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Remarks on a variational problem in Laguerre geometry

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Riassunto: *Si studia la stabilità delle superficie minime di Laguerre, con i metodi della geometria di Lorentz.*

Abstract: *We study the stability of Laguerre minimal surfaces using methods from Lorentzian geometry.*

1 – Introduction

Recently there has been some renewed interest in Laguerre differential geometry [1], [2], [3]. This geometry was to a large extent developed by Blaschke and his school and a large amount of material about it can be found in [4]. Additional material may be found in [5].

Consider the set of oriented spheres in Euclidean 3-space. If we add to this set the space of oriented planes, i.e. the spheres of infinite radii, then we obtain the space of the Moebius (conformal) geometry. This space can be identified with the deSitter space which is a Lorentzian 4-manifold of constant curvature +1. If, on the other hand, we add the spheres of zero radii, i.e. "point spheres" then we obtain the Laguerre space which can be identified with the four dimensional Minkowski space $\mathbb{E}^4_1$.

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Any smooth immersion X of an oriented surface into 3-space has a lift to the unit sphere bundle of $\mathbb{E}^3$ called the Legendre lift. The unit sphere bundle may also be considered as the space of null lines in Minkowski space. Thus the isometry group of Minkowski space acts on the set of Legendre surfaces. The principle aim of Laguerre geometry is to study properties of the immersion which are invariant under this action. The simplest way to do this is to define at each point p of the surface an oriented 2-sphere or point sphere $Y(p)$ which is in some sense invariant under this action. The map Y will be called the $\mathcal{L}$-Gauss map. It is analogous to the conformal Gauss map of Moebius geometry and defines a spacelike immersion of the surface into Minkowski space. The geometric invariants of this spacelike immersion are exactly the Laguerre invariants of the original surface. In particular, its area defines a Laguerre invariant functional:

$$\mathcal{L}[X] = \int_\Sigma dA_Y = \int_\Sigma \frac{H^2 - K}{|K|} dA_X.$$

Blaschke realized that studying the critical points of this functional, which he called *$\mathcal{L}$-minimal surfaces* would lead to surfaces having a particularly interesting properties.

In this note we begin by developing the basics surface theory of Laguerre geometry. We take a new geometric approach based on that used in [7] to study conformal geometry. Our main contribution here is to investigate the stability of the critical points of the functional $\mathcal{L}$. The method we use to do this is analogous to that used to study the stability of Willmore surfaces in [6] and, more generally, to study the stability for zero mean curvature surfaces in Lorentzian manifolds [8]. Since the value of the functional $\mathcal{L}$ is given by the area of the $\mathcal{L}$-Gauss map, we study the second variation of area for the spacelike, zero mean curvature immersion Y. One must restrict the class of variations to those arising as variations of Y through $\mathcal{L}$-Gauss maps. We obtain in Theorem 3.2 a necessary and sufficient condition for the stability of $\mathcal{L}$-minimal surfaces. A consequence of this result is that $\mathcal{L}$ minimal surfaces are indeed local minima of the functional $\mathcal{L}$. For the special case of a minimal surface Σ in $\mathbb{E}^3$, we show that if its Gauss map $\nu : \Sigma \to S^2$ is injective then the surface is a stable $\mathcal{L}$-minimal surface.

2 – Preliminaries

Let $\mathcal{S} \approx \mathbb{E}^3 \times S^2$ denote the unit sphere bundle of $\mathbb{E}^3$. Elements of $\mathcal{S}$ will be called *contact elements*. Each contact element (X, ν) corresponds to a null line in Minkowski space via

$$(X, \nu) \longleftrightarrow \{(X - \lambda\nu, \lambda) | \lambda \in \mathbb{R}\}.$$

For $p \in \mathbb{E}^3$ and $r \in \mathbb{R}$ let $\sigma_r(p)$ denote the *oriented* sphere of radius $|r|$ centered at p which is oriented by its outward (respectively inward) pointing normal if $r > 0$ (respectively $r < 0$) holds. We also allow spheres of zero radius and call them (unoriented)*point spheres*. Note that there is a bijection

$$\Phi : \{\text{oriented spheres and point spheres}\} \longrightarrow \mathbb{E}^4_1$$

defined by $\Phi(\sigma_r(p)) = (p, r)$. Note that if $r > 0$ (resp. $r < 0$) then (p, r) is the vertex of the backward (resp. forward) pointing light cone which intersects $\mathbb{E}^3 \approx \{(x, t) \in \mathbb{E}^4_1 | t = 0\}$ in exactly the the sphere $\sigma_r(p)$. From this one can easily see that two spheres $\sigma_r(p), \sigma'_r(p')$ are in oriented contact (i.e. they are tangent to each other and have the same normal at the point of tangency) if and only if

$$\langle (p, r) - (p', r'), (p, r) - (p', r') \rangle_{\mathbb{E}^4_1} = 0$$

i.e. $(p - p', r - r')$ is a null vector. Thus each contact element (X, ν) determines a null line of 2-spheres all of which are in oriented contact at X with unit normal ν.

Let $X : \Sigma \longrightarrow \mathbb{E}^3$ be an immersion of an oriented surface with normal ν. There is a natural lift $f := (X, \nu) : \Sigma \longrightarrow \mathcal{S}$ satisfying the *contact condition* $\langle dX, \nu \rangle \equiv 0$. In general, a smooth map into $\mathcal{S}$ satisfying this condition is called a *Legendre surface*. Clearly the Poincare group $\mathrm{Iso}(\mathbb{E}^4_1)$ permutes the null lines in Minkowski space. Since we have identified $\mathcal{S}$ with the manifold of null lines in $\mathbb{E}^4_1$ we have that $\mathrm{Iso}(\mathbb{E}^4_1)$ acts on $\mathcal{S}$ and we wish to show that this action permutes Legendre surfaces. Let $f = (X, \nu)$ be a Legendre surface and for each $p \in \Sigma$ let $l_f(p)$ denote the null line $\lambda \to (X - \lambda\nu, \lambda)$. Then the contact condition can be expressed $\langle \partial_\lambda l, dl \rangle \equiv 0$ where d is differentiation on Σ. Since this equation is invariant under the Poincare group, the contact condition is preserved.

Let $X : \Sigma \longrightarrow \mathbb{E}^3$ be an oriented immersion with normal ν. We denote its fundamental forms by $I = \langle dX, dX\rangle$, $II = -\langle dX, d\nu\rangle$ and $III = \langle d\nu, d\nu\rangle$. We also let H and K denote the mean and Gauss curvatures respectively. Following [1] we will say that X is *nondegenerate* if II and III are lineraly independent at each point. Because of the classical equation

$$III = 2HII - KI \tag{1}$$

this implies that K is nowhere vanishing. For a nondegenerate X, define the *$\mathcal{L}$-Gauss map*

$$Y : \Sigma \longrightarrow \mathbb{E}^4_1$$

by $Y := (X + \frac{H}{K}\nu, -\frac{H}{K})$. For each point p, $Y(p)$ represents the *middle sphere* of Σ at p which is the sphere of radius $\frac{1}{2}(1/k_1 + 1/k_2)$ having oriented first order contact with the surface at $X(p)$. Compute

$$\langle dY, dY\rangle_{\mathbb{E}^4_1} = \Big\langle dX + \frac{H}{K}d\nu, dX + \frac{H}{K}d\nu\Big\rangle_{\mathbb{E}^3} = I + \Big(\frac{H^2}{K^2}\Big)III - 2\frac{H}{K}II.$$

Using (1), we obtain

$$ds^2_Y = \frac{H^2 - K}{K^2}III = \frac{1}{4}(1/k_1 - 1/k_2)^2 III.$$

Any map $Z : \Sigma \longrightarrow \mathbb{E}^4_1$ can be interpreted as a two parameter family of spheres. Classically such a map was called a *spherical congruence.* A Legendre surface (X, ν) is said to *envelope* Z if for each $p \in \Sigma$, the sphere $\Phi^{-1}(Z(p))$ is in oriented contact with the Legendre surface at $X(p)$. Analytically, this just means that $Y(p)$ lies on the null line $l_f(p) : \lambda \to (X(p) - \lambda\nu(p), \lambda)$.

PROPOSITION 2.1. *Let $f = (X, \nu) : \Sigma \longrightarrow \mathcal{S}$ be the Legendre surface associated to the nondegenerate immersion X. Then the $\mathcal{L}$-Gauss map Y is the unique spherical congruence satisfying*

(i) *f envelopes Y*
(ii) *$Y : (\Sigma, III) \longrightarrow \mathbb{E}^4_1$ is conformal.*

PROOF. We have shown above that Y satisfies (i) and (ii). Any spherical congruence satisfying (i) is of the form $Z = (X, 0) + u(\nu, -1)$ for some smooth function u. However,

$$\langle dZ, dZ\rangle_{\mathbb{E}^4_1} = I - 2uII + u^2 III.$$

For this to be proportional to III at each point, there must follow

$$I - 2uII \sim III = (-K)\Big(I - 2\Big(\frac{H}{K}\Big)II\Big).$$

Since K is nonvanishing on a nondegenerate surface, we have that $u = \frac{H}{K}$ and so $Z = Y$. □

At this point we need to describe more precisely how $\mathrm{Iso}(\mathbb{E}^4_1)$ transforms Legendre surfaces. Let $f = (X, \nu)$ and $l_f(.)$ be as above. For each $g \in \mathrm{Iso}(\mathbb{E}^4_1)$ and each $p \in \Sigma$, $g(l_f(p))$ is a null line which intersects $\mathbb{E}^3$ in a unique point$X'(p)$. Denote the (null) tangent line to $g(l_f(p))$ by $(\nu'(p), -1)$. Then $gf := (X', \nu')$ defines another Legendre surface.

PROPOSITION 2.2. *Let $f = (X, \nu) : \Sigma \longrightarrow \mathcal{S}$ be the Legendre surface associated to the nondegenerate immersion X. Let Y denote its $\mathcal{L}$-Gauss map and let $g \in \mathrm{Iso}(\mathbb{E}^4_1)$. Then gY is the $\mathcal{L}$-Gauss map of gf.*

PROOF. It is clear that gf envelopes gY and it is also clear that the metrics induced be Y and gY agree. By the previous proposition, it is enough to show that the third fundamental forms of f and gf are conformally related.

Let C denote the light cone in $\mathbb{E}^4_1$ and note that C is a (trivial) real line bundle over S^2. For $g \in \mathrm{Iso}(\mathbb{E}^4_1)$, dg defines a map $[dg]$ so that the following diagram commutes.

$$\begin{array}{ccc} C & \xrightarrow{dg} & C \\ \downarrow & & \downarrow \\ S^2 & \xrightarrow{[dg]} & S^2 \end{array}$$

It is well known that for $g \in \mathrm{Iso}(\mathbb{E}^4_1)$ i.e. $dg \in O(3,1)$, $[dg]$ is a *conformal* map of S^2 to itself. Since $\nu' = [dg](\nu)$ holds the result follows. □

Since the $\mathcal{L}$-Gauss map is invariant in the sense described above, we can use its area to define the invariant *Laguerre functional.* If Σ is compact we define

$$\mathcal{L}[X] = \int_\Sigma dA_Y = \int_\Sigma \frac{1}{4}(1/k_1 - 1/k_2)^2 dA_\nu = \int_\Sigma \frac{H^2 - K}{|K|} dA_X,$$

since $dA_\nu = |K| dA_X$. Note that we have taken the liberty here of normalizing the sign of the Laguerre functional to be non-negative. Since the surface is assumed to be non-degenerate there is no problem with doing this. The critical points of $\mathcal{L}$ are called *Laguerre minimal surfaces.* They are characterized by the property that Y has zero mean curvature in Minkowski space (see [1] or [4]).

3 – Stability of $\mathcal{L}$-minimal surfaces

Let $X : \Sigma \longrightarrow \mathbb{E}^3$ be an immersion of an oriented surface and let $\nu : \Sigma \longrightarrow S^2$ be its Gauss map. We will assume the immersion is nondegenerate in the sense of [1].

Note that nondegeneracy implies that

$$(k_1 - k_2)^2 > 0 \tag{2}$$

holds where $k_j; j = 1, 2$ denote the principal curvatures. Since

$$ds_Y^2 = (1/k_1 - 1/k_2)^2 III \tag{3}$$

we see that the $\mathcal{L}$-Gauss map Y is then a spacelike immersion into $\mathbb{E}^4_1$. Using formula (109), page 318 of [4], one sees that the mean curvature field $\mathcal{H}$ of Y is given by

$$2\mathcal{H} = \Big(\Delta_Y \frac{H}{K}\Big)(\nu, -1), \tag{4}$$

where Δ_Y denotes the Laplacian of the metric ds_Y^2. In particular, for nondegenerate surfaces satisfying (2), the mean curvature field is *isotropic*

$$\langle \mathcal{H}, \mathcal{H} \rangle \equiv 0.$$

Let X be a nondegenerate surface satisfying (3) which is a critical point of $\mathcal{L}$, i.e.

$$\delta\mathcal{L}[X] = 0.$$

This condition is equivalent to the equation $\Delta_Y(H/K) = 0$, [1]. Let X_ϵ be a smooth variation of X in $\mathbb{E}^3$ and let Y_ϵ denote the corresponding $\mathcal{L}$-Gauss maps. Let $\xi := (\partial_\epsilon(Y_\epsilon)_{\epsilon=0})^\perp$ where the superscript $\perp$ denotes projection to the normal bundle of Y. Then with the obvious notation, we have

$$2\partial_\epsilon(\mathcal{H}_\epsilon) = J[\xi] \tag{5}$$

where J denotes the Jacobi operator for the critical immersion Y. On the other hand

$$\begin{aligned} 2\partial_\epsilon(\mathcal{H}_\epsilon)_{\epsilon=0} &= \Big(\partial_\epsilon\Big(\Delta_Y\frac{H}{K}\Big)\Big)(\nu,-1) + \Big(\Delta_Y\frac{H}{K}\Big)\partial_\epsilon(\nu_\epsilon,-1) = \\ &= \Big(\partial_\epsilon\Big(\Delta_Y\frac{H}{K}\Big)\Big)(\nu,-1) \end{aligned} \tag{6}$$

since X is critical. This shows that

$$\langle J[\xi], (\nu,-1)\rangle \equiv 0 \tag{7}$$

for any variation of Y through $\mathcal{L}$-Gauss maps. We will now prove the converse, under the above conditions on X, that is if a section ξ of the normal bundle of Y satisfies (7) then there exists a variation X_ϵ of X such that the corresponding $\mathcal{L}$-Gauss maps satisfy $\xi := (\partial_\epsilon(Y_\epsilon)_{\epsilon=0})^\perp$.

Since Y is spacelike, there is a section q of the normal bundle $T^\perp(Y)$ such that $\langle q,q\rangle \equiv 0$ and $\langle q,\underline{\nu}\rangle \equiv 1$ hold, where $\underline{\nu} := (\nu,-1)$. Define a quadratic form $\mathcal{B}$ on $T^\perp(Y)$ as follows. For any $u, v \in T_p^\perp(Y)$, let $\tilde{u}$ and $\tilde{v}$ be any locally defined, smooth extensions to sections of $T^\perp(Y)$. Let D denote covariant differentiation on $\mathbb{E}^4_1$ and let D^T denote covariant differentiation followed by projection to $dY(T\Sigma)$. Then

$$\mathcal{B}(u,v) := \langle D^T\tilde{u}, D^T\tilde{v}\rangle = \sum_i \langle D^T_{e_i}\tilde{u}, D^T_{e_i}\tilde{v}\rangle$$

for any orthonormal frame $\{e_i\}_{i=1,2}$. The definition is independent of the extension. Then we see that $\mathcal{B}(\underline{\nu},\underline{\nu}) > 0$ holds since the immersion is nondegenerate. We then change frame in the normal bundle by defining

$$a := (\mathcal{B}(\underline{\nu}, \underline{\nu}))^{-1/2}\underline{\nu}, \qquad b := (\mathcal{B}(\underline{\nu}, \underline{\nu}))^{1/2}q.$$

Note that $\mathcal{B}(a, a) \equiv 1$ holds. We now write an arbitrary smooth section of the normal bundle as $\xi =: \sigma a + \tau b$. Following eq. (10) of [8], we see that the condition $\langle J[\xi], \underline{\nu}\rangle \equiv 0$ allows us to write

$$\sigma = -\Lambda[\tau]$$

for a particular second order elliptic operator Λ. Let $\tilde{\tau} := \tau\mathcal{B}(\underline{\nu}, \underline{\nu}))^{1/2}$ and consider the variation of X given by $X_\epsilon := X + \epsilon\tilde{\tau}\nu$. Write the $\mathcal{L}$-Gauss map of X_ϵ as $Y_\epsilon =: (X_\epsilon + \lambda_\epsilon\nu_\epsilon, -\lambda_\epsilon) = (X_\epsilon, 0) + \lambda_\epsilon\underline{\nu}_\epsilon$. We have

$$\langle(\partial_\epsilon Y_\epsilon)_{\epsilon=0}, \underline{\nu}\rangle = (\partial_\epsilon X_\epsilon)_{\epsilon=0}, \nu\rangle = \tilde{\tau}$$

and hence $((\partial_\epsilon Y_\epsilon)_{\epsilon=0})^\perp =: \alpha a + \tau b$ for some function α. Obviously Y_ϵ is a variation of Y through $\mathcal{L}$-Gauss maps and hence must satisfy (7) or equivalently $\langle J[(\partial_\epsilon Y_\epsilon)_{\epsilon=0})^\perp], a\rangle \equiv 0$. Consequently, we have $\alpha = -\Lambda[\tau] = \sigma$ and therefore $(\partial_\epsilon Y_\epsilon)_{\epsilon=0})^\perp = \xi$ as desired. We have shown:

PROPOSITION 3.1. *Let X be as above, let Y be $\mathcal{L}$-Gauss map of X, and let ξ be a section of the normal bundle of Y. Then the necessary and sufficient condition that there exists a variation Y_ϵ of Y through $\mathcal{L}$-Gauss maps such that $(\partial_\epsilon Y_\epsilon)_{\epsilon=0})^\perp = \xi$ is that $\langle J[\xi], \underline{\nu}\rangle \equiv 0$ holds.*

As a consequence, we have the following.

THEOREM 3.2. *Let $X : \Sigma \longrightarrow \mathbb{E}^3$ be a nondegenerate critical point of $\mathcal{L}$. Then X is a stable critical point of $\mathcal{L}$ if and only if*

$$-\int_\Sigma \langle \xi, J[\xi]\rangle dA_Y \geq 0$$

holds for every smooth, compactly supported section of $T^\perp(Y)$ satisfying $\langle J[\xi], \underline{\nu}\rangle \equiv 0$.

The following result shows that critical points of $\mathcal{L}$ do indeed locally minimize the functional. This is really not at all obvious if we consider the fact that $\mathcal{L}$ is given by the area of a surface in a Lorentzian 4-manifold.

COROLLARY 3.3. *Let $X : \Sigma \longrightarrow \mathbb{E}^3$ be a nondegenerate critical point of $\mathcal{L}$. Then each point $p \in \Sigma$ has a neighborhood U on which X minimizes $\mathcal{L}$ among all immersions agreeing with X near ∂U.*

PROOF. The result follows from the previous theorem and the main result of [8]. There it is shown that a spacelike zero mean curvature surface in a Lorentzian 4-manifold satisfying the null convergence condition, $\mathrm{Ric}(N, N) \geq 0$ for all null tangent vectors N, locally minimizes area among all nearby surfaces whose mean curvature is isotropic (null). Since $\mathbb{E}^4_1$ is Einstein, the theorem applies here. $\square$

4 – Minimal surfaces

We apply the preceding to the case of an oriented minimal immersion $X : \Sigma \longrightarrow \mathbb{E}^3$. Note that by (4), any nondegenerate minimal surface is also $\mathcal{L}$-minimal. In this case Y is given by $Y = (X, 0)$. The vectors

$$a := (\nu, -1)/\sqrt{2}, \qquad b := (\nu, +1)/\sqrt{2}$$

define a null framing of the normal bundle $T^\perp(Y)$ of Y in $\mathbb{E}^4_1$. Let A denote the endomorphism field of $T^\perp(Y)$ defined by $\mathcal{B}(u, v) =: \langle Au, v\rangle$. Note that

$$\langle D^T a, D^T a\rangle = \langle D^T a, D^T b\rangle = \langle D^T b, D^T b\rangle = |d\nu|^2/2 = -K,$$

from which it follows that

$$Aa = -Ka - Kb, \quad Ab = -Ka - Kb$$

hold. Let $D^\perp$ denote covariant differentiation in $T^\perp(Y)$. The Jacobi operator is then given by (see [10])

$$J[\xi] = \Delta^\perp + A$$

where

$$\Delta^\perp := \sum_{j=1,2} (D^\perp_{e_j} D^\perp_{e_j} - D^\perp_{\nabla_{e_j} e_j})$$

is the rough Laplacian in $T^\perp(Y)$. Let ξ be a section of $T^\perp(Y)$. Then writing $\xi =: \sigma a + \tau b$ and using that a and b are parallel in $T^\perp(Y)$, we see that $\Delta^\perp \xi = (\Delta\sigma)a + (\Delta\tau)b$, and so

$$J[\xi] = (\Delta\sigma - K\sigma - K\tau)a + (\Delta\tau - K\sigma - K\tau)b.$$

Therefore the condition $\langle J[\xi], \underline{\nu}\rangle \equiv 0$ reduces to

$$\Delta\tau - K\sigma - K\tau \equiv 0. \tag{8}$$

Now the nondegeneracy assumption for X implies that K is nonvanishing on Σ and so we can make the conformal change of metric $d\tilde{s}^2 = -K ds^2$. Then (8) becomes $\tilde{\Delta}\tau + \sigma + \tau \equiv 0$ and we have

$$\begin{aligned}
-\int_\Sigma \langle J[\xi], \xi\rangle dA &= -\int_\Sigma \langle \xi, a\rangle \langle J[\xi], b\rangle dA = \\
&= -\int_\Sigma \tau(\Delta\sigma - K\sigma - K\tau) dA = \\
&= -\int_\Sigma \tau(\tilde{\Delta}\sigma + \sigma + \tau) d\tilde{A} = \\
&= \int_\Sigma \tau((\tilde{\Delta} + 1)(\tilde{\Delta} + 1)\tau - \tau) d\tilde{A} = \\
&= \int_\Sigma \tau(\tilde{\Delta}(\tilde{\Delta} + 2)\tau) d\tilde{A}.
\end{aligned}$$

We obtain the following.

THEOREM 4.1. *Let $X : \Sigma \longrightarrow \mathbb{E}^3$ be a minimal immersion of an oriented surface with injective Gauss map. Then the surface is a stable $\mathcal{L}$-minimal surface.*

Proof. Since the Gauss map is injective the immersion is nondegenerate. The necessary and sufficient condition of the previous theorem will be satified if we succeed in showing that

$$0 \leq \int_\Sigma u(\tilde{\Delta}(\tilde{\Delta}+2)u)d\tilde{A} = \int_\Sigma (\tilde{\Delta}u)^2 - 2|\tilde{\nabla}u|^2 d\tilde{A}.$$

holds for all compactly supported functions u. Since the metric $d\tilde{s}^2$ is the pull back of the metric on S^2 under ν, we may identify $(\Sigma, d\tilde{s}^2)$ with a subset of S^2.

If u is any smooth function with compact support in some open set $\Omega \in S^2$, then we can extend u to be identically zero outside Ω. Denote this extension again by u and expand u in terms of eigenfunctions of the Laplacian on S^2,

$$u = \sum a_i f_i$$

where $\Delta_{S^2} f_i = -\lambda_i f_i$ and $\{f_i\}_1^\infty$ is a complete orthonormal set in $L^2(S^2)$. Then

$$\int (\Delta_{S^2}u)^2 - 2\int |\nabla_{S^2}u|^2 = \sum (a_i^2(\lambda_i(\lambda_i - 2)) \geq 0$$

since the Laplacian on S^2 has no eigenvalues in the interval $(0,2)$. ◻

Enneper's surface and the catenoid are examples of *complete* minimal surfaces in $\mathbb{E}^3$ with injective Gauss map (although no completeness assumption is required in the theorem). Note that both these surfaces are unstable as minimal surfaces in $\mathbb{E}^3$.

5 – Laguerre curvature

Two obvious Laguerre invariants of a nondegenerate immersion X are the curvature of the $\mathcal{L}$-Gauss map and the curvature of its normal bundle. We will denote them by $K_\mathcal{L}$ and $K_\mathcal{L}^\perp$ respectively. In order to state the next result we recall that a 2-dimensional Riemannian manifold is called parabolic if it does not support any nonconstant subharmonic function which is bounded above. In dimension two, parabolicity only depends on the conformal structure of the surface. In particular, any closed surface with a finite number of points removed is parabolic.

Theorem 5.1. *Let* $X : \Sigma \longrightarrow \mathbb{E}^3$ *be a nondegenerate* $\mathcal{L}$*-minimal surface. Assume that* (Σ, III) *is parabolic and that* $K_\mathcal{L} \geq 0$ *holds. Then*

$K_{\mathcal{L}} \equiv 0 \equiv K_{\mathcal{L}}^{\perp}$ holds and all the middle spheres of X are tangent to a fixed plane.

PROOF. It was shown in [9] that If X is a spacelike zero mean curvature conformal immersion of a parabolic surface into $\mathbb{E}_1^4$ with non-negative curvature K, then $K \equiv 0$ holds and the normal bundle is flat. Further, the surface must lie in a null hyperplane. We may assume therefore, by first making a Lorentz transformation, that

$$0 \equiv \langle Y, (1,0,0,1) \rangle_{\mathbb{E}_1^4} = y_1 - y_4 \tag{10}$$

holds. Recall that $Y = (m, r) \in \mathbb{E}^3 \times \mathbb{R}$ represents the 2-sphere in $\mathbb{E}^3$ of radius r centered at $m = (m_1, m_2, m_3)$. Hence a result of (10) implies that the radius is given by $|m_1|$ and so all of the middle spheres are tangent to the plane $\{m_1 = 0\} \subset \mathbb{E}^3$. $\square$

The hypothesis that (Σ, III) is parabolic is a natural one. As mentioned above, in two dimensions parabolicity is a conformal invariant and the conformal class of III is a Laguerre invariant by Proposition 2.2. Also the hypothesis of parabolicity is essential since there exist zero mean curvature surfaces in $\mathbb{E}_1^4$ which have positive curvature and which are conformal to a disc.

The surfaces appearing in the conclusion of the theorem were studied by Blaschke ([4], p. 375).

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