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A remark on the range of differential operators in spaces of flat functions

C. GRAMMATICO – D. GUIDETTI

RIASSUNTO: *In questo lavoro si studiano le proprietà dell'immagine di certi operatori differenziali nello spazio delle funzioni piatte in zero. Il risultato principale è il seguente: se P e Q sono due operatori differenziali lineari, se P gode della proprietà di continuazione unica forte in ogni punto e se l'immagine di Q è contenuta nell'immagine di P, allora esiste un operatore differenziale R tale che risulti: $Q = P \circ R$. Si discutono alcune conseguenze di questo risultato.*

ABSTRACT: *Properties of the range of differential operators in spaces of flat functions are considered; the main result is the following: if P and Q are linear partial differential operators, P has the strong unique continuation property and the range of Q is contained in the range of P, then $Q = P \circ R$, where R is another linear partial differential operator. Some consequences of the main result are discussed.*

1 – Introduction

In paper [1] (Remark 5) ALINHAC and BAOUENDI show that the Laplace operator is not surjective in spaces of flat functions in B_r where $B_r = \{x \in \mathbb{R}^2 |\ |x| < r\}$.

In this note we want to give a simple general result, explaining the mentioned nonsurjectivity. The crucial tool is the notion of differential operator with the strong unique continuation property (see Defini-

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tion 2.2). We show (Theorem 2.1) that, if the dimension is at least 2, P and Q are linear partial differential operators and P has the strong unique continuation property, then the range of Q is contained in the range of P if and only if $Q = P \circ A$ where A is another linear partial differential operator. From this it is possible to characterize partial differential operators with the strong unique continuation property which are onto (Corollary 2.1).

We introduce now some notations we shall use in the following: if $r > 0$ and n is a positive integer, we set $B_r := \{x \in \mathbb{R}^n \mid |x| < r\}$. If α is the multiindex $(\alpha_1, \dots, \alpha_n)$ we set $D^\alpha := (-i)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$. The space $C_\flat^\infty(B_r)$ of flat functions in B_r is, by definition, $\{v \in C^\infty(B_r)$ s.t. $D^\alpha v(0) = 0$ $\forall\, \alpha \in \mathbb{N}^n\}$, where $\mathbb{N}$ is the set of nonnegative integers. $C_\flat^\infty(B_r)$ is a closed subspace of $C^\infty(B_r)$ and inherits therefore the structure of Frechèt space. In general, given $v \in C^\infty(B_r)$, we shall say that v is flat in $x^0 (\in B_r)$ if $D^\alpha v(x^0) = 0$ $\forall\, \alpha \in \mathbb{N}^n$. It is well known that the dual space of $C^\infty(B_r)$ can be identified with the space $\mathcal{E}'(B_r)$ of distributions in B_r with compact support. It is not difficult to show that the dual space of $C_\flat^\infty(B_r)$ can be identified with the quotient space $\frac{\mathcal{E}'(B_r)}{\mathcal{E}'_0}$, where $\mathcal{E}'_0$ is the set of distribution with support in $\{0\}$; if $v \in C_\flat^\infty(B_r)$, $\varphi \in \mathcal{E}'(B_r)$ and $[\varphi]$ is its equivalence class in $\frac{\mathcal{E}'(B_r)}{\mathcal{E}'_0}$, we set

$$\langle v, [\varphi] \rangle = \langle iv, \varphi \rangle, \tag{1.1}$$

where i is the injection of $C_\flat^\infty(B_r)$ into $C^\infty(B_r)$.

If $x_0 \in B_r$, we shall use the notation $\mathcal{E}'_{x_0}$ to indicate the class of distributions with support in $\{x_0\}$ and $\left[\mathcal{E}'_{x_0}\right]$ to indicate $\{[\varphi] \mid \varphi \in \mathcal{E}'_{x_0}\}$.

If P is an operator, we denote with $R(P)$ its range. In case P is a linear differential operator, $\operatorname{ord} P$ stands for its order. If v is a function or a distribution in B_r, $\operatorname{supp} v$ indicates its support.

2 – Results

We begin by characterizing the class of multipliers of $C_\flat^\infty(B_r)$:

Definition 2.1. Let $r > 0$. We set

$$M_r := \Big\{ a \in C^\infty (B_r \setminus \{0\}) \Big| \, \forall \alpha \in \mathbb{N}^n , \exists \, m(\alpha) \in \mathbb{Z} \quad \text{s.t.} \ \limsup_{x \to 0} |x|^{m(\alpha)} |D^\alpha a(x)| < +\infty \Big\} \tag{2.2}$$

Lemma 2.1. *Let* $A := \sum_{|\alpha| \le m} a_\alpha (x) D^\alpha$ *be a linear differential operator, with coefficients* a_α *with domain* $B_r \setminus \{0\}$. *Then* $A (C^\infty_\flat (B_r)) \subseteq C^\infty_\flat (B_r)$ *if and only if* a_α *belongs to* M_r *for every* α.

Proof. The "if"-part is clear. We prove the "only if"-part.

Let $j = \min \{ |\alpha| \text{ s.t. } a_\alpha \not\equiv 0 \}$. Clearly it is sufficient to show that $a_\alpha \in M_r$ if $|\alpha| = j$.

Let $\chi \in C^\infty_0 (B_r)$ be such that $\operatorname{supp} \chi \subseteq \Big\{ x \in \mathbb{R}^n \Big| \, \frac{r}{8} \le |x| \le \frac{7r}{8} \Big\}$ and $\chi(x) = 1$ if $\frac{r}{4} \le |x| \le \frac{3r}{4}$. For $k \in \mathbb{N}$, $|\alpha| = j$ we set $\chi_{k,\alpha}(x) = x^\alpha \chi (2^k x)$ so that $\operatorname{supp} \chi_{k,\alpha} \subseteq \{ x \in \mathbb{R}^n | \quad 2^{-k-3} r \le |x| \le 7 \cdot 2^{-k-3} r \}$.

Let $\beta \in \mathbb{N}^n$. For $v \in C^\infty_\flat (B_r)$ we put

$$p_\beta (v) = \max_{|x| \le \frac{7r}{8}} \left| D^\beta v(x) \right| . \tag{2.3}$$

Then p_β is a continuous seminorm in $C^\infty_\flat (B_r)$. As A is a linear continuous operator in $C^\infty_\flat (B_r)$ (which can be easily verified applying the closed graph theorem), there exists a continuous seminorm q_β in $C^\infty_\flat (B_r)$ such that

$$p_\beta (Av) \le q_\beta (v) \qquad \forall \, v \in C^\infty_\flat (B_r) \, . \tag{2.4}$$

If $q_\beta (v) \le C \max\limits_{\substack{|\gamma| \le \mu \\ x \in H}} |D^\gamma v(x)|$ for certain $C > 0$, $\mu \in \mathbb{N}$, H compact subset of B_r for every $v \in C^\infty_\flat (B_r)$, we have:

$$\begin{aligned} \max_{2^{-k-2} r \le |x| \le 3 \cdot 2^{-k-2} r} \left| D^\beta a_\alpha (x) \right| &\le \frac{p_\beta (A \chi_{k,\alpha})}{\alpha!} \le \\ &\le C \max_{\substack{|\gamma| \le \mu \\ x \in H}} |D^\gamma \chi_{k,\alpha} (x)| \le C' 2^{k\mu} , \end{aligned}$$

where C' is positive and independent of k. It follows, for $|x| \leq \frac{3r}{8}$,

$$|D^{\beta} a_{\alpha}(x)| \leq C' \left(\frac{3r}{4}\right)^{\mu} |x|^{-\mu} . \tag{2.5}$$ □

DEFINITION 2.2. Let P be a linear partial differential operator with coefficients in M_r; we shall say that P has the strong unique continuation property (SUC property) in B_r if for every $u \in C_\flat^\infty(B_r)$, for every $x_0 \in B_r$, if u is flat in x_0 and Pu vanishes in a neighbourhood of x_0, then u also vanishes in a neighbourhood of x_0.

REMARK 2.1. Examples of operators satisfying the SUC property are hypoelliptic operators of principal type with analytic coefficients (see [4]), elliptic operators of second order with real principal part (see [1], Corollary 2); other examples are given in Lerner's paper (see [3]).

Observe also that, if $a \in M_r$ and $\{x \in B_r \setminus \{0\} \mid a(x) = 0\}$ has empty interior, the operator of multiplication by a has the SUC property. Finally the composition of operators with the SUC property has the SUC property.

We pass now to the main result of the paper:

THEOREM 2.1. *Let P and Q be linear differential operators with coefficients in M_r; assume that $n \geq 2$ and P has the strong unique continuation property. Consider P and Q as linear continuous operators in the space $C_\flat^\infty(B_r)$. Then, the following conditions are equivalent:*

1. $R(Q) \subseteq R(P)$
2. *there exists a linear partial differential operator A with coefficients in M_r such that* $\mathrm{ord}(A) \leq \mathrm{ord}(Q)$ *and* $Q = P \circ A$.

PROOF. It is obvious that 2 implies 1. We show that 1 implies 2.

First of all, we remark that, as a consequence of the SUC property, P is injective (if $Pv \equiv 0$ then $\{x \in B_r \mid v\ is\ flat\ in\ x\}$ is open, closed, not empty in B_r). It follows from 1 that the operator $A := P^{-1}Q$ is well defined in $C_\flat^\infty(B_r)$. Using the closed graph theorem, it is not difficult to verify that A is continuous. We want to show that A is a linear partial differential operator.

We start by proving that, if $x_0 \in B_r \setminus \{0\}$, $0 < \rho < \min\{|x_0|, r - |x_0|\}$, $v \in C^\infty_\flat(B_r)$ and $\operatorname{supp} v \subseteq \overline{B_\rho(x_0)}$, then $\operatorname{supp} Av \subseteq \overline{B_\rho(x_0)}$. In fact if $u := Av$, we have $Pu = Qv$.

As Qv vanishes in $B_r \setminus \overline{B_\rho(x_0)}$ and P has the SUC property, the set $\{x \in B_r \setminus \overline{B_\rho(x_0)} | \ u \text{ is flat in } x\}$ is open, closed and not empty. As this set is connected, u is flat at every point of $B_r \setminus \overline{B_\rho(x_0)}$.

As a consequence, for every $x_0 \in B_r \setminus \{0\}$ we have

$$A^t\left(\left[\mathcal{E}'_{x_0}\right]\right) \subseteq \left[\mathcal{E}'_{x_0}\right]$$

where A^t is the transpose operator of A.

In fact, let $x_1 \in B_r \setminus \{0\}$ $x_1 \neq x_0$ and let $\varphi \in \mathcal{E}'_{x_0}$, we choose $\rho > 0$ such that $\rho < \min\{|x_1|, r - |x_1|, |x_1 - x_0|\}$; if $v \in C^\infty_\flat(B_r)$ and $\operatorname{supp} v \subseteq \overline{B_\rho(x_1)}$, we have

$$\langle v, A^t[\varphi]\rangle = \langle iAv, \varphi\rangle = 0 \tag{2.6}$$

as $x_0 \notin \operatorname{supp} Av$. It follows that for every $v \in C^\infty_\flat(B_r)$, $x \in B_r \setminus \{0\}$

$$Av(x) = \langle v, A^t[\delta_x]\rangle = \sum_{|\alpha| \le m(x)} a_\alpha(x) D^\alpha v(x) . \tag{2.7}$$

Now we show that $a_\alpha \in C^\infty(B_r \setminus \{0\})$ for every $\alpha \in \mathbb{N}^n$. This can be seen applying the operator A to the elements of $C^\infty_\flat(B_r)$ which are locally equal to $1, x_j\ (1 \le j \le n), \dots, x^\alpha, \dots$. Next we show that there exists $m \in \mathbb{N}$ such that $a_\alpha \equiv 0$ if $|\alpha| > m$. We start by proving that, if $x_0 \in B_r \setminus \{0\}$ and $0 < \rho < \min\{|x_0|, r - |x_0|\}$, there exists $m(x_0, \rho)$ such that $a_\alpha(x) = 0$ for $|\alpha| > m(x_0, \rho)$ and $x \in B_\rho(x_0)$. In fact, as A is continuous, there exist a compact subset H of B_r , a positive constant M and a nonnegative integer $m(x_0, \rho)$ such that for every $u \in C^\infty_\flat(B_r)$

$$\max_{x \in \overline{B_\rho(x_0)}} |Au(x)| \le M \max_{\substack{x \in H \\ |\alpha| \le m(x_0,\rho)}} |D^\alpha u(x)| . \tag{2.8}$$

If $v \in C^\infty_\flat(B_r)$ and $\operatorname{supp} v \subseteq \overline{B_\rho(x_0)}$, also $\operatorname{supp} Av \subseteq \overline{B_\rho(x_0)}$. So

$$\max_{x \in \overline{B_\rho(x_0)}} |Av(x)| \le M \max_{\substack{x \in \overline{B_\rho(x_0)} \\ |\alpha| \le m(x_0,\rho)}} |D^\alpha v(x)| . \tag{2.9}$$

This inequality implies that a_α vanishes in $B_\rho(x_0)$ if $|\alpha| > m(x_0, \rho)$.

We show now that $m(x) \leq q \quad \forall x \in B_r \setminus \{0\}$, where $q = \operatorname{ord} Q$. In fact let $x_0 \in B_r \setminus \{0\}$ and $\rho > 0$, $\rho < \min\{|x_0|, r - |x_0|\}$. We observe that there exists $x \in B_\rho(x_0)$ such that $A_{m(x_0,\rho)}(x,\cdot) \not\equiv 0$ and $P(x,\cdot) \not\equiv 0$. To see this, choose $x \in B_\rho(x_0)$ such that $A_{m(x_0,\rho)}(x,\cdot) \not\equiv 0$. By continuity, $\exists\, \delta > 0$ s.t. $A_{m(x_0,\rho)}(y,\cdot) \not\equiv 0 \quad \forall y \in B_\delta(x)$.

If $P(y,\cdot) \equiv 0 \quad \forall y \in B_\delta(x)$, P cannot have the strong unique continuation property.

We indicate with $P^\sharp(x,D)$ the principal part of $P(x,D)$. Then

$$P^\sharp(x,\cdot) \not\equiv 0 \quad \text{and} \quad \operatorname{ord}\left(P^\sharp(x,\cdot)\right) \geq 0\,.$$

Let $\xi \in \mathbb{R}^n$ be such that $A_{m(x_0,\rho)}(x,\xi) \neq 0$. We can choose ξ in such a way that also $P^\sharp(x,\xi) \neq 0$. In fact, $A_{m(x_0,\rho)}(x,\eta) \neq 0$ for η sufficiently nearby ξ. As $P^\sharp(x,\cdot)$ cannot vanish identically in an open set, there certainly exists η s.t. $P^\sharp(x,\eta) \neq 0$.

Let now $t \in \mathbb{R}$ and $v_t \in C_\flat^\infty(B_r)$ such that

$$v_t(y) = e^{iy\cdot t\xi}$$

in some neighbourhood of x. Then, for $t \to \infty$

$$Qv_t(x) = \mathcal{O}(t^q) \tag{2.10}$$

and

$$\begin{aligned} P(Av_t)(x) = P^\sharp(x,\xi)\, A_{m(x_0,\rho)}(x,\xi)\, t^{m(x_0,\rho)+\operatorname{ord}P^\sharp(x,\xi)} &+ \\ + \mathcal{O}\left(t^{m(x_0,\rho)+\operatorname{ord}P^\sharp(x,\xi)-1}\right)&\,. \end{aligned} \tag{2.11}$$

It follows

$$m(x_0,\rho) \leq m(x_0,\rho) + \operatorname{ord}P^\sharp(x,\cdot) \leq q\,. \tag{2.12}$$

So, for any $x \neq 0$

$$Av(x) = \sum_{|\alpha|\leq q} a_\alpha(x)\, D^\alpha v(x) \qquad \forall v \in C_\flat^\infty(B_r) \tag{2.13}$$

and the result follows from Lemma 2.1. ∎

Corollary 2.1. *Let* $n \geq 2$. *Let* P *be an operator with coefficients in* M_r *with the strong unique continuation property. Then, the following statements are equivalent:*

1. P *is onto* $C_\flat^\infty(B_r)$
2. $\forall\, v \in C_\flat^\infty(B_r) \quad Pv = av$ *with* a *and* $a^{-1} \in M_r$.

Proof. It is obvious that $2 \Rightarrow 1$.

To show the converse, observe that from Theorem 2.1 there exists $b \in M_r$ such that

$$v = P(bv) \qquad \forall\, v \in C_\flat^\infty(B_r)\,.$$

As $P : C_\flat^\infty(B_r) \longrightarrow C_\flat^\infty(B_r)$ is bijective, necessarily the multiplication operator by b is onto $C_\flat^\infty(B_r)$. This implies that $b(x_0) \neq 0 \quad \forall\, x_0 \in B_r \setminus \{0\}$ (otherwise, if $b(x_0) = 0$ for some $x_0 \in B_r$, $b(x_0)\, v(x_0) = 0 \quad \forall\, v \in C_\flat^\infty(B_r)$).

So necessarily $Pu = b^{-1}u \quad \forall\, u \in C_\flat^\infty(B_r)$. Set $a := b^{-1}$; then the result follows from Lemma 2.1. ∎

Remark 2.3. The result stated in Corollary 2.1 is false if P has not the strong unique continuation property.

For example, for $n = 2$ the operator in B_r

$$Pu(x,y) = x\frac{\partial u}{\partial x}(x,y) + y\frac{\partial u}{\partial y}(x,y) \tag{2.14}$$

is a bijection of $C_\flat^\infty(B_r)$ onto itself (as can be seen passing to polar coordinates).

Remark 2.3. An easy consequence of Corollary 2.1 is the fact that the Laplace operator is not surjective in $C_\flat^\infty(B_r)$, as clearly it does not exist $a \in M_r$ such that $\Delta u = au$ for every $u \in C_\flat^\infty(B_r)$. In fact, taking $v \in C_\flat^\infty(B_r), v \equiv 1$ in an open subset U of $B_r \setminus \{0\}$ one gets $a \equiv 0$ in U, so that, as U is arbitrary, $a \equiv 0$ but this means that the Laplace operator vanishes identically in $C_\flat^\infty(B_r)$, which is obviously false. This is an alternative proof of the result of Alinhac and Baouendi mentioned in introduction.

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Indirizzo degli Autori:

Cataldo Grammatico – Davide Guidetti – Dipartimento di Matematica Università – Piazza di Porta S.Donato 5 – 40126 Bologna, Italia
E-mail: grammati@dm.unibo.it – guidetti@dm.unibo.it