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Inverse and Direct Problem of the Dynamics of Central Motions

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RIASSUNTO: *Si studia il problema inverso della Dinamica (nel senso di Szebehely) e si stabilisce per esso un'equazione lineare del primo ordine alle derivate parziali in cui la funzione incognita è la componente radiale $F = F(r,\theta)$ di un campo di forze centrali (generalmente non conservativo) capace di generare, come orbite, le curve di un'assegnata famiglia monoparametrica. Si mostra che tale equazione è utile anche per il problema diretto ed in tal caso è interpretabile come un'equazione non lineare del secondo ordine del tipo di Monge-Ampère. Si ottiene inoltre una condizione necessaria e sufficiente affinché la data famiglia di orbite possa essere creata da un campo di forze conservativo $F = F(r)$. Inoltre si determina come la forza $F = F(r)$, il momento angolare e l'energia totale dipendano dalla assegnata famiglia di orbite.*

ABSTRACT: *For a given monoparametric family of orbits, a Szebehely-type inverse problem is solved i.e. a linear partial differential equation of the first order is written giving the radial component $F = F(r,\theta)$ of a central force (in general not conservative) creating the family. It is shown how this equation can be used also for direct problem considerations and that, in this case, it reads as second order nonlinear partial differential equation of the Monge-Ampère type. The equation is also used to provide conditions so that a preassigned monoparametric or two-parametric family of orbits can be generated by a conservative central force $F = F(r)$. The force $F(r)$ as well as the expressions for the angular momentum and the total energy dependence on the given family are found.*

KEY WORDS AND PHRASES: *Inverse Szebehely-type Problem of Dynamics – Central Motions – Equation of Monge-Ampère type – Monoparametric and two-parametric families of curves.*
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1 – Introduction

A simple version of the inverse problem of Dynamics referring to the motion in the xy plane of one material point P of unit mass is formulated as follows: *A monoparametric family of curves*

$$f(x, y) = c \tag{1}$$

is given in advance and required are all potentials $V = V(x, y)$ *which can allow for the creation of orbits with equation* (1) *traced by the point* P.

For a preassigned dependence

$$E = E(f(x, y)) = E(c) \tag{2}$$

of the total energy E of the moving point P on each specific orbit corresponding to constant c, these potentials are given as solutions of Szebehely's first order linear partial differential equation

$$V_x + \gamma V_y + \frac{2\Gamma}{1+\gamma^2}(E - V) = 0\,, \tag{3}$$

where

$$\gamma = \frac{f_y}{f_x}, \quad \Gamma = \gamma\gamma_x - \gamma_y\,. \tag{4}$$

Lettered subscripts, throughout this paper, denote partial differentiation with respect to the pertinent variables. The derivation of equation (3) was offerred by Szebehely [14] and later by Broucke and Lass [7], Puel [11] and, with the notation used here, by Bozis [2].

A free of energy second order partial differential equation, relating merely orbits and potentials was obtained from equation (3), in view of (2) also, by requiring that $E_y = \gamma E_x$, (Bozis [3]). This equation reads

$$V_{xx} + kV_{xy} - V_{yy} + \lambda V_x + \mu V_y = 0\,, \tag{5}$$

with

$$k = \frac{\gamma^2 - 1}{\gamma}, \quad \lambda = \frac{\Gamma_y - \gamma\Gamma_x}{\gamma\Gamma}, \quad \mu = \lambda\gamma + \frac{3\Gamma}{\gamma}\,. \tag{6}$$

On the other hand Whittaker [16] defines *central forces* as forces (not necessarily conservative) acting on a particle and *"directed towards* *or from a fixed centre"* that can be assumed as origin of the cartesian coordinate system. He then proceeds to put the question of obtaining the *magnitude* of the central force which can create *the single orbit*

$$f(x,y)=0\,, \tag{7}$$

traced with a (constant) angular momentum value L. Whittaker, comes up with the result

$$F(x,y)=-L^2 r\frac{f_y^2 f_{xx}-2f_x f_y f_{xy}+f_x^2 f_{yy}}{(xf_x+yf_y)^3} \tag{8}$$

where $r=(x^2+y^2)^{\frac{1}{2}}$. Actually Whittaker wrote this formula without the sign minus because he considered the magnitude $|\overrightarrow{F}|$ of an attractive force. Our formula (8), instead of the magnitude, give the *radial component* $F(x,y)=\pm|\overrightarrow{F}|$ of the central force and take into account both repulsive or attractive forces.

It is understood that the single orbit (7) can be classified as a member of various monoparametric families (1) and, in this sense, different force fields (8) may be obtained, all creating the same orbit (7) traced with the same value of L (Bozis and Blaga, [5]). Besides, it is meaningless to look at formula (8) as offerring $F(x,y)$ *only at the points* (x,y) *of the* Oxy *plane satisfying equation* (7). It would also be erroneous to think of (8) as giving the *radial component* of the central force *at all points of the xy plane*, unless we have in mind that *all the orbits* (1) *are traced with the same angular momentum value.* So, Whittaker's result (8) is valid for all members of the family (1), provided that it is interpreted as if the dependence

$$L=L(f(x,y))=L(c) \tag{9}$$

is also given in advance.

The basic findings of this paper are the following:

(i) we shall show that the function $F(x,y)$, given by (8) and supplemented by (9), is the general solution of a (free of L) first order linear partial differential equation for $F(x,y)$ (equation (12) and in polar coordinates equation (20)), with coefficients depending only on the function $f(x,y)$ giving the orbits (1). This equation then is of the same meaning as the second order equation (5) of the inverse problem, with the understanding that the radial component $F(x,y)$ of the central force now replaces the potential $V(x,y)$ (which may not even exist).

(ii) The significance of equation (12), as it will be explained, lies in that it can be used for the inverse problem of Dynamics of central motions as well as for the direct problem, i.e. to search for the totality of families of orbits which can be generated by a central force field $F(x, y)$. In this case equation (12) rearranged becomes (14) that is a second order nonlinear partial differential equation of the Monge-Ampère type.

(iii) As an application of the new equation we shall find *a necessary and sufficient condition* (23) which a preassigned family (19) has to satisfy in order that this family can result from a *conservative* central force field. It will become clear that such a family is generated by *one* central force field $F(r)$ determined from (24), factored by an arbitrary constant, and that motion of P on it takes place with a specific angular momentum $L(f)$ and also a specific energy dependence on the given family.

Finally we consider two-parametric families of curves and we find conditions in order that the families can be generated by a certain central force field.

2 – A partial differential equation for a central force field

Starting from equation (8), we can obtain a free of angular momentum, linear, first order, partial differential equation for the radial component $F = F(x, y)$ of the central forces (not necessarily conservative) generating the given family of orbits (1), with coefficients depending merely on the orbits. This is effectuated as follows:

In view of (4) we show directly that

$$f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_x^2 f_{yy} = -\Gamma f_x^3 \,.$$

Thus, we write equation (8) as

$$F(x, y) = L^2 r \frac{\Gamma}{(x + y\gamma)^3} \,. \tag{10}$$

Because of (9), we can write

$$(L^2)_y = \gamma (L^2)_x \,. \tag{11}$$

Solving (10) for L^2 and replacing in (11), we obtain after some straightforward algebra our first result

$$\gamma F_x - F_y + \Theta F = 0 \,, \tag{12}$$

where

$$\Theta = \frac{\gamma(x\lambda + y\mu)}{x + y\gamma} + \frac{y - x\gamma}{x^2 + y^2}\,. \tag{13}$$

For any given family (1), the functions γ, λ, μ appearing in (13) are known. In view of the manner used to obtain the differential equation (12), its general solution is given by (10), where L is an arbitrary function of $f(x, y)$. Of course, this fact can also be verified directly.

Apart from being linear, equation (12) is also homogeneous. Thus, if (f, F) is a compatible pair for (12), so is the pair $(f, F_0 F)$ with $F_0 =$ positive constant. This latter constant must be positive because with $F(x, y)$ given by (10), we must have $\frac{X+\gamma Y}{\Gamma} = F_0 \frac{L^2}{(x+y\gamma)^2} \geq 0$, (Bozis [4]), where $X = X(x, y)$, $Y = Y(x, y)$ are the cartesian components of the force field $F(x, y)$.

Although equation (12) is of the first order and it is applicable generally for nonconservative central forces, its meaning is similar to that of the second order equation (5). Indeed:

(i) *Both equations relate orbits to what produces these orbits, i.e. central force fields in* (12), *potentials in* (5).

(ii) *They were both derived on the grounds of a known integral of motion, i.e. angular momentum for* (12), *energy for* (5).

We must observe, of course, that equation (12), as it stands, serves for inverse-problem considerations, i.e. given the orbits (1) to find the central force $F(x, y)$. In this sense equation (12) seems, at first sight, of limited significance because we already know the general solution of (12): it is given by (10) with L an arbitrary function of *the given family* $f(x, y)$.

But this is not so. Because equation (12) can be regarded also from the direct problem viewpoint, if it is rearranged so as to have unknown the function $\gamma(x, y)$ i.e. the orbits (1), and known the central force field $F(x, y)$. Indeed, in view of relations (13) and (6), equation (12) is written as follows:

$$\gamma^2 \gamma_{xx} - 2\gamma\gamma_{xy} + \gamma_{yy} = H(x, y, \gamma, \gamma_x, \gamma_y, F, F_x, F_y) \tag{14}$$

where

$$H = (\gamma\,\gamma_x - \gamma_y)\Big[\frac{\gamma\,F_x - F_y}{F} + \frac{3y(\gamma\,\gamma_x - \gamma_y)}{x + \gamma\,y} + \frac{y - \gamma\,x}{x^2 + y^2} - \gamma_x\Big]\,.$$

Equation (14) is our second result and answers the following direct problem: *find all orbits created by a given central force field* $F(x, y)$

(equation (10) cannot answer because the angular momentum L, present in (10), depends on the unknown families of orbits). This task of course is far from being trivial. It amounts to solving a second order nonlinear partial differential equation in $\gamma = \gamma(x, y)$ of the Monge-Ampère type [8].

In fact any solution $\gamma = \gamma(x, y)$ of equation (14) is a possible answer and, in view of the first of (4), it leads to one monoparametric family of orbits (1). After finding (1) we can use the relation (10) in order to calculate the corresponding angular momentum $L = L(f)$.

The above situation, in a different but similar context, reminds us of the dispute between Sakellariou [12] and Hatzidakis [9], about central forces of the type $F = F(r, v)$ depending both on the radius r and velocity v, to which Levi-Civita [10] also intervened to offer his opinion.

As an example consider the nonconserative force field

$$F = \frac{y^2 - x^2}{(x^2 + y^2)^2} \,. \tag{15}$$

Then equation (14) reads

$$\begin{aligned} &\gamma^2 \gamma_{xx} - 2\gamma\gamma_{xy} + \gamma_{yy} = \\ &= (\gamma\gamma_x - \gamma_y)\Big[\frac{1 + 2x^2 - 6y^2}{x^2 + y^2}(y - \gamma x) + 3y\frac{\gamma\gamma_x - \gamma_y}{x + \gamma y} - \gamma_x\Big] \,. \end{aligned} \tag{16}$$

One solution of this equation is

$$\gamma = \frac{y(3x^2 + y^2)}{r^3 - x(x^2 + 3y^2)} \,, \tag{17}$$

leading to the monoparametric family

$$f(x, y_{)} = x - \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = c \,. \tag{18}$$

Replacing (15) and (17) into (10) we obtain the angular momentum dependence from orbit to orbit

$$L^2 = \frac{1}{3} c \,.$$

The example was taken, for specific selection of certain constants, from Appell [1].

If the family of orbits

$$f(r,\theta)=c \tag{19}$$

is given in polar coordinates r,θ, equation (12) for the central force $F = F(r,\theta)$ becomes

$$c_1F_r + c_2F_\theta + c_0F = 0\,, \tag{20}$$

with

$$\begin{aligned}
c_1 &= -rf_rf_\theta[f_r(r^2f_r^2+2f_\theta^2)+r(f_\theta^2f_{rr}-2f_rf_\theta f_{r\theta}+f_r^2f_{\theta\theta)}]\,,\\
c_2 &= -c_1\frac{f_r}{f_\theta}\,,
\end{aligned}$$

and

$$\begin{aligned}
c_0 = r^2f_r[f_\theta^3f_{rrr} &- 3f_rf_\theta(f_\theta f_{rr\theta}-f_rf_{r\theta\theta})-f_r^3f_{\theta\theta\theta}]-\\
&- 8f_rf_\theta[f_r(r^2f_r^2+2f_\theta^2)+r(f_\theta^2f_{rr}-2f_rf_\theta f_{r\theta}+f_r^2f_{\theta\theta})]-\\
&- 3r^2[f_\theta^3f_{rr}^2+2f_r^2f_\theta f_{r\theta}^2-3f_rf_\theta^2f_{rr}f_{r\theta}+f_r^2f_\theta f_{rr}f_{\theta\theta}-f_r^3f_{r\theta}f_{\theta\theta}]+\\
&+ f_r^2f_\theta(5r^2f_r^2+6f_\theta^2)\,.
\end{aligned}$$

The general solution of equation (20), as can be verified directly also, is the expression (10) in polar coordinates

$$F(r,\theta)=\frac{c_1}{r^6f_r^4f_\theta}L^2(f(r,\theta))\,, \tag{21}$$

where L is an arbitrary function of $f(r,\theta)$ standing for the angular momentum which is variable, in general, from orbit to orbit.

The monoparametric family (18) and the force field (15) may be written in polar coordinates as

$$f(r,\theta)=r(\cos\theta-\cos 2\theta)=c,\quad F(r,\theta)=-\frac{\cos 2\theta}{r^2}$$

and they satisfy, as expected, equation (20). The value of L^2 is found from (21) and it is of course $L^2=\frac{1}{3}c$.

Comment. Although (10) is the general solution of equation (12), for direct problem considerations equation (12) is indispensable. Indeed, (12) is the natural tool to answer questions of the following type: to find conditions on the families (1) so that these families can be created by central forces given in advance or for which (forces) some sort of additional information is given in advance.

Let us try, for instance, to find all families (1) which can be generated by constant central forces $F(x, y) = F_0$. (Let us have in mind the same constant F_0 for all members of the family). It is immediately seen from (12) that Θ must vanish, so $\Theta = 0$ is the required condition for the orbits. (Such is, e.g., the case with the family of concentric circles $x^2 + y^2 = c$ for which $\gamma = \frac{y}{x}$, $\lambda = \frac{3}{x}$, $\mu = -\frac{3}{y}$, $\Theta = 0$. For this case we find $L^2 = -F_0\,(x^2 + y^2)^{\frac{3}{2}}$ from (10).

As another example, we consider as additional information about the central force $F(x, y)$, its divergence is zero (as e.g. in the case of the planar force due to the presence of a finite planar material concentration, outside the matter). In this case the problem is to find if there exist orbits for which both equations (12) and $\operatorname{div}\overrightarrow{F} = 0$, i.e. $xF_x + yF_y + F = 0$ can be compatible. Notice that this latter equation cannot be written in view of (10) because of the presence of $L = L(f)$ in it.

Remark. The well known Binet's formula for a unit mass

$$\frac{d^2}{d\theta^2}\Big(\frac{1}{r}\Big) + \frac{1}{r} = -\frac{r^2}{L^2} F(r, \theta)$$

is unquestionably used to offer the radial component of the central force $F(r, \theta)$ creating *one* specific orbit $r = r(\theta)$ traced with a certain value of L. To cover the case of a preassigned *family* (19), formula (21) is suitable.

3 – Conservative central forces

Whittaker's definition for the central force, that we have assumed in this paper, is also adopted e.g. by Appell [1] and Levi-Civita [10]. In many textbooks, however, e.g. Symon [13] only *conservative forces* directed to a fixed center are considered as central. In this later case, of course, the radial component of the force is of the form $F = F(r)$. Thus, equation (20) becomes

$$\frac{F'(r)}{F(r)} = -\frac{c_0}{c_1}\,, \tag{22}$$

and it can be valid only for families of orbits (19) for which is verified the condition

$$\left(\frac{c_0}{c_1}\right)_\theta = 0\,. \tag{23}$$

But true is also the opposite: i.e. if the condition (23) is satisfied, then equation (20) *always has one solution* $F = F(r)$. In fact, for a given family (19) for which (23) is good, we have $\frac{c_0}{c_1}$ depending only of the variable r, therefore we can write always a relation as (22), or equivalently $c_1 F_r + c_0 F = 0$, and the equation (20) admit the solution $F = F(r)$.

In conclusion we have our third result: *a preassigned family of orbits* (19) *can be generated by a conservative central force field* $F = F(r)$ *if and only if the condition* (23) *is satisfied.*

In this case, the force field is found from equation (22), in terms of the given orbits only, by the formula

$$F(r) = F_0 \exp\left[-\int \frac{c_0}{c_1} dr\right], \tag{24}$$

up to a multiplicative constant F_0.

The angular momentum, found in view of (21), is given by

$$L^2 = \frac{r^6 f_r^4 f_\theta}{c_1} F(r)\,.$$

Finally we can calculate the total energy E of the particle using the equation analogue to (3), in terms of polar coordinates [15], that is

$$f_r V_r + \frac{1}{r^2} f_\theta V_\theta = \frac{2(E-V)}{f_\theta^2 + r^2 f_r^2}\Big(f_{rr} f_\theta^2 - 2 f_r f_\theta f_{r\theta} + f_{\theta\theta} f_r^2 + r f_r^3 + \frac{2}{r} f_r f_\theta^2\Big)\,,$$

and taking into account that $V(r) = -\int F(r)dr$ we obtain the energy

$$E = \beta F(r) - \int F(r) dr\,,$$

where

$$\beta = \frac{r^2 f_r^2 f_\theta}{2c_1}(r^2 f_r^2 + f_\theta^2)\,.$$

Thus, in this case, both L and E are expressed on the grounds of the given function $f(r,\theta)$.

A common case in Celestial Mechanics is that of a family of *geometrically similar orbits*, that is orbits of the type

$$f(r,\theta) = rg(\theta) = c \tag{25}$$

generated by conservative central forces $F = F(r)$; in this case the condition (23) becomes, after some straightforward algebra

$$\frac{g(g''' + g')}{g'(g'' + g)} = k_0 = \text{const}\,, \tag{26}$$

where the primes denote differentiation with respect the variable θ and where $k_0 \neq 0$, $g'' + g \neq 0$, because $g'' + g = 0$ leads to family of straight lines.

From the previous equation, introducing three constants c_1, c_2, c_3, we get subsequently:

$$\frac{g'' + g}{g^{k_0}} = c_1$$

$$g'^2 = \frac{2c_1}{k_0+1} g^{k_0+1} - g^2 + c_2$$

and integrating we found

$$\theta + c_3 = \int \frac{dg}{\left(\frac{2c_1}{k_0+1} g^{k_0+1} - g^2 + c_2\right)^{\frac{1}{2}}}\,. \tag{27}$$

Remarks.

1. For $k_0 = 0, 1, 2, -3$, the integral (27) can be done, and also for $k_0 = 3$ and $c_1 c_2 = \frac{1}{2}$, the integral (27) can be found.

2. In the case at hand the family (25) is created of a homogeneous potential $V = r^m G(\theta)$. Such is, for instance, the case of all conics $r[1 + e_0 \cos(\theta - \theta_0)] = c$ of constant eccentricity e_0 and orientation θ_0 with varying semi-major axis a resulting from Newton's potential $V = -\frac{k}{r}$. Equation (5), write in polar coordinates, with the

above potential, becomes an ordinary second order differential equation in $G(\theta)$ (equation (9.8) of BOZIS [6]). The result (26) is in agreement with equation (9.8) of BOZIS [6], if we put $G(\theta) = \text{const.}$, i.e. if we assume further that the central potential $V(r)$ is also homogeneous in x, y which is not generally the case, of course. Actually from (26) and equation (9.8), above mentioned, we find $k_0 = -(m+1)$, where m is the degree of homogeneity of V.

3. Equation (27) gives the totality of families of *geometrically similar orbits* created in the central field given by

$$\frac{F'(r)}{F(r)} = -\frac{c_0}{c_1}\,.$$

Condition (23) is valid e.g. for the monoparametric family of the spirals

$$r\theta = c\,,$$

leading to

$$F = -\frac{1}{r^3}, \quad L^2 = 1, \quad E = -\frac{1}{2c^2}\,.$$

The same condition (23) is not valid for the spirals

$$r\theta^2 = c\,,$$

i.e. there is no conservative central force creating this family.

REMARK. One might reach the same condition (23) as follows: write equation (5) in polar coordinates, apply it for a potential $V = V(r)$, supposedly depending only on $r = (x^2 + y^2)^{1/2}$, and express the ratio $\frac{V''(r)}{V'(r)}$ in terms of the orbital elements.

4 – Two-parametric families

Let a family of curves

$$f(r, \theta, b) = c \tag{28}$$

parametrized by b and c given in advance. Apparently, formula (21) is not applicable because the resulting force must be independent of the two parameters and in particular of the parameter b which is, generally, expected to survive as it appears in the function $f(r,\theta,b)$ and also in the term $L^2(f(r,\theta,b))$. Only families (28) satisfying certain conditions may be generated by a certain central force field $F = F(r,\theta)$. To find these conditions we proceed as follows:

We differentiate equation (20) with respect to b and we write the system

$$c_1 F_r + c_2 F_\theta = -c_0 F\,, \tag{29}$$

$$c_{1,b} F_r + c_{2,b} F_\theta = -c_{0,b} F\,. \tag{30}$$

Denoting by

$$\epsilon = \frac{f_r}{f_\theta}\,, \tag{31}$$

taking into account that $c_2 = -\epsilon c_1$ and solving the system (29), (30) for $\frac{F_r}{F}$, $\frac{F_\theta}{F}$, we find

$$\frac{F_r}{F} = \frac{\delta_1}{\delta_0}, \quad \frac{F_\theta}{F} = \frac{\delta_2}{\delta_0}\,, \tag{32}$$

where

$$\begin{aligned} &\delta_0 = -c_1^2 \epsilon_b \neq 0\,, \quad \delta_1 = c_0(\epsilon_b c_1 + \epsilon c_{1,b}) - \epsilon c_1 c_{0,b}\,, \\ &\delta_2 = -c_1 c_{0,b} + c_0 c_{1,b}\,. \end{aligned} \tag{33}$$

For the system (29), (30) be compatible, the following conditions must be satisfied:

$$\text{(i):} \quad \Big(\frac{\delta_1}{\delta_0}\Big)_b = 0\,, \qquad \text{(ii):} \quad \Big(\frac{\delta_2}{\delta_0}\Big)_b = 0\,, \qquad \text{(iii):} \quad \Big(\frac{\delta_1}{\delta_0}\Big)_\theta = \Big(\frac{\delta_2}{\delta_0}\Big)_r\,. \tag{34}$$

Then the force $F(r,\theta)$ is determined from (32) up to a multiplicative constant.

Thus, e.g. for the two-parametric family

$$f(r,\theta,b) = \frac{1}{\cos\theta}\Big(\frac{1}{r} - b\cos 2\theta\Big) = c$$

we obtain, from (33), $\delta_0 = -\frac{9b^2\cos^2 2\theta(3\sin\theta+\sin 3\theta)}{2r^{10}\cos^{11}\theta} \neq 0$ and $\frac{\delta_1}{\delta_0} = -\frac{2}{r}$, $\frac{\delta_2}{\delta_0} = -2\tan 2\theta$ which satisfy the conditions (34).

We obtain the nonconservative central force $F(r,\theta) = \frac{\cos 2\theta}{r^2}$ from (32), and we find $L^2 = \frac{1}{3b}$ from (21).

Remarks.

1. In view of equations (33) it can be shown by straightforward algebra that if the condition (34) (ii) is valid, then so is the first condition (34) (i) and vice versa. In conclusion: *if and only if, for a given two-parametric family of orbits, the last two conditions are satisfied, then all members of the family result from a central force field determined from the system* (32).

2. If $\delta_2 = 0$, then $F_\theta = 0$ and the force is conservative. Besides, from (34) (iii) we understand that $\frac{\delta_1}{\delta_0}$ depends merely on r and we obtain formula (24), from the first of (32), because $\frac{\delta_1}{\delta_0} = -\frac{c_0}{c_1}$.

This is e.g. the case with the two-parametric family (Appell [1])

$$\frac{1}{r^2} - b\cos 2\theta = c$$

which leads to $F = -r$ and in the sequel to $L^2 = \frac{1}{c^2-b^2}$, $E = \frac{c}{c^2-b^2}$.

3. The parameter b appearing in (28) is neither additive nor multiplicative for the function f, because then (28) would be essentially monoparametric. So, in view of (31), it is $\epsilon_b \neq 0$. Then the case $\delta_0 = 0$ leads to $c_1 = 0$ and also to $\delta_1 = \delta_2 = 0$. Then formulae (32) become meaningless. Such is e.g. the case of the two parametric family

$$\frac{1}{\cos\theta}\Big(\frac{1}{r} - b\sin\theta + \cos 2\theta\Big) = c\,.$$

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