatorname{rank}(U)$. Since $\mu(M') < \mu(F')$ and M' is stable, U is not contained in M'. Hence the image, R, of U in L' has rank 1. Since $\mu(L') - 1 < \mu(F')$, we have $R = L'$. Since $\operatorname{rank}(U \cap M') = u - 1 \le r - 2$, we have $\deg(U \cap M') < (u-1)\mu(M')$.

(c) Here we give the only case which we are able to prove 1.1 using only numerical tricks with the slope of all sheaves involved in the construction. We assume to be in the set-up of step (b). In particular we assume that we have the exact sequences (8) and (9) with $\operatorname{rank}(M) = \operatorname{rank}(M') = r-1$. We assume $\deg(M') \equiv 0 \bmod (r-1)$. This assumption is equivalent to the existence of an integer y such that $\deg(M') = y(r-1)$ and $\deg(L') = y+1$, i.e. to the condition $\deg(F') \equiv 1 \bmod (r)$. Take any F' fitting in a non-split extension (9) with these numerical invariants; such a sheaf exists because $\operatorname{Ext}^1(X; L', M') \ne 0$ by Riemann-Roch. We claim that F' is stable. Assume the existence of a saturated subsheaf U of M' with $\mu(U) \ge \mu(F') = y + 1/r$ and $1 \le \operatorname{rank}(U) < \operatorname{rank}(F')$. Taking U minimal, we may assume U stable. Hence the induced map $U \to L'$ is either 0 or surjective. In the latter case (9) splits, contradiction. In the former case U' is a subsheaf of M' and hence $\mu(U') \le y$, contradiction. Hence in this way we obtain the result for all large numbers, say z, of general positive elementary transformations supported at Q and with $z + \deg(F) \equiv 1 \bmod (r)$ (see Remark 4.1, too). In particular for $r = 2$ we proved the theorem for the case $t + \deg(F)$ odd.

(d) By local duality (see e.g. [5], Prop. 3.1.6, part 1) for every torsion free sheaf G on X the natural map $G \to \operatorname{Hom}(\operatorname{Hom}(G, \omega_X), \omega_X)$ is an isomorphism. Furthermore, we have $\operatorname{Ext}^i(G, \omega_X) = 0$ for every $i \ge 1$ ([5], Lemma 2.5.3). Hence the functor $\operatorname{Hom}(-, \omega_X)$ sends short exact sequences of torsion free sheaves on X into short exact sequences. Furthermore, we have $\deg(\mathrm{Hom}(G,\omega_X)) = -\deg(G) + (2g-2)\mathrm{rank}(G)$ even when ω_X is not locally free ([5], Prop. 3.1.6, part 2). Hence the functor $\mathrm{Hom}(-,\omega_X)$ preserves the conditions of stability and semistability.

(e) Let A be a rank r torsion free sheaf and $P \in X_{\mathrm{reg}}$. Then A is stable if and only if $A(P)$ is stable. Furthermore, $A(P)$ is obtained from A making r positive elementary transformations supported at P, while A is obtained from $A(P)$ making r negative elementary transformations supported at P. Hence, for fixed F with $\mathrm{rank}(F) = r$, the proof of 1.1 is finished if we may find r stable sheaves $F_1, \dots, F_r$, each of them obtained from F making positive and negative elementary transformations supported at points of X_{reg} and such that the integers $\deg(F_i)$, $1 \le i \le r$, cover all the congruence classes of integers modulo r. We assume that the theorem is false for some rank r torsion free sheaf and among such “bad” sheaves we take one, called F, with the property that there is a congruence class modulo r such that for all large integers t with $t+\deg(F)$ in that congruence class the general sheaf, G, obtained from F making t general positive elementary transformations supported at $Q_1, \dots, Q_t$ has a proper subsheaf, U, with $\mu(U) \ge \mu(G)$ and with $w := \mathrm{rank}(U)$ minimal among all such “bad” rank r sheaves. Since $\mathrm{rank}(U)$ is minimal, U is stable. By part (d) we have $\mu(\mathrm{Hom}(G,\omega_X)) \ge \mu(\mathrm{Hom}(U,\omega_X))$. By part (d) $\mathrm{Hom}(G,\omega_X)$ has a rank w torsion free quotient sheaf isomorphic to $\mathrm{Hom}(U,\omega_X)$. Hence by the minimality of w we obtain the inequality $w \le r-w$. We distinguish two subcases according to the value of w.

(e1) Here we assume $2w < r$. We start with an exact sequence (8) with L torsion free and with $\mathrm{rank}(L) = w$. Thus here we have $\mathrm{rank}(M) = r-w$. We make the construction of part (b) starting from this filtration. After a large number, t, of positive elementary transformations supported at $Q_1, \dots, Q_t$ we arrive at an exact sequence (9) with $\mathrm{rank}(M') = r-w$, $\mathrm{rank}(L') = w$, M' obtained from M making some of the t positive elementary transformations applied to F and L' obtained from L making the remaining positive elementary transformations. Indeed, the positive elementary transformations which send L into L' may be considered general, while each of the positive elementary transformation, say supported at Q_i, which is used to transform M into M' are very particular positive elementary transformations of F: a positive elementary transformation of F induces a positive elementary transformation of M if and only if the kernel of the associated map $F^*|\{Q_i\} \to \mathbf{K}_{Q_i}$ contains $L^*|\{Q_i\}$. Notice that $\deg(F') = t + \deg(F) = \deg(\mathrm{G})$ and in particular $\deg(F')$ falls in a prescribed in advance congruence class modulo r. Since F is a "bad" sheaf, F' is not stable. Hence there is a proper saturated subsheaf U of F' with $\mu(U) \geq \mu(F')$. Taking $u{:=}\ \mathrm{rank}(U)$ minimal we may assume U stable. Since the positive elementary transformations we did are not general, we have $u \leq w$, but in general we cannot claim the equality $u = w$. We distinguish two subcases, (e1.1) and (e1.2), according to the value of u.

(e1.1) Here we assume $u = w < r/2$ and distinguish two further possibilities.

(e1.1.1) Here we are in the situation of (e1.1) with the further assumption $\mu(U) > \mu(F'/U)$, i.e. F' not semistable. We have $\mu(M') < \mu(L') \leq \mu(U)$ and $\mathrm{rank}(U) = \mathrm{rank}(L')$. We obtain a contradiction to the stability of U, M' and L' unless the map $U \to L'$ induced by the composition of the inclusion of U in F' and of the surjection $F' \to L'$ is an isomorphism, i.e. unless the extension (9) splits. For general positive elementary transformations we may easily avoid this case.

(e1.1.2) Here we are in the situation of (e1.1) with the further assumption $\mu(U) = \mu(F'/U)$, i.e. F' semistable. Here we stop at an exact sequence (9) with $\deg(L') = \deg(U)+1$, i.e. $\mu(L')-1/w = \mu(F')+1/(r-w)$. We obtain that U is a subsheaf of L' with L'/U skyscraper sheaf with $h^0(X, L'/U) = 1$. Hence the inclusion $U \to L$ shows that F' has a subsheaf, F'', with $F'' \cong M' \oplus U$. Set $P{:=}\ \mathrm{Supp}(F'/F'') \in X$. We may divide the set $\{Q_1, \dots, Q_t\}$ into two disjoint subsets, say $\{Q_1, \dots, Q_k\}$ and $\{Q_{k+1}, \dots, Q_t\}$, according to the fact that the corresponding positive elementary transformation of F induces a positive elementary transformation of M or not. Since X_{reg} is irreducible, moving the points Q_i, $i \geq 1$, we reduce to the case $P \notin \{Q_1, \dots, Q_t\}$. Now we use that every positive elementary transformation supported at a smooth point, Q_i, of X has as inverse a negative elementary transformation supported at Q_i and that two positive elementary transformations supported at distinct points commute (in the sense made clear in [7], Prop. 2.2). Hence making backward t suitable negative elementary transformations at $Q_t, \dots, Q_1$ we obtain that F has a subsheaf, A, with $\mathrm{rank}(A) = r$, $\deg(A) = \deg(F) - 1$ and $A \cong M \oplus L''$ with L'' subsheaf of L with L/L'' skyscraper sheaf. Obviously, this is a very restrictive condition on F.

We may find a contradiction in the following way. To cover simultaneously the corresponding part of case (e1.2) and the subcase (e2.1.2) which we will met later we write $s:=\operatorname{rank}(M)$. It is sufficient to find a contradiction taking instead of F the sheaf, H, obtained from F making n suitable positive elementary transformations supported at generic points of X. We take as positive elementary transformations $n-s$ general positive elementary transformations and s positive elementary transformations which induce positive elementary transformations of M and are general with this restriction. Call T the sheaf obtained from M making these positive elementary transformations. Hence H has a subsheaf, B, with $B \cong T \oplus R$ and T/B skyscraper sheaf with $h^0(X, T/B) = 1$. The composition of the inclusions of A in F and of F in T induces an inclusion $M \oplus L'' \cong A \to B \cong T \oplus R$ whose restriction to $\{0\} \oplus L''$ is the inclusion of $\{0\} \oplus L''$ into $\{0\} \oplus R$. However, L'' is saturated in the sheaf obtained from F making s positive elementary transformations inducing positive elementary transformations of M and it remains saturated when we make $n-s$ further general positive elementary transformations. Hence the saturation of L'' in H cannot contain R, contradiction.

(e1.2) Here we assume $u < w < r/2$. We start from a filtration $M_1 \subset M_2 \subset F$ with M_2 saturated subsheaf of F, M_1 saturated subsheaf of M_2 and F and with $\operatorname{rank}(M_1) = u$, $\operatorname{rank}(M_2) = w$. Set $u_1 := u$, $M := M_1$ and $L := F/M$. Hence we have an exact sequence (8) and in the usual way we obtain an exact sequence (9) making t positive elementary transformations. Call again U a saturated proper subsheaf of F' with maximal slope and minimal rank. Set $u_2 := \operatorname{rank}(U)$. The t positive elementary transformations we would make starting with M_2 are more general than the ones we can make starting from M_1. Hence we have $u_2 \le u_1$ (e.g. by the properness of relative Quot-schemes). If $u_2 = u_1$, then we may repeat the proof of part (e1.1). If $u_2 < u_1$ we start with a 3-steps filtration of F by saturated subsheaves of F respectively with rank u_2, u_1 and w. Again we find an integer $u_3 \le u_2$ and such that if $u_3 = u_2$ we may repeat the proof given in (e1.1). If $u_3 < u_2$, we repeat the construction starting from a 4-step filtration of F by saturated subsheaves of F respectively with rank u_3 u_2, u_1 and w. After at most w steps this construction must stop with an integer u_{i+1} with $u_{i+1} = u_i$ and hence we conclude.

(e2) Here we assume $2w = r$. In particular r is even. Using the functor $\mathrm{Hom}(-, \omega_X)$ and the minimality of w we obtain easily that G/U is stable. Now we make the construction of part (c) starting from a filtration (8) with $\mathrm{rank}(M) = \mathrm{rank}(L) = r/2$. After a large number, t, of positive elementary transformations supported at $Q_1, \dots, Q_t$ we arrive at an exact sequence (9). Notice that $\deg(F') = t + \deg(F) = \deg(G)$ and in particular $\deg(F')$ falls in a prescribed in advance congruence class modulo r. Since F is a "bad" sheaf, F' is not stable. Hence there is a proper saturated subsheaf U of F' with $\mu(U) \geq \mu(F')$. Taking $u{:=}\ \mathrm{rank}(U)$ minimal we may assume U stable. Since the positive elementary transformations we did are not general, we have $u \leq w = r/2$, but in general we cannot claim the equality $u = w$. Hence we distinguish again two subcases.

(e2.1) Here we assume $u = w = r/2$. Call $f\colon U \to L'$ the composition of the inclusion of U in F' and the surjection $F' \to L'$ given by (9). We distinguish two subsubcases.

(e2.1.1) Here we assume $\mu(U) > \mu(F')$, i.e. $\deg(U) \geq \deg(L')$. Since $\mathrm{rank}(U) = \mathrm{rank}(L')$ and U is stable, either $f = 0$ or f induces a splitting of (9). The first case is impossible because M' is stable and $\mu(M') < \mu(U)$. We may easily avoid the second case taking not too special the t positive elementary transformations.

(e2.1.2) Here we assume $\mu(U) = \mu(F')$. Hence $\deg(F')$ is even. By the maximality of $\mu(U)F'$ is semistable and F'/U is semistable. Here we stop at an exact sequence (9) with $\deg(L') = \deg(M') + 2$. We obtain that U is a subsheaf of L' with L'/U skyscraper sheaf with $h^0(X, L'/U) = 1$. Hence the inclusion $U \to L$ shows that F' as a subsheaf, F'', with $F'' \cong M' \oplus U$. We conclude as in subcase (e1.1.2) taking $s = r/2$. □

Since for each torsion free sheaf F we are interested only in sheaves obtained from F making a fixed number (larger but bounded) of positive elementary transformations, in the statement of Theorem 1.1 we may take the word "generic" for the sequence in the sense that for every integer $t > 0$ there is a Zariski open dense subset Ω of the cartesian product $(X_{\mathrm{reg}})^t$ such that the statement is true if $(Q_1, \dots, Q_t) \in \Omega$.

Proof of 1.2. Just use Remark 4.1 and the statement of 1.1. □

Remark 4.2. Assume $g{:=}\ p_a(X) = 1$ and X singular, i.e. let X be a rational curve with one ordinary node or one ordinary cusp. Then the proofs of Theorems 1.1 and 1.2 give the existence of semistable torsion free sheaves with a fixed formal isomorphism type at the singular point of X and with prescribed degree. Obviously, taking r and d coprime, we obtain also the existence of a stable torsion free sheaf with prescribed formal isomorphism type at the singular point of X. These results are well-known.

5 – Proof of 1.3

Here we classify low degree spanned non special torsion free sheaves on an integral curve X with $g := p_a(X) \geq 2$. Thus in this section we are interested in "general" non-special sheaves, i.e. in sheaves for which Clifford's type theorems are not true. We will prove Theorem 1.3. Let F be a rank r torsion free sheaf on X spanned by its global sections and with $h^1(X, F) = 0$. Set $d := \deg(F)$. We have $d > 0$ because F is not the trivial bundle $\mathbf{O}_X^{\oplus r}$ since $g > 0$. For the same reason F has not $\mathbf{O}_X$ as a direct factor. Since F is not trivial, we have $h^0(X, F) \geq r + 1$. Furthermore, we have $h^0(X, F) \geq \mathrm{emb}(F)$. By Riemann-Roch we have $h^0(X, F) = d + 1 - g$. Hence $d \geq \max\{r - g, \mathrm{emb}(F) + 1 - g\}$. Theorem 1.3 shows that for every X this inequality is the only numerical restriction. To obtain this result we need the following well-known lemma.

LEMMA 5.1. *Let H be a torsion free sheaf and H' (resp. H'') the general sheaf obtained from H making a general negative (resp. positive) elementary transformation. Then we have:*
(*a*) $h^0(X, H') = \max\{h^0(X, H) - 1, 0\}$;
(*b*) $h^1(X, H'') = \max\{h^1(X, H) - 1, 0\}$.

PROOF. Take $\sigma \in h^0(X, H)$, $\sigma \neq 0$ and let R be the saturation in H of the subsheaf $\sigma(\mathbf{O}_X)$. Take $Q \in X_{\mathrm{reg}}$ such that R is spanned by σ at Q; for instance take Q general. Thus σ in not a section of the sheaf obtained from H making a general negative elementary transformation supported at Q. Hence by semicontinuity we have part (*a*). Part (*b*) follows from Serre duality and part (*a*) applied to the torsion free sheaf $\mathrm{Hom}(H, \omega_X)$. □

REMARK 5.2. Let F and G be rank r torsion free sheaves on X and $f\colon F \to G$ a morphism such that for every $P \in \mathrm{Sing}(X) f$ induces an isomorphism $f_P^\wedge\colon F_P^\wedge \to G_P^\wedge$ of $\mathbf{O}_{X,P}^\wedge$-modules. Using the corresponding result on the normalization of X we obtain that f is the composition of positive and negative elementary transformations supported at points of $X - \mathrm{reg}$.

Proof of 1.3. Call $\mathbf{T}(x)$ the irreducible variety parametrizing all sheaves of degree x obtained from G making positive and negative elementary transformations supported at points of X_{reg}. For every integer $i \geq i$, set $\mathbf{T}(x,i) := \{F \in \mathbf{T}(x)\colon\ h^1(X,F) \geq i\}$ and $\mathbf{Y}(x,i) := \{F \in \mathbf{T}(x)\colon\ h^1(X,F) = i\}$. By semicontinuity $\mathbf{T}(x,i)$ is a closed subset of $\mathbf{T}(x)$. By part (b) of Lemma 5.1 for every integer $u \geq r(g-1)$ the general sheaf $F \in \mathbf{T}(u)$ has $h^1(X,F) = 0$. For every integer $u \geq r(g-1)$ set $\mathbf{T}(u)' := \{F \in \mathbf{T}(u)\colon\ h^1(X,F) = 0\}$. Hence for every $u \geq r(g-1)$ $T(u)'$ is a Zariski open dense subset of $T(u)$. We divide the remaining part of the proof into 6 steps.

Step 1. Here we will check that for every integer $x \geq rg$ a general $F \in \mathbf{T}(x)'$ is generically spanned, i.e. $H^0(X,F)$ spans a subsheaf, $G(F)$, of F with $\mathrm{rank}(G(F)) = r$. The result is obvious if $r = 1$. Hence we assume $r \geq 2$ and use induction on the rank to check this assertion. There is a rank 1 torsion free sheaf L such that H fits in an exact sequence

$$0 \to L \to G \to H \to 0 \tag{10}$$

We apply to G several positive and negative elementary transformations supported at points of X_{reg}, some of them very particular. The non-general positive elementary transformations correspond to the subspace $L|\{P\}$ of $G|\{P\}$ and hence each of them induces a positive elementary transformation of L. The non-general negative elementary transformations have as kernel the kernel of the surjection $G|\{P\} \to H|\{P\}$ and hence induce a negative transformation of H. The general positive elementary transformations preserve L and induce positive elementary transformations of H. The general negative elementary transformations preserve H and induce negative elementary transformations of L. At the end of this process we may obtain an exact sequence

$$0 \to L'' \to G'' \to H'' \to 0 \tag{11}$$

with $\deg(L'') = g-1$, $\deg(H'') = x-g+1$, $h^1(X,L'') = h^1(X,H'') = 0$ and with L'' and H'' generically spanned. Since $h^1(X,L'') = 0$, G'' is generically spanned.

Step 2. Here we assume $r = 1$ and prove that for every integer $d' \geq g+1$ we may find a degree d' torsion free sheaf, D, obtained from G making positive and negative elementary transformations supported at points of X_{reg}, with $h^1(X,D) = 0$ and such that $H^0(X,D)$ spans D at every point of X_{reg}. By Serre Theorem B for any $Q \in X_{\mathrm{reg}}$ there is a large integer y such that for every $z \geq y$ we have $h^1(X, G(zQ)) = 0$. Hence the general sheaf, B, obtained from G making z general positive elementary transformations has $h^1(X,B) = 0$ and hence $h^0(X,B) = \deg(G)+z+1-g$. By part (b) of Lemma 5.1 for every integer $u \geq g-1$ the general sheaf $F \in \mathbf{T}(u)$ has $h^1(X,F) = 0$. We need to check that for the general such G the vector space $H^0(X,G)$ spans G at every point of X_{reg}. We consider the case $u = g+1$, the general case following by induction on u using exact sequences

$$0 \to G \to G'' \to \mathbf{K}_P \to 0 \tag{12}$$

with $P \in X_{\mathrm{reg}}$, G spanned at every point of X_{reg} and $h^1(X,G) = 0$. If this is false, then for every $F \in \mathbf{T}(g+1)$ with $h^1(X,F) = 0$ and $h^0(X,F) = 2$ there is $P \in X_{\mathrm{reg}}$ such that $h^0(X,F) = h^0(X,F(-P))$. We have $h^1(X,F(-P)) = 1$. Since $\dim(X_{\mathrm{reg}}) = 1$, in order to obtain a contradiction it is sufficient to check that $\mathbf{T}(g,1)$ has codimension ≥ 2 in $\mathbf{T}(g)$. By the proof of Lemma 5.1 we obtain that $\mathbf{T}(g-1,1)$ is a proper closed subset of $\mathbf{T}(g-1)$ and that every element of $\mathbf{Y}(g,i)$ induces a one-dimensional family of elements of $\mathbf{T}(g-1,i)$. Vice versa, every element of $\mathbf{Y}(g-1,i)$ is induced by at most finitely many elements of $\mathbf{Y}(g,i)$. Hence it is sufficient to check that $\mathbf{T}(g-1,2)$ has codimension at least two in $\mathbf{T}(g-1)$. Since $\mathbf{T}(g-1,1)$ is a proper closed subset of the irreducible variety $\mathbf{T}(g-1)$, it is sufficient to prove that for all integers $i \geq 2$ every element of $\mathbf{Y}(g-1,i)$ is the flat limit of an irreducible family of elements of $\mathbf{Y}(g-1,i-1)$. Fix $A \in \mathbf{Y}(g-1,i)$. For any $Q \in X_{\mathrm{reg}}$ we have $A \cong A(-Q)(Q)$. Hence A is the flat limit of a family bundles, say $\{A_m\}_{m \in M}$, obtained from A first making a one-dimensional family of "moving" general negative elementary transformations and then a one-dimensional family of general positive elementary transformations (i.e. each A_m is obtained from A making one sufficiently general negative elementary transformation and one sufficiently general positive elementary transformation). By the proof of Lemma 5.1 we have $A_m \in \mathbf{Y}(g-1,i-1)$ for general $m \in M$; here we use the assumption $g \geq 2$.

STEP 3. Here we assume $r \geq 1$. We will show that for every integer $x \geq r+1+r(g-1)$ there is a sheaf, Z, obtained from G making positive and negative elementary transformations supported at points of X_{reg}, with $h^1(X,Z)=0$ and such that $H^0(X,Z)$ spans Z at every point of X at which Z is locally free (i.e. at the points at which G is locally free) and in particular at every point of X_{reg}. By Step 2 this assertion is true if $r=1$. Hence we may assume $r>1$ and use induction on r. As in Step 2 using the exact sequence (12) we reduce to the case $x=r+1+r(g-1)$. As in Step 2 we start from an exact sequence (10) and obtain an exact sequence (11) with $F'' \in \mathbf{T}(rg)'$, $\deg(L'')=g$, $\mathrm{rank}(L'')=1$, $\deg(H'')=(r-1)g$, $\mathrm{rank}(H'')=r-1$. We make a similar filtration for H'' and obtain an increasing filtration, say $\{G_i\}_{0\leq i\leq r}$, of F'' with $G_0=\{0\}$, $G_r=F''$, $\mathrm{rank}(G_i)=i$, each G_i saturated subsheaf of F'', $L_j := G_j/G_{j-1}$, $1\leq j\leq r$, torsion free rank 1 sheaf of degree g with $h^1(X,L_j)=0$. Furthermore, it is easy to check as in the proof of Step 2 the following fact: there is an integer $z\geq 0$ such that for every j, $1\leq j\leq r$, the unique (up to a constant) non-zero section of L_j has z simple zeroes on X_{reg}, say $P(j,w)$, $1\leq w\leq z$. Notice that $h^0(X,F'')=r$, that F'' is spanned outside $\mathrm{Sing}(X)$ and the zr points $P(j,w)$, $1\leq j\leq r$, $1\leq w\leq z$, and at each $P(j,z)$ the vector space $H^0(X,F'')$ spans a linear subspace of dimension $r-1$ of the fiber $F''|\{P(j,w\}$. Every positive elementary transformation, M, of F'' has $h^1(X,M)=0$ and it is spanned outside $\mathrm{Sing}(X)$ and the set $\{P(j,w)\}_{1\leq j\leq r, 1\leq w\leq z.}$ Let $W\cong \mathbf{P}(F''|X_{\mathrm{reg}})$ be the parameter space for one positive elementary transformation supported at one point of X_{reg}. We need to check that the general such M is spanned at every point of $P(j,w)$. Call $B(j,z)$ the subset W formed by the positive elementary transformations such that the corresponding M is not spanned at $P(j,z)$. Every $B(j,z)$ is a closed subset of the irreducible variety W. Since a general positive elementary transformation of L_j is spanned (for general L_j) by Step 2 and $h^1(X,Gj)=0$ for every j, one can see by induction on j that $B(j,w)$ is a proper closed subset of W. Hence $\bigcup_{j,w} B(j,w)\neq W$, as wanted.

STEP 4. Fix $P\in \mathrm{Sing}(X)$ and $Q\in X_{\mathrm{reg}}$. Here we will show that for every integer $x\geq \mathrm{emb}_P(F)+r(g-1)$ a general element of $\mathbf{T}(x)'$ is spanned at P. Fix $F\in \mathbf{T}(r(g-1))'$. By Riemann-Roch we have $h^0(X,F)=0$. By Serre Theorem A there is a large integer t such that $F(tQ)$ is spanned by its global sections and $h^1(X,F(tQ))=0$. Hence a general bundle, H, obtained from F making rt general positive elementary transformations supported at smooth points of X is spanned at P. Call $e(1), \dots, e(rt)$ a choice of such sufficiently general positive elementary transformations. Any two positive elementary transformations supported at distinct points commute (in the sense made clear in [7], Prop. 2.2). Hence there must be at least one such elementary transformation, say e(1), such that the sheaf F_1 obtained from F making the positive elementary transformation e(1) has a section, s_1, whose value $s_1(P)$ in the fiber $F_1|\{P\} \cong F|\{P\} \cong \mathbf{K}^e$, $e{:=}\ \mathrm{emb}_P(F)$, is not zero; here we use that e(1) is supported at a point of $X-\mathrm{reg}$ and hence not by P and thus making e(1) induces a fixed isomorphism of $F|\{P\}$ with $F_1|\{P\}$. If $e = 1$ (i.e. $r = 1$ and G is locally free at P) we stop. Assume $e > 0$. For the same reason there is at least one among the positive elementary transformations e(2), $\dots, e(rt)$, say e(2), such that the sheaf F_2 obtained from F_1 making the elementary transformation e(2) has a section s_2 whose value $s_2(P)$ in the fiber $F_2|\{P\} \cong F_1|\{P\}$ is linearly independent from $s_1(P)$. And so on. After $\mathrm{emb}_P(F)$ steps we conclude.

Step 5. Since $\mathrm{Sing}(X)$ is finite, we obtain that for every integer $x \geq \mathrm{emb}(F) + r(g-1)$ a general element of $\mathbf{T}(x)'$ is spanned by its global sections at each point of $\mathrm{Sing}(X)$ and in particular at each point of $\mathrm{Sing}(F)$.

Step 6. The proof of Step 4 gives without any modification the last assertion of 1.3. □

Remark 5.3. Notice that "spannedness" is an open condition in a flat family of torsion free sheaves on X which have constant cohomology. Hence by semicontinuity if the sheaf A whose existence is claimed in Theorem 1.3 is obtained from A making t positive elementary transformations and $\deg(G) - d + t$ negative elementary transformations, the thesis of 1.3 is true for the general sheaf obtained from G making t general positive elementary transformations and $\deg(G) - d + t$ general negative elementary transformations.

Notice that the sheaves G and A considered in the statement of Theorem 1.3 have the same formal isomorphism type at every point of $\mathrm{Sing}(X)$. Hence Theorem 1.3 implies the following result.

Corollary 5.4. *Let t be a formal singularity type for* rank r *torsion free sheaves at* $\mathrm{Sing}(X)$. *Set* $e{:=}\ \mathrm{emb}(\tau)$. *Fix an integer* $\nu \geq \max\{r+1, e\}$

and set $d := v + r(g-1)$*. Then there exists a* rank r *torsion free sheaf* A *on* X *with* $\deg(A) = d$*,* $h^1(X, A) = 0$*,* A *spanned by its global sections and with formal isomorphism type* τ *along* $\mathrm{Sing}(X)$*.*

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