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Existence and partial regularity results for the gradient flow for a variational functional of degenerate type

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RIASSUNTO: *Si dimostra l'esistenza e la regolarità parziale della soluzione del problema di evoluzione associato ad un funzionale variazionale di tipo degenere. Si usa il metodo di penalizzazione di Chen e Struwe per ottenere una approssimazione delle soluzioni e si riconosce che per queste vale uniformemente un teorema di ε-regolarità. Si ottiene una formula di monotonia per la crescita quadratica delle densità di energia all'infinito e si ricava una stima di Harnack dalla formula di Bochner.*

ABSTRACT: *We establish the existence and partial regularity of a weak solution to the evolution problem associated to a variational functional of degenerate type. We use the penalty method due to Chen and Struwe to make approximation of solutions. The main ingredient is to show that the ϵ-regularity theorem holds uniformly for approximating solutions. For this purpose, we derive the monotonicity formula using the quadratic growth of the energy density at infinity. We also obtain the Bochner formula in somewhat technical way and take some cares to derive the Harnack estimate from it.*

1 – Introduction

Let M, N be compact, smooth Riemannian manifolds of dimension m, l with metrics g, h respectively and suppose that ∂M, $\partial N = \emptyset$.

KEY WORDS AND PHRASES: *Degenerate parabolic system – p-harmonic map – Gradient flow.*
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Since N is compact, N may be isometrically embedded into a Euclidean space R^n for some n. For a C^1-map $u : M \to N \subset R^n$, we define a variational functional $I(u)$ by

$$I(u) = \int_M f(|Du|)dM\,, \tag{1.1}$$

where, in local coordinates on M, with $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$, $|g| = |\det(g_{\alpha\beta})|$ and $D_\alpha = \partial/\partial x^\alpha$ $(\alpha = 1, \cdots, m)$,

$$dM = \sqrt{|g|}dx, \quad |Du|^2 = \sum_{\alpha,\beta=1}^{m}\sum_{i=1}^{n} g^{\alpha\beta}D_\alpha u^i D_\beta u^i$$

and f is the real valued convex C^2-function defined on $[0, \infty)$ such that, for $p > 2$,

$$f(\tau) = \begin{cases} \dfrac{1}{p}\tau^p, & 0 \le \tau \le 1 \\ \dfrac{p-1}{2}\tau^2 - (p-2)\tau + \dfrac{(p-1)(p-2)}{2p}, & \tau \ge 1. \end{cases}$$

The Euler-Lagrange equation of the variational functional I is

$$-\triangle_M^f u + A^f(u)(Du, Du) = 0\,, \tag{1.2}$$

where $\triangle_M^f$ denotes the differential operator on M

$$\triangle_M^f u = \frac{1}{\sqrt{|g|}} D_\alpha\Big(\sqrt{|g|}g^{\alpha\beta}\frac{f'(|Du|)}{|Du|}D_\beta u\Big)$$

which is a degenerate elliptic operator and, by the second fundamental form $A(u)(Du, Du)$ of N in R^n at u, $A^f(u)(Du, Du)$ is given by

$$A^f(u)(Du, Du) = \frac{f'(|Du|)}{|Du|}g^{\alpha\beta}A(u)(D_\alpha u, D_\beta u)\,.$$

Here and in what follows, the summation notation over repeated indices is adopted.

We are concerned with critical points defined as solutions of (1.2). One method to look for critical points is to make use of the gradient flow for the functional I. The gradient flow is described by a system of second order nonlinear degenerate parabolic partial differential equations with initial condition

$$\partial_t u - \triangle_M^f u + A^f(u)(Du, Du) = 0 \quad \text{in } (0,\infty)\times M\,, \tag{1.3}$$

$$u(0,x) = u_0(x) \qquad \text{for } x \in M\,. \tag{1.4}$$

The partial regularity of minimizing harmonic maps was achieved in [15], [23]. The results were generalized to obtain the partial regularity of minimizing p-harmonic maps ($p > 1$) in [16] and similar results were also treated in [14] (also see references in [16], [14]). These results become fundamental to the regularity theory of harmonic maps. The partial regularity of p-harmonic maps ($p \geq 2$) was also investigated in [22], [12]. On the other hand, Chen and Struwe established the global existence and partial regularity for gradient flows for harmonic maps by combining Struwe's monotonicity formula with the penalty method (see [4], [24]). The gradient flows for p-harmonic maps are given by nonlinear degenerate parabolic system. The regularity of weak solutions of degenerate parabolic systems with only principal terms was discussed and the $C^{1,\alpha}$-regularity of solutions was accomplished in [2], [9], [10], [11] (also see [6], [7] and [25], [26] for corresponding elliptic systems). The global existence of a weak solution to the gradient flow for p-harmonic maps was shown provided the target manifold is a sphere [1]. This result was also extended to the case that the target manifold is a homogeneous space [17], [18], [19]. The m-harmonic flow is investigated in [20] and the global existence and the partial $C^{1,\alpha}$-regularity of a weak solution are obtained. However the partial regularity of gradient flows for p-harmonic maps remains a difficult problem to be settled. The key estimate to investigate a partial regularity is the so-called monotonicity formula (refer to [4], [24]). In contrast with harmonic flows, we only know that, for p-harmonic flows, even if $p = m$, the monotonicity type inequality holds under some "smallness" condition (see [20], Theorem 1). In this paper we make an extension of Struwe's results [4], [24], which may be of some use for attacking the partial regularity problem for p-harmonic flows.

To state our results, we need some preminalies: Let us define the parabolic metric δ_q by $\delta_q(z_1,z_2)=\max\{|t_1-t_2|^{1/q},\ |x_1-x_2|\}$ for any $z_i=(t_i,x_i)\in(0,\infty)\times R^m$, $i=1,2$, and denote by $\mathcal{H}^k(\cdot)$ the k-dimensional Hausdorff measure with respect to δ_2. For any bounded domain $B\subset R^m$, we use the usual function spaces: $C^k_\alpha(B,R^n)$, $L^q(B,R^n)$ and $W^1_q(B,R^n)$. For any bounded domain $Q\subset(0,\infty)\times R^m$, we denote by $C^{\alpha/q,\alpha}(Q,R^n)$ the space of locally Hölder continuous functions in Q with exponent α, $0<\alpha<1$, on the metric δ_q. We now give the definition of the energy space, on which we consider the variational functional I:

$$W^{1,2}(M,N)=\{u\in W^{1,2}(M,R^n): u(x)\in N \text{ for almost all } x\in M\}. \tag{1.5}$$

The global weak solutions of (1.3) and (1.4) is defined as follows: $u\in L^\infty((0,\infty);\,W^{1,2}(M,N))\cap W^{1,2}((0,\infty);\,L^2(M,R^n))$ satisfying, for all $\phi\in L^2((0,\infty);\ W^{1,2}(M,R^n))\cap L^\infty((0,\infty)\times M,R^n)$ the support of which is compactly contained in $(0,\infty)\times U$ for a coordinate chart U on M

$$\int\limits_{(0,\infty)\times M}\left\{\phi\cdot\partial_t u+\frac{f'(|Du|)}{|Du|}g^{\alpha\beta}D_\beta u\cdot D_\alpha\phi+\phi\cdot A^f(u)(Du,Du)\right\}dM\,dt=0 \tag{1.6}$$

and satisfying the initial condition

$$|u(t)-u_0|_{L^2(M)}\to 0 \quad \text{as } t\to 0\,.$$

Then our main theorem is the following:

THEOREM 1. *Suppose $u_0\in W^{1,2}(M,N)$. Then there exists a global weak solution $u\in L^\infty((0,\ \infty);W^{1,2}(M,N))\cap W^{1,2}((0,\infty);L^2(M,R^n))$ with the energy inequality*

$$\int_{(0,\infty)\times M}|\partial_t u|^2 dM\,dt+\sup_{0\le t<\infty}I(u(t))\le I(u_0)\,. \tag{1.7}$$

Moreover there exist an open set $Q_0\subset(0,\infty)\times M$ (with respect to a metric δ_2) and a positive number $\alpha, 0<\alpha<1$, such that $u\in C^{\alpha/p,\alpha}(Q_0,R^n)$ and $Du\in C^{\alpha/2,\alpha}(Q_0,R^{mn})$ and it holds that

$$\partial_t u-\triangle^f_M u+A^f(u)(Du,Du)=0 \quad \textit{almost everywhere in } Q_0 \tag{1.8}$$

and that

$$\mathcal{H}^m((0,T) \times M \setminus Q_0) < \infty \quad \textit{for any} \;\; T > 0 \,. \tag{1.9}$$

Some standard notations: For $z_0 = (t_0, x_0) \in (0, \infty) \times R^m$ and $r, \tau > 0$,

$$B_r(x_0) = \{x \in R^m : |x - x_0| < r\}\,, \quad Q_{r,\tau}(z_0) = (t_0 - \tau,\, t_0) \times B_r(x_0) \tag{1.10}$$

and $Q_r(z_0) = Q_{r,r^2}(z_0)$. The center points x_0, z_0 are omitted when no confusion may arise. For vectors $u, v \in R^n$ and $P, Q \in R^{mn}$, put

$$u \cdot v = \sum_{i=1}^n u^i v^i, \quad \langle P, Q \rangle = g^{\alpha\beta} P_\alpha \cdot Q_\beta, \quad |P|^2 = \langle P, P \rangle \,. \tag{1.11}$$

This paper is arranged as follows: Section 2 contains some auxiliary lemmata. In Section 3, we exploit an approximating scheme, called the "penalty method", similar to [4] to construct weak solutions to (1.3) and (1.4). To prove an existence of weak solutions to the penalized equations (3.2) and (3.3), we use Galerkin's method and the "monotonicity trick" similar as in [1]. We also investigate the smoothness of weak solutions of the penalized equations. All estimates obtained there are depending on an approximating number and, however, are fundamental to showing a priori estimate for them. Section 4 is a most significant part in this paper. Here we obtain a priori estimate for weak solutions of the penalized equations. To show the validity of the monotonicity formula, we crucially use the fact that the density f has quadratic growth at infinity. We choose the constant C, which is the coefficient of the penalty term of the penalized functional (3.1), to be large and then we obtain the Bochner formula. Finally, we show that the ϵ-regularity theorem holds uniformly for approximating solutions. Here we derive a Harnack type estimate from the Bochner formula. Because of the degeneracy of the equation, we take some cares to obtain a Harnack type inequality. In Section 5, we argue similarly as in [4] to pass to the limit of approximating solutions. Here we use the compactness results for degenerate parabolic systems.

2 – Preliminaries

First we simply state the estimates for f, which is used in later sections:

$$\frac{1}{p}(\tau^2-1) \le f(\tau) \le (p-1)\tau^2 \quad \text{for all } \tau \ge 0\,, \tag{2.1}$$

$$\frac{1}{p}\tau^2 \le f(\tau) \quad \text{for all } \tau \ge 1\,. \tag{2.2}$$

For brevity, we set

$$F(\tau) = \frac{f'(\tau)}{\tau}, \quad H(\tau) = -\frac{f'(\tau)}{\tau} + f''(\tau) \quad \text{for all } \tau \ge 0\,. \tag{2.3}$$

By direct calculation, we see that

$$\begin{aligned} F(\tau) &= \begin{cases} \tau^{p-2} & \text{for } 0 \le \tau \le 1 \\ (p-1) - \dfrac{p-2}{\tau} & \text{for } \tau \ge 1, \end{cases} \\ H(\tau) &= \begin{cases} (p-2)\tau^{p-2} & \text{for } 0 \le \tau \le 1 \\ \dfrac{p-2}{\tau} & \text{for } \tau \ge 1. \end{cases} \end{aligned} \tag{2.4}$$

Then we note that $(F(\tau))' = H(\tau)/\tau \quad \text{for all } \tau \ge 0$.

We now gather some algebraic inequalities.

Lemma 2.1. *There exist positive constants γ_i $(i=1,2)$ depending only on p and $(g_{\alpha\beta})$ such that, for any vector $P = (P^i_\alpha) \in R^{mn}$, $Q = (Q^i_\alpha) \in R^{mn}$,*

$$\min\{|P|^{p-2}, 1\} \le F(|P|) \le (p-1)\min\{|P|^{p-2}, 1\}\,, \tag{2.5}$$

$$|H(|P|)| \le (p-2)\min\{|P|^{p-2}, 1\}\,, \tag{2.6}$$

$$\begin{aligned} \gamma_1 \min\{|P|^{p-2}, 1\}|Q|^2 &\le |Q|^2 F(|P|) + \Big\langle Q, \frac{P}{|P|}\Big\rangle^2 H(|P|) \le \\ &\le \gamma_2 \min\{|P|^{p-2}, 1\}|Q|^2\,. \end{aligned} \tag{2.7}$$

LEMMA 2.2. *There exist positive constants γ, $\bar\gamma$ depending only on p and $(g_{\alpha\beta})$ such that, for all vectors $P, Q \in R^{mn}$ with $V(s) = Q + s(P-Q)$ for any s, $0 \le s \le 1$,*

$$\langle P-Q, F(|P|)P - F(|Q|)Q\rangle \ge \gamma |P-Q|^2 \int_0^1 \min\{|V(s)|^{p-2}, 1\} ds\,, \tag{2.8}$$

$$|F(|P|)P - F(|Q|)Q| \le \bar\gamma |P-Q| \int_0^1 \min\{|V(s)|^{p-2}, 1\} ds\,, \tag{2.9}$$

$$|F(|P|)|P| - F(|Q|)|Q|| \le \bar\gamma |P-Q| \int_0^1 \min\{|V(s)|^{p-2}, 1\} ds\,. \tag{2.10}$$

PROOF. Noting that

$$\begin{aligned} F(|P|)P - F(|Q|)Q = \int_0^1 \Big\{ & F(|V(s)|)(P-Q) + \\ & + H(|V(s)|)\Big\langle P-Q, \frac{V(s)}{|V(s)|}\Big\rangle \frac{V(s)}{|V(s)|}\Big\} ds\,, \end{aligned} \tag{2.11}$$

we have, by (2.7) and (2.6), (2.8) and (2.9). Since

$$\begin{aligned} & F(|P|)|P| - F(|Q|)|Q| = \\ & \qquad = (p-1)\int_0^1 \Big\langle P-Q, \frac{V(s)}{|V(s)|}\Big\rangle \min\{|V(s)|^{p-2}, 1\} ds\,, \end{aligned} \tag{2.12}$$

we immediately have (2.10) by Schwarz's inequality.

We use the distance function to the target manifold N to approximate the equation (1.3). Since N is smooth and compact, there exists a uniform tubular neighborhood $\mathcal{O}(N) \subset R^n$ of N of width $2\delta_N$ such that each point $p \in \mathcal{O}(N)$ has a unique nearest point $q = \pi_N(p)$ with a distance $\mathrm{dist}(p, N) = |p-q|$ and the projection π_N from $\mathcal{O}(N)$ to N is smooth.

We use the same regularization as in [4], p. 87. Let χ be a smooth, non-decreasing function such that $\chi(s) = s$ for $s \le \delta_N^2$ and $\chi(s) = 2\delta_N^2$ for $s \ge 4\delta_N^2$. Then the function $\chi(\mathrm{dist}^2(p, N))/2$ is C^2-function on every $p \in R^n$ and, at points p with $\mathrm{dist}(p, N) \le \delta_N$, its gradient is orthogonal to the tangential space $T_{\pi_N(p)}N$ of N at $\pi_N(p) \in N$ in R^n.

Let us give an estimate for $\chi(\mathrm{dist}^2(u, N))$ with $u \in R^n$ without the proof.

Lemma 2.3. *There exists a positive constant γ depending only on N such that, for all vectors $u, v \in R^n$,*

$$|\chi(\mathrm{dist}^2(u,N)) - \chi(\mathrm{dist}^2(v,N))| \le \gamma(|u|+|v|)|u-v|\,, \tag{2.13}$$

$$\Big|\frac{d}{du}\chi(\mathrm{dist}^2(u,N))\Big| \le 4\delta_N\,, \tag{2.14}$$

$$\Big|\frac{d}{du}\chi(\mathrm{dist}^2(u,N)) - \frac{d}{dv}\chi(\mathrm{dist}^2(v,N))\Big| \le \gamma|u-v|\,. \tag{2.15}$$

We recall the result concerning to the compactness of Sobolev embedding (for the proof, refer to [1], p. 28, Lemma 1.4).

Lemma 2.4. *Let $T > 0$ be fixed. Suppose that $\{u_l\}$ is bounded in $L^\infty((0,T);W^{1,q}(M,R^n))$, $q \ge 1$, and $\{\partial_t u_l\}$ is bounded in $L^2((0,T);L^2(M,\ R^n))$. Then there exist subsequence $\{u_l\}$ and a function $u \in L^\infty((0,T);W^{1,q}(M,R^n)) \cap W^{1,2}((0,T);L^2(M,R^n))$ such that $\{u_l\}$ converges to u strongly in $L^r((0,T)\times M,R^n)$ for each r, $q \le r < mq/(m-q)$.*

3 – Approximating solutions

In this section we explain the approximate scheme to construct solutions to (1.3) and (1.4). For a C^1-map v defined on M with value in R^n, we define the penalized functional (refer to [4], p. 87) with parameters k, $k \to \infty$, by

$$I^k(v) = \int_M \{f(|Dv|) + Ck\chi(\mathrm{dist}^2(v,N))\}dM\,, \tag{3.1}$$

where C is a positive constant stipulated later. We approximate a solution of (1.3) and (1.4) by gradient flows for the penalized functionals (3.1). For each positive number k, the gradient flows for (3.1) are described by a system of nonlinear degenerate parabolic partial differential equations with initial data

$$\partial_t u - \Delta_M^f u + Ck\frac{d}{du}\chi(\mathrm{dist}^2(u,N)) = 0 \quad \text{in } (0,\infty)\times M\,, \tag{3.2}$$

$$u(0,x) = u_0(x) \qquad \text{for } x \in M\,. \tag{3.3}$$

We call the equation (3.2) the penalized equation.

Now we prove the existence of weak solutions to the penalized equation with initial data $u_0 \in W^{1,2}(M, N)$.

Lemma 3.5. *For every $k \geq 1$, there exists a weak solution for* (3.2) *and* (3.3).

Proof. We argue similarly as in [1], pp. 29-31, Theorem 1.5, to construct weak solutions (3.2) and (3.3). Choose a fundamental system of functions $\{w_k\} \subset W^{1,2}(M, R^n)$, which is orthonormalized in $L^2(M, R^n)$. For $u, v \in L^2(M, R^n)$ and $U, V \in L^2(M, R^{mn})$, we put

$$[u, v] = \int_M u \cdot v dM, \quad [U, V] = \int_M \langle U, V\rangle dM .$$

Fix $k \geq 1$. For any positive integer l, we find an approximate solution

$$u_l(t) = \sum_{i=1}^{l} c_l^i(t) w_i \tag{3.4}$$

such that $c_l^j(t)$, $1 \leq j \leq l$, satisfy the system of ordinary differential equations

$$\frac{d}{dt} c_l^j + [F(|Du_l|) Du_l, Dw_j] + Ck\Big[\frac{d}{du}\chi(\mathrm{dist}^2(u_l, N)), w_j\Big] = 0 , \tag{3.5}$$

$$c_l^j(0) = [u_0, w_j] . \tag{3.6}$$

Observe from Schwarz's inequality and (2.9), (2.14) and (2.15) that the second and third terms in (3.5) are locally bounded and Lipschitz continuous on the real variables c_l^i $(i = 1, \cdots, l)$. Thus we can choose a positive number $T = T(l)$ such that (3.5) and (3.6) have solutions $c_l^i \in C^1((0, T))$, $1 \leq i \leq l$, satisfying

$$\int_{(0,T)\times M} |\partial_t u_l|^2 dM dt + I^k(u_l(T)) = I^k(u_l(0)) . \tag{3.7}$$

By (2.13), we make estimation

$$\int_M \chi(\mathrm{dist}^2(u_l(0), N)) dM \leq 2\gamma(N) \int_M |u_0|^2 dM . \tag{3.8}$$

By (2.1), we also have, for any $V \in R^{mn}$,

$$\frac{1}{p}(|V|^2 - 1) \leq f(|V|) \leq (p-1)|V|^2 . \tag{3.9}$$

By substitution of (3.8) and (3.9) to (3.7), we have the uniform boundedness with respect to $T > 0$ and $l \geq 1$:

$$\begin{aligned}\int_{(0,T)\times M} |\partial_t u_l|^2 dM dt + \int_M (|Du_l(T)|^2 + Ck\chi(\mathrm{dist}^2(u_l(T), N))dM \leq \\ \leq \gamma\Big(|M| + \int_M (|Du_0|^2 + Ck|u_0|^2)dM\Big)\,.\end{aligned} \tag{3.10}$$

By the usual extensive methods, we obtain a time-global solution $u_l \in L^\infty((0,\infty); W^{1,2}(M, R^n)) \cap W^{1,2}((0,\infty); L^2(M, R^n))$ satisfying the energy equality (3.7) for any $T > 0$.

We consider how to pass to the limit of u_l as $l \to \infty$ and obtain a solution to the penalized equation (3.2) and (3.3). Argue with (3.10) similarly as in [1], pp. 29-31, to have a subsequence $\{u_l\} \subset L^\infty((0,\infty); W^{1,2}(M, R^n)) \cap W^{1,2}((0,\infty), L^2(M, R^n))$ such that

$$\partial_t u_l \to \partial_t u \quad \text{weakly in } L^2((0,\infty) \times M, R^n)\,, \tag{3.11}$$

$$Du_l \to Du \quad \text{weakly* in } L^\infty((0,\infty); L^2(M, R^n)) \tag{3.12}$$

and

$$\begin{aligned}&\frac{d}{du}\chi(\mathrm{dist}^2(u_l, N)) \to \frac{d}{du}\chi(\mathrm{dist}^2(u, N)) \\ &\text{weakly in } L^2_{\mathrm{loc}}((0,\infty); L^2(M, R^n))\,.\end{aligned} \tag{3.13}$$

Noting that $f(|P|)$ is convex on $P \in R^{mn}$, by (3.11), (3.12) and (3.13), we pass to the limit as $l \to \infty$ in (3.5) and (3.7) to find that u satisfies (3.2) in the weak sense and

$$\int_{(0,T)\times M} |\partial_t u|^2 dM dt + I^k(u(T)) \leq I(u_0) \quad \text{for any } T > 0\,. \tag{3.14}$$

Now we consider the smoothness of weak solutions u_k, $k \geq 1$, to (3.2). For brevity, put $u = u_k$. To make local estimation for u, we fix some notations (refer to [4], p. 96). Let $R_M > 0$ be a lower bound for the injective radius of the exponential map on M such that, for any R, $0 < R < R_M$, the geodesic ball $\mathcal{B}_R(x_0)$ of radius R around x_0 is well-defined and diffeomorphic to the Euclidean ball $B_R(0) \subset R^m$ through the exponential map. Take a point $(t_0, x_0) \in (0,\infty) \times M$ arbitrarily and fix it. Then we find that, for any $t \in [0, t_0]$, the map

$$u(t, \exp_{x_0} \cdot) : R^m \supset B_{R_M}(0) \ni x \to u(t, \exp_{x_0} x) \in R^n \tag{3.15}$$

is well-defined. Replace $u(t, \exp_{x_0} x)$ by $u(t,x)$ for any $(t,x) \in [0,t_0] \times B_{R_M}(0)$ and, moreover, by translation and an appropriate extension of u to $R^m \setminus B_{R_M}(0)$, regard u as a map defined on $[-t_0, 0] \times \mathbb{R}^m$ with values in R^n. Put $P_{R_M} = (-t_0, 0) \times B_{R_M}$.

First we state the result concerning the twice differentiability of solutions.

LEMMA 3.6. *The function* $\min\{|Du|^{p/2-1}, 1\}Du$ *has weak spatial derivatives which lie in* $L^2_{\mathrm{loc}}(P_{R_M})$ *and there exists a positive constant* γ *depending only on* C, p, M *and* N *such that, for all* $Q_{2r} \subset P_{R_M}$,

$$\sup_{-r^2 \le t \le 0} \int_{\{t\}\times B_r} |Du|^2 dM + \int_{Q_r} \min\{|Du|^{p-2}, 1\}|D^2u|^2 dMdt \le \tag{3.16}$$
$$\le \gamma(k + r^{-2}) \int_{Q_{2r}} |Du|^2 dMdt\,.$$

PROOF. For a positive number h, we denote, by $\Delta_{h,i}u(t,x) = (u(t,x+he^i) - u(t,x))/h$ and $\overline{\Delta}_{h,i}u(t,x) = (u(t,x) - u(t,x-he^i))/h$, the difference quotients in the i-th direction $(i = 1, \cdots, m)$. Let $\eta \in C^1_0(B_{3r/2})$ be $0 \le \eta \le 1$, $\eta = 1$ in B_r and $|D\eta| \le 4/r$, and $\sigma \in C^1((-\infty, 0])$ be $0 \le \sigma \le 1$, $\sigma = 1$ on $[-r^2, 0]$, $\sigma = 0$ on $(-\infty, -(2r)^2)$ and $|\partial_t \sigma| \le 4/r^2$. We choose h, $0 < h < \mathrm{dist}\{\mathrm{supp}\,\eta, \partial B_{2r}\}$. Taking a test function $(\sqrt{|g|})^{-1}\overline{\Delta}_{h,i}$ $(\sqrt{|g|}\sigma\eta^2\Delta_{h,i}u)$ for (3.2), we have, by change of variables,

$$\begin{aligned} 0 = \int_{Q_{2r}} \Big\{ &\sigma\eta^2 \Delta_{h,i}(\partial_t u) \cdot \Delta_{h,i}u + \\ &- F(|Du|)g^{\alpha\beta}D_\beta u \cdot D_\alpha\Big(\frac{1}{\sqrt{|g|}}\overline{\Delta}_{h,i}\Big(\sqrt{|g|}\sigma\eta^2\Delta_{h,i}u\Big)\Big) + \\ &+ Ck\sigma\eta^2(\Delta_{h,i}u) \cdot \Delta_{h,i}\Big(\frac{d}{du}\chi(\mathrm{dist}^2(u,N))\Big)\Big\} dMdt = \\ = I + II + III\,. \end{aligned} \tag{3.17}$$

We now estimate the each term in (3.17). Simply,

$$I = \frac{1}{2}\int_{\{t=0\}\times B_{2r}} |\Delta_{h,i}u|^2\eta^2 dMdt - \frac{1}{2}\int_{Q_{2r}} |\Delta_{h,i}u|^2\eta^2\partial_t\sigma dMdt\,. \tag{3.18}$$

Use the abbreviation: $u_\lambda = u + \lambda h \Delta_{h,i}u$ for $h > 0$. Noting that

$$\begin{aligned} &\Delta_{h,i}(F(|Du|)g^{\alpha\beta}D_\beta u) = \\ &= (\Delta_{h,i}g^{\alpha\beta})(F(|Du|)D_\beta u)(t, x + he^i) + g^{\alpha\beta}\Delta_{h,i}(F(|Du|)D_\beta u), \end{aligned} \tag{3.19}$$

we have, by (2.8), (2.9) with vectors $P = Du(t, x+he^i)$ and $Q = Du(t, x)$, Schwarz's and Young's inequality,

$$\begin{aligned}
II =& -\int_{Q_{2r}} F(|Du|) g^{\alpha\beta} D_\beta u D_\alpha \Big(\frac{1}{\sqrt{|g|}} \overline{\Delta}_{h,i}\Big(\sqrt{|g|}\sigma\eta^2 \Delta_{h,i} u\Big)\Big) dM dt = \\
=& -\int_{Q_{2r}} (F(|Du|) g^{\alpha\beta} D_\beta u)(t, x+he^i) \Delta_{h,i} u \Delta_{h,i} \Big(\frac{1}{\sqrt{|g|}} D_\alpha \sqrt{|g|}\Big) \sigma\eta^2 dM dt + \\
&+ \int_{Q_{2r}} \Delta_{h,i}(F(|Du|) g^{\alpha\beta} D_\beta u) \cdot D_\alpha(\sigma\eta^2(\Delta_{h,i} u)) dM dt \geq \\
\geq& -\widetilde{\gamma} \int_{Q_{2r}} \sigma(\eta^2 + |D\eta|^2 + |\Delta_{h,i} u|^2)(|Du(t, x+he^i)|^2 + |Du|^2) \cdot \\
&\cdot \int_0^1 \min\{|Du_\lambda|^{p-2}, 1\} d\lambda dM dt + \\
&+ \gamma \int_{Q_{2r}} \sigma\eta^2 |\Delta_{h,i}(Du)|^2 \int_0^1 \min\{|Du_\lambda|^{p-2}, 1\} d\lambda dM dt \,,
\end{aligned} \tag{3.20}$$

where the positive constant $\widetilde{\gamma}$ is depending on the bounds for the metric $(g_{\alpha\beta})$ and the derivative and we used that, by (2.10) with $P = Du(t, x + he^i)$ and $Q = Du_\lambda$, $0 \leq \lambda \leq 1$,

$$\begin{aligned}
&(F(|Du|)|Du|)(t, x + he^i) = \\
&= \int_0^1 F(|Du_\lambda|)|Du_\lambda| d\lambda + \int_0^1 (F(|Du|)|Du|(t, x+he^i) - F(|Du_\lambda|)|Du_\lambda|) d\lambda \leq \\
&\leq \gamma \int_0^1 \min\{|Du_\lambda|^{p-2}, 1\}|Du_\lambda| d\lambda + \\
&\qquad + \gamma h |\Delta_{h,i}(Du)| \int_0^1 (1-\lambda) \int_0^1 \min\{|Du_{\lambda+s(1-\lambda)}|^{p-2}, 1\} ds d\lambda = \\
&= \gamma \int_0^1 \min\{|Du_\lambda|^{p-2}, 1\}|Du_\lambda| d\lambda + \gamma h |\Delta_{h,i}(Du)| \int_\lambda^1 \min\{|Du_s|^{p-2}, 1\} ds \,.
\end{aligned}$$

Using (2.15) with $u = u(t, x + he^i)$ and $v = u(t, x)$, we obtain, from Young's inequality,

$$|III| \leq \gamma(N) C k \int_{Q_{2r}} \sigma\eta^2 |\Delta_{h,i} u|^2 dM dt \,. \tag{3.21}$$

Gathering (3.18)-(3.21) and recalling $|D\eta| \le 4/r$ and $|\partial_t \sigma| \le 4/r^2$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\{t=0\}\times B_{2r}} \sigma\eta^2 |(\Delta_{h,i}u)|^2 dM + \\
&\qquad + \int_{Q_{2r}} \sigma\eta^2 |\Delta_{h,i}(Du)|^2 \int_0^1 \min\{|Du_\lambda|^{p-2},1\} d\lambda dM dt \le \\
&\le \gamma r^{-2}\int_{Q_{2r}} \eta^2 |\Delta_{h,i}u|^2 dM dt + \gamma \int_{(t_0-(2r)^2,t_0)\times B_{3r/2}} \sigma(\eta^2+|D\eta|^2)\cdot \\
&\qquad \cdot (|Du(t,x+he^i)|^2 + |Du|^2 + Ck|\Delta_{h,i}u|^2) dM dt\,.
\end{aligned}$$

Choosing a cut off function σ such that $\operatorname{supp}\sigma = [-(2r)^2, \tau]$ for any $\tau > 0$, we arrive at (3.16).

We now derive local boundedness of the spatial derivative of u in P_{R_M}.

Lemma 3.7 (Local boundedness). *We have $Du \in L^\infty_{\mathrm{loc}}(P_{R_M})$.*

Proof. We claim that a partial differential inequality for $|Du|^2$ holds:

$$\begin{aligned}
&\partial_t\Big(\frac{1}{2}|Du|^2\Big) - \frac{1}{\sqrt{|g|}} D_\alpha\Big\{\sqrt{|g|}\Big(g^{\alpha\gamma}F(|Du|) + \\
&\qquad + H(|Du|)\frac{g^{\alpha\beta}g^{\gamma\bar\gamma}D_\beta u\cdot D_{\bar\gamma}u}{|Du|^2}\Big) D_\gamma\Big(\frac{1}{2}|Du|^2\Big)\Big\} + \\
&\qquad + \gamma\min\{|Du|^{p-2},1\}|D^2u|^2 \le \\
&\le \gamma(M,N)(\min\{|Du|^{p-2},1\}|Du|^2 + k|Du|^2)\,.
\end{aligned} \tag{3.22}$$

Proof of Claim. First we make formal estimation, which is justified later. We assume that $(g_{\alpha\beta}) = Id$, since, in the general case, we have only the term containing derivatives of the metric $(g_{\alpha\beta})$ which is bounded by $\gamma\min\{|Du|^{p-2},1\}|Du|^2$ with a positive constant γ depending only on M, p. We calculate, by (3.2),

$$\begin{aligned}
&\partial_t\Big(\frac{1}{2}|Du|^2\Big) - D_\alpha\Big\{\Big(\delta^{\alpha\gamma}F(|Du|) + H(|Du|)\frac{D_\alpha u\cdot D_\gamma u}{|Du|^2}\Big) D_\gamma\Big(\frac{1}{2}|Du|^2\Big)\Big\} = \\
&\qquad = -F(|Du|)|D^2u|^2 - H(|Du|)\frac{\langle D^2u, Du\rangle^2}{|Du|^2} - \\
&\qquad - Du\cdot D\Big\{Ck\frac{d}{du}\chi(\operatorname{dist}^2(u,N))\Big\} = I + II\,.
\end{aligned} \tag{3.23}$$

By (2.7)

$$I \leq -\gamma \min\{|Du|^{p-2}, 1\}|D^2u|^2 . \tag{3.24}$$

Noting that $\chi' = 0$ if $\operatorname{dist}(u, N) \geq 2\delta_N$ and that, for $u \in \mathcal{O}(N)$,

$$\Big|\operatorname{dist}(u, N)\frac{d^2}{du^2}\operatorname{dist}(u, N)\Big| \leq \gamma(N) ,$$

we have

$$\begin{aligned} II &= -2Ck\Big((\chi' + 2\operatorname{dist}(u, N)\chi'')\big|D\operatorname{dist}(u, N)\big|^2 + \\ &\quad + \chi'\operatorname{dist}(u, N)\sum_{\alpha=1}^{m}\sum_{i,j=1}^{n} D_\alpha u^i D_\alpha u^j \frac{d^2}{du^i du^j}\operatorname{dist}(u, N)\Big) \leq \\ &\leq \gamma(N)Ck|Du|^2 . \end{aligned} \tag{3.25}$$

Substituting (3.24) and (3.25) into (3.23), we obtain the desired estimate.

We proceed with our estimates by Moser's iteration [9]. Put $v = |Du|^2$. Take the test function $\phi = \sigma\eta^2 v^a$, $a \geq 0$, in (3.22), where let r be $0 < r < R_M/2$ and $\eta \in C_0^1(B_{2r})$ be $0 \leq \eta \leq 1$, $\eta = 1$ in B_r and $|D\eta| \leq 4/r$ and σ be a function defined in the proof of Lemma 3.6. Make routine estimate to have, for any τ, $0 < \tau < 1$,

$$\begin{aligned} &\sup_{t_0-(1-\tau^2)r^2<t<t_0} \int_{B_{2r}} v^{a+1}\eta^2 dM + \int_{Q_{2r}} \Big|D\Big(\min\{|Du|^{\frac{p-2}{2}}, 1\}v^{\frac{a+1}{2}}\Big)\Big|^2 \eta^2 dM dt \leq \\ &\leq \gamma(M, N, k, C)\int_{Q_{2r}} v^{a+1}\Big(\min\{|Du|^{(p-2)}, 1\}|D\eta|^2 + \eta|\partial_t\eta|\Big)dM dt. \end{aligned} \tag{3.26}$$

Hölder's and Sobolev's inequality give, for a positive number δ determined later,

$$\begin{aligned} &\int_{B_{2r}} \min\{|Du|^{p-2}, 1\}v^{1+a+\delta}\eta^{2(1+\frac{2}{m})}dM \leq \\ &\leq \Big(\int_{B_{2r}} \Big|D\Big(\min\{|Du|^{\frac{p-2}{2}}, 1\}v^{\frac{a+1}{2}}\eta\Big)\Big|^2 dM\Big)\Big(\int_{B_{2r}} v^{\frac{m\delta}{2}}\eta^2 dM\Big)^{\frac{2}{m}} . \end{aligned} \tag{3.27}$$

Integrating (3.27) in $(-2r^2, 0)$ and applying (3.26) to the resulting inequality, we have

$$
\begin{aligned}
&\int_{Q_{2r}} \min\{|Du|^{p-2},1\} v^{1+a+\delta} \eta^{2(1+\frac{2}{m})} dM dt \leq \\
(3.28)\quad &\leq \gamma(M,N,k,C) \Big\{ \int_{Q_{2r}} v^{a+1} \Big(\min\{|Du|^{p-2},1\} |D\eta|^2 + \eta |\partial_t \eta| \Big) dM dt \Big\}^{1+\frac{2}{m}}.
\end{aligned}
$$

We take $\delta = 2(1+a)/m$ with $a \geq 0$. Then, by (3.28), we have, for any τ, $0 < \tau < 1$,

$$
\begin{aligned}
&\int_{Q_{r(1-\tau)}} v^{(1+a)(1+\frac{2}{m})} dM dt \leq \\
&\leq |Q_{r(1-\tau)}| + \int_{Q_r \cap \{v \geq 1\}} \min\{|Du|^{p-2},1\} v^{1+a+\delta} dM dt \leq \\
(3.29)\quad &\leq |Q_r| + \gamma \Big((\tau r)^{-2} \int_{Q_r} \Big(v^{1+a} + \min\{|Du|^{p-2},1\} v^{1+a} \Big) dM dt \Big)^{1+\frac{2}{m}} \leq \\
&\leq \gamma \Big((\tau r)^{-2} \int_{Q_r} (1 + v^{1+a} + \min\{|Du|^{p-2},1\} v^{1+a}) dM dt \Big)^{1+\frac{2}{m}}.
\end{aligned}
$$

Thus

$$
(3.30)\quad \int_{Q_{r(1-\tau)}} \Big(1 + v^{(1+a)(1+\frac{2}{m})} \Big) dM dt \leq \gamma \Big((\tau r)^{-2} \int_{Q_r} (1 + v^{1+a}) dM dt \Big)^{1+\frac{2}{m}}.
$$

We now define sequences $\{b_\nu\}$, $\{r_\nu\}$ and $\{\phi_\nu\}$ for $\nu = 0, 1, 2, \cdots$, by

$$
(3.31)\quad b_\nu = \Big(1 + \frac{2}{m} \Big)^\nu, \quad r_\nu = \frac{R_M}{2}(1 + 2^{-\nu}), \quad \phi_\nu = \Big(\int_{Q_{r_\nu}} (1 + v^{b_\nu}) dM dt \Big)^{\frac{1}{b_\nu}}.
$$

Then, from (3.30), we obtain

$$
(3.32)\quad \phi_{\nu+1} \leq (\gamma R_M^{-2})^{(1+\frac{2}{m})^{-\nu}} 2^{2(\nu+1)(1+\frac{2}{m})^{-\nu}} \phi_\nu,
$$

where γ depends only on k, C, m, M and N. Starting from $\nu = 0$ and iterating (3.32), we have

$$
(3.33)\quad \sup_{Q_r} v \leq \gamma(M,N,k,C) \frac{1}{|Q_{2r}|} \int_{Q_{2r}} (1+v) dM dt.
$$

Now we formally completed the proof of Lemma 3.7. It remains to justify the above calculations. We argue similarly as in [9], p. 95. By Lemma 3.5, we note that $u \in L^\infty((-t_0,0); W^{1,2}(B_{R_M}, R^n)) \cap W^{1,2}((-t_0,0); L^2(B_{R_M}, R^n))$ and that

$$\int_{Q_{R_M}} \phi \cdot \partial_t u + F(|Du|)\langle Du, D\phi\rangle + Ck\phi \cdot \frac{d}{du}\chi(\mathrm{dist}^2(u,N))dMdt = 0 \tag{3.34}$$

holds for any $\phi \in L^2((-t_0,0); W^{1,2}(B_{R_M}, R^n))$. Since $\partial_t u \in L^2(P_{R_M}, R^n)$ and $\frac{d}{du}\chi(\mathrm{dist}^2(u,N)) \in L^\infty(P_{R_M}, R^n)$, we find from (3.34) that $\Delta_M^f u \in L^2(P_{R_M}, R^n)$ and that

$$\partial_t u - \Delta_M^f u + Ck\frac{d}{du}\chi(\mathrm{dist}^2(u,N)) = 0 \tag{3.35}$$

holds almost everywhere in P_{R_M}. Let take a difference-quotient of (3.35) with respect to a variable x^α, $1 \le \alpha \le m$, and multiply the resulting equation by $\sigma\eta^2\Delta_{h,\alpha}u$ and integrate it in Q_r. Then we make estimation similarly as in (3.26) to have an analogue of (3.26) with $a = 0$. In this stage, since the right hand in (3.26) with $a = 0$ is finite, we take the limit as $h \to 0$ to see that (3.26) is valid for $a = 0$. Using Sobolev embedding theorem, similarly as in (3.26)-(3.30) with $a = 0$, we have (3.30) with $a = 0$ and then find that $v = |Du|^2$ belongs to $L_{\mathrm{loc}}^{1+\frac{2}{m}}(P_{R_M})$. Thus we are able to repeat the above process for $a > 0$, where, for $a, 0 < a < 1$, we use a test function $\sigma\eta^2(v+\epsilon)^a\Delta_{h,\alpha}u$ with a positive number ϵ tending to 0 and we estimate the lower order term similarly as in (3.25). Step by step, we proceed with the above procedure to obtain (3.33).

We claim that u and Du is locally Hölder continuous in P_{R_M}.

LEMMA 3.8. *$u \in C^{\alpha/p,\alpha}(P_{R_M}, R^n)$ for any $\alpha, 0 < \alpha < 1$, where the Hölder constant depends only on $M, p, I(u_0), t_0, \alpha$ and the approximating number k. Also $Du \in C^{\beta/2,\beta}(P_{R_M}, R^{mn})$ with β, $0 < \beta < 1$, depending on m, p and the Hölder constant depending only on M, p, t_0, the approximating number k and L^∞-norm of Du.*

PROOF. We argue similarly as in [2], p. 104, Theorem 1 and [8], p. 245, Theorem 1.1 (see also [7], [9], [10]). Here we can regard our penalized equation as a degenerate parabolic system of p-harmonic type with lower order term of a bounded function, since the function $\frac{d}{du}\chi(\mathrm{dist}^2(u,N))$ defined for $u \in R^n$ is bounded.

4 – Bochner and monotonicity formula

Let $u = u_k$ be a weak solution to (3.2) and (3.3). Let us take $z_0 = (t_0, x_0) \in (0, \infty) \times M$ and argue in the same settings as in Section 3. We recall the definitions of u and R_M and use the notation

$$
\begin{aligned}
e_k(u) &= f(|Du|) + Ck\chi(\mathrm{dist}^2(u, N)), \\
\tilde{e}_k(u) &= f(|Du|) + k\chi(\mathrm{dist}^2(u, N)) .
\end{aligned}
\tag{4.1}
$$

First we state the Bochner formula for the energy density $\tilde{e}(u) = \tilde{e}_k(u_k)$.

Lemma 4.9. *It holds, with a uniform positive constant $\tilde{\gamma}$, γ and $\widetilde{C}$, for any $\phi \in L^2((-t_0, 0); W_0^{1,2}(B_{R_M})) \cap W^{1,2}((-t_0, 0); L^2(B_{R_M}))$ with $\phi \geq 0$ in P_{R_M} and all t_1, t_2, $-t_0 < t_1, t_2 \leq 0$,*

$$
\begin{aligned}
&\int\limits_{\{t\}\times B_{R_M}} \tilde{e}(u)\phi dM\Big|_{t=t_1}^{t=t_2} - \int\limits_{(t_1,t_2)\times B_{R_M}} \tilde{e}(u)\partial_t\phi \, dM dt + \\
&\quad + \int\limits_{(t_1,t_2)\times B_{R_M}} \Big(g^{\alpha\gamma} F(|Du|) + \\
&\quad + H(|Du|)\frac{g^{\alpha\beta} g^{\gamma\bar{\gamma}} D_\beta u \cdot D_{\bar{\gamma}} u}{|Du|^2}\Big) D_\gamma \tilde{e}(u) D_\alpha \phi \, dM dt + \\
&\quad + \tilde{\gamma} \int\limits_{(t_1,t_2)\times B_{R_M}} 2\phi (F(|Du|))^2 g^{\alpha\beta} g^{\gamma\bar{\gamma}} D_\gamma D_\beta u \cdot D_{\bar{\gamma}} D_\alpha u \, dM dt + \\
&\quad + \tilde{\gamma} \int\limits_{(t_1,t_2)\times B_{R_M}} \frac{1}{2}\phi F(|Du|) H(|Du|) \frac{g^{\gamma\bar{\gamma}} D_\gamma |Du|^2 D_{\bar{\gamma}} |Du|^2}{|Du|^2} dM dt + \\
&\quad + \widetilde{C} k^2 \int\limits_{(t_1,t_2)\times B_{R_M}} \phi \Big| \frac{d}{du} \chi(\mathrm{dist}^2(u, N)) \Big|^2 dM dt \leq \\
&\leq \gamma(M, N) \int\limits_{(t_1,t_2)\times B_{R_M}} \phi (\min\{|Du|^{p-2}, 1\})^2 |Du|^2 (1 + |Du|^2) dM dt .
\end{aligned}
\tag{4.2}
$$

Proof. Similarly as the proof of (3.22), we assume that $(g_{\alpha\beta}) = Id$ and make formal calculation. Using Lemmata 3.6, 3.7 and 3.8, we can justify the formal calculation similarly as in [12], pp. 390-393, Lemmata 2.3

and 2.4. For brevity, we put $w = \mathrm{dist}(u, N)$ and, if no confusion arises, we omit the arguments of F, H and χ. First we multiply the both hand side of (3.23) in the proof of Lemma 3.7 by $F(|Du|)$ to have

$$
\begin{aligned}
&\partial_t f(|Du|) - D_\alpha(A^{\alpha\gamma} D_\gamma f(|Du|)\} = \\
&= -F(|Du|)\Big(F(|Du|)|D^2u|^2 + H(|Du|)\frac{\langle D^2u, Du\rangle^2}{|Du|^2}\Big) - \\
&\quad - \frac{H(|Du|)}{|Du|^2} A^{\alpha\gamma} D_\alpha\Big(\frac{1}{2}|Du|^2\Big) D_\gamma\Big(\frac{1}{2}|Du|^2\Big) - \\
&\quad - F(|Du|) Du \cdot D\Big\{Ck\frac{d}{du}\chi\Big\},
\end{aligned}
\tag{4.3}
$$

where we put

$$
A^{\alpha\gamma} = \delta^{\alpha\gamma} F(|Du|) + H(|Du|)\frac{D_\alpha u \cdot D_\gamma u}{|Du|^2}.
$$

Estimate similarly as in (3.25) to see that the third term is bounded by

$$
\gamma(N) F(|Du|) Ck\chi' w |Du|^2 . \tag{4.4}
$$

Use Young's inequality so that (4.4) is bounded by

$$
\frac{C}{2}(k^2(\chi')^2 w^2 + \gamma(N)(F(|Du|))^2|Du|^4). \tag{4.5}
$$

A substitution of (4.5) into (4.3) gives

$$
\begin{aligned}
&\partial_t f(|Du|) - D_\alpha(A^{\alpha\gamma} D_\gamma f(|Du|)\} + \\
&\quad + F(|Du|)\Big(F(|Du|)|D^2u|^2 + H(|Du|)\frac{\langle D^2u, Du\rangle^2}{|Du|^2}\Big) \le \\
&\le \frac{C}{2}\Big(k^2(\chi')^2 w^2 + \gamma(N)(F(|Du|))^2|Du|^4\Big),
\end{aligned}
\tag{4.6}
$$

where we used that

$$
\frac{H\,(|Du|)}{|Du|^2} A^{\alpha\gamma} D_\alpha\Big(\frac{1}{2}|Du|^2\Big) D_\gamma\Big(\frac{1}{2}|Du|^2\Big) \ge 0 .
$$

Next

$$\begin{aligned}\partial_t(k\chi(w^2)) - \operatorname{div}(F(|Du|)D(k\chi(w^2))) = \\ = -Ck^2\Big|\frac{d\chi}{du}\Big|^2 - 2kF(|Du|)(\chi' + 2w^2\chi'')|Dw|^2 - \\ - 2kF(|Du|)w\chi' \sum_{ij=1}^{n} Du^i \cdot Du^j \frac{d^2w}{du^i du^j}\,.\end{aligned} \tag{4.7}$$

By direct calculation, we have

$$\begin{aligned}&\operatorname{div}\Big\{H(|Du|)\frac{1}{|Du|^2}Du \cdot \sum_{\gamma=1}^{m} D_\gamma u D_\gamma(k\chi)\Big\} = \\ &= \sum_{\gamma=1}^{m}\Big(D_\gamma u \cdot \sum_{\alpha=1}^{m} D_\alpha u D_\alpha |Du| (\tau^{-2}H(\tau))'|_{\tau=|Du|} + H(|Du|)\times \\ &\qquad\qquad \times \frac{\operatorname{div}(Du \cdot D_\gamma u)}{|Du|^2}\Big)D_\gamma(k\chi) + \\ &+ H(|Du|)\sum_{\alpha,\gamma=1}^{m}\frac{D_\alpha u \cdot D_\gamma u}{|Du|^2} 2\Big(k(\chi' + 2w^2\chi'')|Dw|^2 + \\ &+ k\chi' w \sum_{k,l=1}^{n} D_\alpha u^k D_\gamma u^l \frac{d^2 w}{du^k du^l} + k\chi' w D_\alpha D_\gamma u \cdot \frac{dw}{du}\Big)\,.\end{aligned} \tag{4.8}$$

By substitution of (4.8) into (4.7), we obtain, from simple calculation,

$$\begin{aligned}&\partial_t(k\chi(w^2)) - D_\alpha(A^{\alpha\gamma}D_\gamma(k\chi(w^2))) = \\ &= -Ck^2\Big|\frac{d\chi}{du}\Big|^2 - 2k(\chi' + 2w^2\chi'')A^{\alpha\gamma}D_\alpha w D_\gamma w - \\ &\quad - 2k\chi' w \sum_{i,j=1}^{n} A^{\alpha\gamma} D_\alpha u^i D_\gamma u^j \frac{d^2 w}{du^i du^j} - \\ &\quad - k\sum_{\gamma=1}^{m}\Big(D_\gamma u \cdot \sum_{\alpha=1}^{m} D_\alpha u D_\alpha |Du| (\tau^{-2}H(\tau))'|_{\tau=|Du|} + \\ &\quad + H\frac{\operatorname{div}(Du \cdot D_\gamma u)}{|Du|^2}\Big)D_\gamma \chi - kH\sum_{\alpha,\gamma=1}^{m}\frac{D_\alpha u \cdot D_\gamma u}{|Du|^2} D_\alpha D_\gamma u \cdot \frac{d\chi}{du} = \\ &= I_1 + I_2 + I_3 + I_4 + I_5\,.\end{aligned} \tag{4.9}$$

Let us evaluate the each term in the right hand in (4.9). If $\operatorname{dist}(u, N) \geq 2\delta_N$, then $\chi' = 0$, and the right hand of (4.9) is equal to zero. We deal with the remaining case. By (2.7), we have, similarly as (4.5),

$$
\begin{aligned}
|I_3| &\le \gamma(N)k\chi' w|Du|^2\min\{|Du|^{p-2},1\} \le \\
&\le \frac{1}{4}k^2(\chi')^2w^2+\gamma(N)(\min\{|Du|^{p-2},1\})^2|Du|^4\,,
\end{aligned}
\tag{4.10}
$$

where we used that $\chi'\ge 0$ and that, for $u\in\mathcal{O}(N)$ and $u\notin N$, with a positive constant $\gamma(N)$ depending on a bound for the curvature of N, $\frac{d^2w}{du^2}\le\gamma(N)$. Noting the growth of f, we have, by Schwarz's and Young's inequality,

$$
\begin{aligned}
|I_4+I_5| &\le k\Big|\frac{d\chi}{du}\Big|(3H|D^2u|+|(\tau^{-2}H(\tau))'|_{\tau=|Du|}||Du|^3|D|Du||)\le \\
&\le \gamma k\Big|\frac{d\chi}{du}\Big||D^2u|\min\{|Du|^{p-2},1\}\le \\
&\le \frac{1}{4}(\min\{|Du|^{p-2},1\})^2|D^2u|^2+\gamma k^2\Big|\frac{d\chi}{du}\Big|^2\,.
\end{aligned}
\tag{4.11}
$$

Combining (4.10) and (4.11) with (4.9) and adding the resulting inequality by (4.6), we arrived at (4.2), where, since the third term in the left hand side of (4.6) is bounded below by

$$
\gamma(\min\{|Du|^{p-2},1\})^2|D^2u|^2\,,
$$

the first term in (4.11) is absorbed into this quantity and we choose a positive constant C to be large dependently only of M,N and p and then absorb the second term in (4.11) into the first term I_1 in (4.9).

Secondly we derive the monotonicity type inequality (refer to [4], [24]). This is a crucial estimate to obtain the partial regularity. Let $\phi\in C_0^\infty(B_{R_M})$ be a cut-off function such that $0\le\phi\le 1$ and $\phi=1$ in some neighborhood of B_{R_M}. Then we define

$$
\begin{aligned}
\Phi(R,u) &= R^2\int_{\{t=-R^2\}\times R^m} e_k(u)G\phi^2\sqrt{|g|}dx \quad \text{for}\quad 0<R\le\sqrt{t_0}\\
\Psi(R,u) &= \int_{-(2R)^2}^{-R^2}\int_{\{t\}\times R^m} e_k(u)G\phi^2\sqrt{|g|}dxdt \quad \text{for}\quad 0<R\le\sqrt{\frac{t_0}{2}}
\end{aligned}
$$

where, with $\omega=2(p-1)$,

$$
G(t,x)=(4\pi(-t))^{-\frac{m}{2}}\exp\Big(-\frac{|x|^2}{2\omega(-t)}\Big)\quad\text{for}\ -\infty<t<0\,.
$$

Lemma 4.10. *There exist positive constants γ and ϵ, $0<\epsilon<1$, depending only on M, N and p such that, for any R_0, R_1, $0<R_0\le R_1<\sqrt{t_0}$,*

$$\Phi(R_0,u)\le \exp(\gamma(R_1^{1-\epsilon}-R_0^{1-\epsilon}))\Phi(R_1,u)+\gamma I(u_0)(R_1-R_0) \tag{4.12}$$

and, for any R_0, R_1, $0<R_0\le R_1<\sqrt{t_0/2}$,

$$\Psi(R_0,u)\le \exp(\gamma(R_1^{1-\epsilon}-R_0^{1-\epsilon}))\Psi(R_1,u)+\gamma I(u_0)(R_1-R_0)\,. \tag{4.13}$$

Proof. We give the proof of (4.12). (4.13) is similarly proven. Let R be $0<R<\sqrt{t_0}$. For brevity, we assume that $(g_{\alpha\beta})=Id$. In the general case, we can argue similarly as in [4], pp. 97-99. Using a scaling transformation: $t=R^2s$, $x=Ry$ and putting $u_R(s,y)=u(R^2s,Ry)$, and $\phi_R(y)=\phi(Ry)$, the equation (3.2) in $(-t_0,0)\times B_{R_M}$ is rewritten as follows: In $(-t_0/R^2,0)\times B_{R_M/R}$,

$$\partial_s u_R-\operatorname{div}(F(R^{-1}|Du_R|)Du_R)+CkR^2\frac{d}{du}\chi(\operatorname{dist}^2(u_R,N))=0\,. \tag{4.14}$$

Also note that

$$\begin{aligned}\Phi(R,u)=R^2\int_{\{s=-1\}\times R^m}&\{f(R^{-1}|Du_R|)+\\ &+Ck\chi(\operatorname{dist}^2(u_R,N))\}G\phi_R^2dy\,.\end{aligned} \tag{4.15}$$

We now calculate $\frac{d}{dR}\Phi(R,u)|_R$ for any R, $0<R<\sqrt{t_0}$. We demonstrate only formal calculation, since we can justify it by testing (3.35) by $Gx\cdot Du$ and $Gt\partial_t u$ with the usual cutoff function (refer to [13], Proposition 10). If no confusion arises, we omit writing the arguments of functions χ, f and F.

$$\begin{aligned}&\frac{d}{dR}\Phi(R,u)|_R=\\ &=2R\int_{\{s=-1\}\times R^m}\Big\{f-\frac{R^{-1}|Du_R|}{2}f'+Ck\chi\Big\}G\phi_R^2\,dy+\\ &\quad+\int_{\{s=-1\}\times R^m}\Big\{FD\frac{du_R}{dR}\cdot Du_R+CkR^2\frac{du_R}{dR}\cdot\frac{d\chi}{du}\Big\}G\phi_R^2\,dy+\\ &\quad+2R^2\int_{\{s=-1\}\times R^m}e_k(u)G\phi_R\,y\cdot(D\phi)(R\cdot)dy=I_1+I_2+I_3\,.\end{aligned} \tag{4.16}$$

We now make estimation of I_1. Split the integrations into two parts:

$$I_1 = 2R\Big(\int_{\{s=-1\}\times R^m\cap\{R^{-1}|Du_R|<1\}} \cdots + \int_{\{s=-1\}\times R^m\cap\{R^{-1}|Du_R|\geq 1\}} \cdots\Big) \tag{4.17}$$

By the definition of f and Young's inequality with (2.2), the second term of (4.17) is bounded below by

$$-\frac{p(p-2)}{2}R^2\int_{\{s=-1\}\times R^m} fG\phi_R^2 dy - \frac{p-2}{2}\int_{\{s=-1\}\times R^m} G\phi_R^2 dy\,. \tag{4.18}$$

Thus we have

$$\begin{aligned} I_1 \geq & -\frac{p(p-2)}{2}R^2\int_{\{s=-1\}\times R^m} fG\phi_R^2 dy - \\ & -\Big(R+\frac{p-2}{2}\Big)\int_{\{s=-1\}\times R^m} G\phi_R^2 dy\,. \end{aligned} \tag{4.19}$$

Using (4.14) and $D_\alpha G = -y^\alpha G/\omega(-s)$, we have, by integration by parts,

$$\begin{aligned} I_2 = & \int_{\{s=-1\}\times R^m} \frac{d}{dR}u_R\cdot\Big\{-\Delta_M^f u_R + CkR^2\frac{d\chi}{du}\Big\}G\phi_R^2 dy - \\ & -\int_{\{s=-1\}\times R^m} F\frac{du_R}{dR}\cdot Du_R\cdot D(G\phi_R^2)dy \geq \\ \geq & \frac{1}{2R}\int_{\{s=-1\}\times R^m} |2s\partial_s u_R + y\cdot Du_R|^2 G\phi_R^2 dy - \\ & -\frac{1}{4R}\int_{\{s=-1\}\times R^m}\Big|1-\frac{2F}{\omega}\Big|^2 |y\cdot Du_R|^2 G\phi_R^2 dy\,. \end{aligned} \tag{4.20}$$

Noting that

$$\Big|\tau\Big(1-\frac{2F(\tau)}{\omega}\Big)\Big| \leq \begin{cases} \dfrac{p-2}{p-1}, & \tau\geq 1 \\ \dfrac{p}{p-1}, & 0\leq\tau\leq 1\,, \end{cases}$$

we have

$$I_2 \geq -\frac{p^2}{2(p-1)^2}R\int_{\{s=-1\}\times R^m} |y|^2 G\phi_R^2 dy\,. \tag{4.21}$$

Exactly similarly as in [3], p. 10 (also refer to [4], pp. 98-99), we evaluate (4.21) and I_3.

$$(4.22)\quad I_2 \geq -\gamma(p,m,R_M)R(1+R^{-2\delta})\,.$$
$$|I_3| \leq \Phi(R,u) + R^2\int_{\{s=-1\}\times R^m} e_k(u)G|y\cdot(D\phi)(R\cdot)|^2\sqrt{|g_R|}dy \leq$$
$$\leq \Phi(R,u)+\gamma(m,M)I(u_0)\,.$$

From combination of (4.19) and (4.22) with (4.16), we obtain

$$(4.23)\quad \frac{d}{dR}\Phi(R,u) \geq -\gamma(p)\Phi(R,u) - \gamma(m,p,M)(1+I(u_0)+R(1+R^{-\delta}))$$

from which the desired estimate follows, if δ is selected to be so small.

Finally we prove ϵ-regularity theorem for weak solutions u to (3.2) and (3.3). We note Lemmata 3.7 and 3.8 and recall R_M, $u=u_k$ and the notation: $\tilde{e}_k(u) = f(|Du|) + k\chi(\mathrm{dist}^2(u,N))$.

LEMMA 4.11 (ϵ-regularity theorem). *There exist a positive constant ϵ_0 depending only on $I(u_0)$, M, N and p such that, for any weak solution u to* (3.2) *and* (3.3), *the following holds: If, for R, $0<R<\min\{R_M,1\}$, there holds*

$$(4.24)\qquad \Psi(R,u) = \int_{-(2R)^2}^{-R^2}\int_{\{t\}\times R^m} e_k(u)G\phi^2 dMdt < \epsilon_0\,,$$

then, with a uniform positive constant γ,

$$(4.25)\qquad \sup_{Q_{R/2}} e_k(u) \leq \gamma R^{-2}\,.$$

PROOF. We argue similarly as in [4], [24]. By our monotonicity formula (4.12) and (4.13) and the smallness condition (4.24), we have, for positive numbers r,σ, $0<r,\sigma<R$ and $r+\sigma<R$, and $z_0\in P_r$,

$$(4.26)\qquad \sigma^{-m}\int_{Q_\sigma(z_0)} e_k(u)dMdt \leq \epsilon\,.$$

Since $u \in C^0\left((-t_0, 0); C^1(B_{R_M})\right)$, there exists $\sigma_0, 0 \le \sigma_0 < R$, such that

$$(R-\sigma_0)^2 \sup_{Q_{\sigma_0}} \tilde{e}_k(u) = \max_{0 \le \sigma \le R} \{(R-\sigma)^2 \sup_{Q_\sigma} \tilde{e}_k(u)\}\,. \tag{4.27}$$

Here, if $\sigma_0 = R$, the desired estimate (4.25) follows. We can also choose $(t_0, x_0) \in |\,\overline{Q}_{\sigma_0}\,|$ such that

$$\sup_{Q_{\sigma_0}} \tilde{e}_k(u) = \tilde{e}_k(u)(t_0, x_0)\,.$$

Set $\tilde{e}_0 = \tilde{e}_k(u)(t_0, x_0)$ and $\rho_0 = (1/2)(R-\sigma_0)$. By choice of σ_0 and (t_0, x_0),

$$\sup_{Q_{\rho_0}(t_0,x_0)} \tilde{e}_k(u) \le \sup_{Q_{\sigma_0+\rho_0}} \tilde{e}_k(u) \le 4\tilde{e}_0\,. \tag{4.28}$$

Introduce

$$r_0 = \rho_0\sqrt{\tilde{e}_0}, \quad v(s,y) = u\Big(t_0 + \frac{s}{\tilde{e}_0}, x_0 + \frac{y}{\sqrt{\tilde{e}_0}}\Big) \tag{4.29}$$

and we use the original notation $g_{\alpha\beta}(\cdot)$ for $g_{\alpha\beta}(\cdot/\sqrt{\tilde{e}_0})$. Let us show that $r_0 \le 1$. First note by (3.2) in Q_{ρ_0} that v satisfies, almost everywhere in Q_{r_0},

$$\partial_s v - \frac{1}{\sqrt{|g|}} D_\alpha\Big(F(\sqrt{\tilde{e}_0}|Dv|)\sqrt{|g|}g^{\alpha\beta}D_\beta v\Big) + \frac{Ck}{\tilde{e}_0}\frac{d}{dv}\chi(\mathrm{dist}^2(v,N)) = 0. \tag{4.30}$$

Moreover (4.28) and (4.29) imply that

$$\bar{e}_{\tilde{k}}(v)(0,0) = 1, \quad \sup_{Q_{r_0}} \bar{e}_{\tilde{k}}(v) \le 4\,, \tag{4.31}$$

where we put

$$\bar{e}_{\tilde{k}}(v) = \frac{1}{\tilde{e}_0} f\Big(\sqrt{\tilde{e}_0}|Dv|\Big) + \tilde{k}\chi\left(\mathrm{dist}^2(v,N)\right), \quad \tilde{k} = \frac{k}{\tilde{e}_0}\,.$$

Similarly as in the proof of Lemma 4.2, we have Bochner type estimate for $\bar{e}_{\tilde{k}}(v)$. For brevity, we put $\bar{e}(v) = \bar{e}_{\tilde{k}}(v)$. We see that v satisfies, for

$\phi \in L^2((-(r_0)^2, 0);\, W_0^{1,2}(B_{r_0})) \cap W^{1,2}\ ((-(r_0)^2, 0);\, L^2(B_{r_0}))$ with $\phi \geq 0$ in Q_{r_0} and all intervals $(t_1, t_2) \subset (-(r_0)^2, 0)$,

$$
\begin{aligned}
&\int_{\{s\}\times B_{r_0}} \bar{e}(v)\phi dM\Big|_{s=t_1}^{s=t_2} - \int_{(t_1,t_2)\times B_{r_0}} \bar{e}(v)\partial_s\phi dM ds +\\
&\quad + \int_{(t_1,t_2)\times B_{r_0}} \Big(g^{\alpha\gamma}F(V) +\\
&\qquad\qquad + H(V)\frac{g^{\alpha\beta}g^{\gamma\bar{\gamma}}D_\beta v\cdot D_{\bar{\gamma}}v}{|Dv|^2}\Big)D_\gamma\bar{e}(v)D_\alpha\phi dM ds +\\
&\quad + \tilde{\gamma}\int_{(t_1,t_2)\times B_{r_0}} \phi F(V)\Big(2F(V)g^{\alpha\beta}g^{\gamma\bar{\gamma}}D_\gamma D_\beta v\cdot D_{\bar{\gamma}}D_\alpha v +\\
&\qquad\qquad + H(V)\frac{g^{\gamma\bar{\gamma}}D_\gamma|Dv|^2 D_{\bar{\gamma}}|Dv|^2}{2|Dv|^2}\Big)dM ds +\\
&\quad + \widetilde{C}\tilde{k}^2\int_{Q_{2r}} \phi\Big|\frac{d\chi}{dv}\Big|^2 dM ds \leq\\
&\leq \gamma(M,N)\int_{(t_1,t_2)\times B_{r_0}} \phi(\min\{V^{p-2},1\})^2|Dv|^2\Big(\frac{1}{\tilde{e}_0} + |Dv|^2\Big)dM ds\,,
\end{aligned}
\tag{4.32}
$$

where $V = \sqrt{\tilde{e}_0}|Dv|$ and we omit writing the arguments of χ. Assume that $r_0 > 1$. Then we can derive Harnack type estimate from (4.32) (see Appendix for the proof).

Lemma 4.12 (Harnack estimate). *There exists a positive constant γ depending only on M,N and p such that the inequality holds:*

$$
\sup_{Q_{1/2}} \bar{e}(v) \leq \gamma\Big(\frac{1}{|Q_1|}\int_{Q_1} \bar{e}(v) dM ds\Big)^{\frac{1}{1+A}}\,,
\tag{4.33}
$$

where $A = (m+1)(1-2/p) > 0$.

Noting that $r_0 > 1$ implies $\sigma_0 + 1/\sqrt{\tilde{e}_0} \leq \sigma_0 + \rho_0 < R$ and adapting (4.26) with $\sigma = 1/\sqrt{\tilde{e}_0}$, we have, by scaling back,

$$
\int_{Q_1} \bar{e}(v) dM ds = (\tilde{e}_0)^{\frac{m}{2}}\int_{Q_{1/\sqrt{\tilde{e}_0}(t_0,x_0)}} \tilde{e}_k(u) dM dt \leq \epsilon\,,
\tag{4.34}
$$

where note that the constant C in the density $e_k(u)$ is sufficiently large, but depending only on M, N and p. We can choose a positive number ϵ

to be small dependently only of m, p, M and N and then, obtain the contradiction from (4.31), (4.33) with (4.34). Finally, exactly similarly as in [4], pp. 92-93, we conclude (4.25).

5 – Proof of Theorem

From our energy inequality (3.14), we know that

$$\{Du_k\} \text{ is bounded in } L^\infty((0,\infty); L^2(M, R^{mn}))\,, \tag{5.1}$$

$$\{\partial_t u_k\} \text{ is bounded in } L^2((0,\infty) \times M, R^n)\,. \tag{5.2}$$

Thus we choose a subsequence $\{u_k\}$ and a map $u \in L^\infty((0,\infty);\ W^{1,2}(M,R^n)) \cap W^{1,2}((0,\infty); L^2(M, R^n))$ such that, as $k \to \infty$,

$$Du_k \to Du \quad \text{weakly* in } L^\infty((0,\infty); L^2(M, R^{mn}))\,, \tag{5.3}$$

$$\partial_t u_k \to \partial_t u \quad \text{weakly in } L^2((0,\infty) \times M, R^n)\,. \tag{5.4}$$

By (5.1), (5.2) and Lemma 2.4 with $q = 2$, we find that

$$\begin{aligned} u_k \to u \ \text{ strongly in } & L^2_{\mathrm{loc}}((0,\infty); L^2(M, R^n)) \\ & \text{and almost everywhere in } (0,\infty) \times M\,. \end{aligned} \tag{5.5}$$

Again, by (3.14), we have

$$\operatorname{dist}(u_k, N) \to 0 \quad \text{in } L^2_{\mathrm{loc}}((0,\infty); L^2(M))\,. \tag{5.6}$$

From (5.5) with (5.6) and (3.14) with (5.3) and (5.4), we obtain that

$$u \in N \quad \text{almost everywhere in } (0,\infty) \times M\,, \tag{5.7}$$

$$\sup_{0 \le T < \infty} \left(\int_{(0,T)\times M} |\partial_t u|^2 dM dt + I(u(T)) \right) \le I(u_0)\,, \tag{5.8}$$

where we used the convexity of f.

We now define a singular set for the weak limit u which is obtained as above. Let ϵ_0 be a constant determined in Lemma 4.11. Then let

$$\Sigma = \Big\{ z_0 = (t_0, x_0) \in (0,\infty) \times M :$$

$$\liminf_{k\to\infty} \int_{t_0-(2R)^2}^{t_0-R^2} \int_{\mathcal{B}_{R_M}} e_k(u_k) G_{z_0} \phi^2 dM dt \geq \epsilon_0 \tag{5.9}$$

$$\text{for any } R, 0 < R < \sqrt{t_0}/2 \Big\}.$$

For $t_0 \in (0,\infty)$ and R, $0 < R < \sqrt{t_0}/2$, let

$$\begin{aligned} \Sigma_R^{t_0} &= \Big\{ x_0 \in M : \liminf_{k\to\infty} \int_{t_0-(2R)^2}^{t_0-R^2} \int_{\mathcal{B}_{R_M}} e_k(u_k) G_{z_0} \phi^2 dM dt \geq \epsilon_0 \Big\}, \\ \Sigma^{t_0} &= \cap_{0<R<\sqrt{t_0}/2} \Sigma_R^{t_0}. \end{aligned} \tag{5.10}$$

Then $\Sigma = \cup_{t_0 \in (0,\infty)} \Sigma^{t_0}$. We argue with Lemma 4.10 similarly as in [24] and [5] to find that

$$\Sigma \text{ and } \Sigma^{t_0} \text{ are closed for any } t_0 \in (0,\infty) \tag{5.11}$$

and to obtain an estimation on Hausdorff measure of Σ and Σ^{t_0}.

$$\mathcal{H}^m_{\mathrm{loc}}(\Sigma) < \infty, \quad \mathcal{H}^{m-2}(\Sigma^{t_0}) < \infty \text{ for any } t_0 \in (0,\infty). \tag{5.12}$$

Now we prove that the limit u is weak solution to (1.3) and (1.4). We argue similarly as in [4], pp. 93-96.

First we show that u satisfies (1.3) almost everywhere in a local region Q around a point z_0 in the complement of the singular set Σ.

For $z_0 \notin \Sigma$, we take a positive number R, $0 < R < \sqrt{t_0}/2$, and an infinite sequence u_k such that

$$\int_{(t_0-(2R)^2, t_0-R^2)\times\mathcal{B}_{R_M}} e_k(u_k) G_{z_0} \phi^2 dM dt \leq \epsilon_0. \tag{5.13}$$

We see from Lemma 4.11 that, with $Q = Q_{R/2}(z_0)$ and a positive constant γ depending only on R,

$$|Du_k|, \quad k\,\mathrm{dist}^2(u_k, N) \leq \gamma \;\text{ in }\; Q, \tag{5.14}$$

which implies that there exists a subsequence u_k such that, as $k \to \infty$,

$$u_k \to u \quad \text{in } C^0(Q, R^n)\,, \tag{5.15}$$

$$Du_k \to Du \quad \text{weakly* in } L^\infty_{\mathrm{loc}}(Q, R^{mn})\,. \tag{5.16}$$

We claim that

$$\Big\{k\frac{d}{du}\chi(\mathrm{dist}^2(u_k, N))\Big\} \ \text{ is bounded in } L^2_{\mathrm{loc}}(Q, R^n)\,. \tag{5.17}$$

Choosing a test function ϕ^2 with $\phi \in C_0^\infty(Q)$ in the Bochner formula (4.2), we have, by Young's inequality, with a uniform positive constant γ,

$$\begin{aligned}
&2\int_Q \phi^2 \min\{|Du|^{p-2}, 1\}|D^2u|^2 dz+\\
&\quad+\frac{\widetilde{C}}{2}\int_Q \phi^2 k^2\Big|\frac{d}{du}\chi(\mathrm{dist}^2(u,N))\Big|^2 dMdt \le\\
&\le \gamma\int_Q \min\{|Du|^{p-2}, 1\}|Du|^2\{(1+|Du|^2)\phi^2 + |D\phi|^2\}dz+\\
&\quad+\int_Q \tilde{e}(u)\partial_t\phi^2 dMdt\,,
\end{aligned} \tag{5.18}$$

where we used (2.8). Adopting (5.14) for (5.18), we obtain (5.17), where note that the positive constant $\widetilde{C}$ does not depend on k. From (3.35) with (5.2) and (5.17), it follows that

$$\{\Delta^f_M u_k\} \text{ is bounded in } L^2_{\mathrm{loc}}(Q, R^n)\,. \tag{5.19}$$

By (5.19), we choose a subsequence $\{u_k\}$ and a function $\mathcal{A} = (\mathcal{A}^i) \in L^2_{\mathrm{loc}}(Q, R^n)$ such that

$$\Delta^f_M u_k \to \mathcal{A} \quad \text{weakly in } L^2_{\mathrm{loc}}(Q, R^n)\,. \tag{5.20}$$

Then we observe from (5.20) that

$$\Delta^f_M u = \mathcal{A}^i, \quad 1 \le i \le n, \quad \text{almost everywhere in } Q\,. \tag{5.21}$$

We here use the compactness result.

LEMMA 5.13. *Let $\{u_k\}$ be a family of functions u_k, $k = 1, 2, \cdots$, defined on Q with values in R^n. Suppose that each u_k satisfies, in the weak sense,*

$$\partial_t u_k - \Delta_M^f u_k = f_k \,. \tag{5.22}$$

Moreover, assume that $\{u_k\}$ is bounded in $L^\infty((t_0 - (R/2)^2, t_0); W^{1,2}(B_{R/2}(x_0), R^n))$, $\{\partial_t u_k\}$ is bounded in $L^2(Q, R^n)$ and $\{f_k\}$ is bounded in $L^2(Q, R^n)$. Then we can choose a subsequence $\{u_k\}$ and a function u defined on Q with value in R^n such that

$$\Delta_M^f u_k \to \Delta_M^f u \text{ weakly in } \big(L^2\big((t_0-(R/2)^2, t_0); W^{1,2}(B_{R/2}(x_0), R^n)\big)\big)^*. \tag{5.23}$$

PROOF OF LEMMA 5.13. Noting that $f \in C^2([0,\infty))$ is a convex function, we use Lemmata 2.1 and 2.2 and argue similarly as [1], pp. 31-33, Theorem 2.1, to obtain a subsequence $\{u_k\}$ and a vector-valued function u such that, with $V_k(s) = Du + s(Du_k - Du)$, $0 \le s \le 1$,

$$\begin{aligned} &\int_{Q'} |Du_k - Du|^q \times \\ &\qquad \times \Big(\int_0^1 \min\{|V_k(s)|^{p-2}, 1\} ds\Big) dM dt \to 0 \quad \text{as } k \to \infty \end{aligned} \tag{5.24}$$

holds for any q, $1 \leqq q < 2$, and for all Q' compactly contained in Q. By (2.9) in Lemma 2.2 and Hölder's inequality, we have, for any $\phi \in C_0^\infty(Q, R^n)$ with $Q' = \operatorname{supp}\phi$ and all q, $1 < q < 2$,

$$\begin{aligned} &\int_Q (F(|Du_k|)Du_k - F(|Du|)Du) \cdot D\phi \, dM dt \le \\ &\le \gamma\Big(\int_Q |D\phi|^{\frac{q}{q-1}} dM dt\Big)^{1-\frac{1}{q}} \Big(\int_{Q'} |Du_k - Du|^q \times \\ &\qquad \times \Big(\int_0^1 \min\{|V_k|^{p-2}, 1\} ds\Big) dM dt\Big)^{\frac{1}{q}} \longrightarrow 0 \quad \text{as } k \to 0\,. \end{aligned} \tag{5.25}$$

Thus we have (5.23). Using (5.4), (5.17), (5.20) and (5.21), we can take the limit as $k \to \infty$ in (3.2). Then there exist a smooth unit vector field ν_N defined on N with $\nu_N(v) \perp T_v(N)$ in R^n for all $v \in N$ and a real valued function λ defined on Q with $\lambda \in L^2(Q, R)$ such that u satisfies

$$\partial_t u - \Delta_M^f u + \lambda \nu_N(u) = 0 \quad \text{almost everywhere in } Q\,. \tag{5.26}$$

We now observe that u and Du are locally Hölder continuous in Q. Multiplying the equation (5.26) by $\nu_N(u)$ and noting that $\partial_t u \cdot \nu_N(u) = 0$ and $Du \cdot \nu_N(u) = 0$ almost everywhere in Q, we have, for almost every $z \in Q$,

$$\begin{aligned} \lambda &= -\partial_t u(z) \cdot \nu_N(u(z)) + \Delta_M^f u(z) \cdot \nu_N(u(z)) = \\ &= -F(|Du|)\langle Du, D\nu_N(u(z))\rangle\,, \end{aligned} \tag{5.27}$$

where we used that $D_\alpha(\sqrt{|g|}g^{\alpha\beta}D_\beta u \cdot \nu_N(u)) = 0$ almost everywhere in Q. Thus we have

$$|\lambda| \leq \gamma(N)F(|Du|)|Du|^2\,. \tag{5.28}$$

Combining (5.28) with (5.26) and noting the growth of F, we argue similarly as in [2], p. 104, Theorem 1, [8], p. 245, Theorem 1 and p. 291 (see also [9], [10]) to find that $u \in C^{\alpha/p,\alpha}(Q)$ and $Du \in C^{\alpha/2,\alpha}(Q)$ with a positive number α, $0 < \alpha < 1$, where we use the energy inequality (5.8) and the uniform local boundedness of Du in Q, which is obtained from combination of (5.14) with (5.16).

Noting that, for any $w \in N$,

$$\sum_{i,j=1}^{n} \frac{d\nu^j}{dw^i}(w)V^iW^j = -\nu(w) \cdot A(w)(V, W) \quad \text{for any } V, W \in T_wN\,,$$

we see from (5.27) that

$$\lambda \nu_N(u) = F(|Du|)g^{\alpha\beta}A(u)(D_\alpha u, D_\beta u) \text{ almost everywhere in } Q. \tag{5.29}$$

Finally, we conclude, exactly similarly as in [4], pp. 95-96, that u is a weak solution to (1.3) and (1.4).

6 – Appendix

In this section we prove Lemma 4.12. We use the same notations as in the proof of Lemma 4.11.

PROOF OF LEMMA 4.12. Take a cylinder $Q_{r,\tau} \subset Q_{r_0}$ of vertex at origin and positive numbers σ_1, σ_2, $0 < \sigma_1, \sigma_2 < 1$. Let $\eta \in C_0^1(B_r)$ be $\eta = 1$ in $B_{(1-\sigma_1)r}$ with $|D\eta| \leq 2/(\sigma_1 r)$ and $\sigma \in C_0^1((-\tau, 0])$ be $\sigma = 1$ in $(-(1-\sigma_2)\tau, 0]$ with $\partial_t \sigma \leq 2/(\sigma_2 \tau)$. For positive numbers l, ϵ, $0 < \epsilon + l < \min\{\sigma_2, 1-\sigma_2\}$, we define $\sigma_l^\epsilon \in C_0^1((-\tau+\epsilon, -\epsilon))$ to be $\sigma_l^\epsilon = 1$ in $(-(1-\sigma_2)\tau, -\epsilon - l)$. Let us denote the regularization on time variable by

$$d_h\, f(t) = \frac{1}{h}\int_h^{t+h} f(\tau)d\tau\,, \quad \overline{d_h} f(t) = \frac{1}{h}\int_{t-h}^{t} f(\tau)d\tau\,.$$

For an arbitrary small number h, $0 < h < \epsilon$, we substitute a valid test function $\phi = \bar{d}_h\, (\eta^2 \sigma_l^\epsilon\, (d_h \bar{e}(v))^a)$ with $a > 0$ into (4.32) and, similarly as in the proof of Lemma 3.7, we let h tend to zero and then l, ϵ to zero in the resulting inequality to have, with $\bar{e} = \bar{e}(v)$,

$$\begin{aligned}
&\frac{1}{a+1}\int_{B_r\times\{t=0\}} \bar{e}^{a+1}\eta^2 dM - \frac{1}{2}\int_{Q_{r,\tau}} \bar{e}^{a+1}\eta^2 \partial_t \sigma dM dt + \\
&\quad + \int_{Q_{r,\tau}} \Big(g^{\alpha\gamma}F(V) + H(V)\frac{g^{\alpha\beta}g^{\gamma\bar{\gamma}}D_\beta v\cdot D_{\bar{\gamma}} v}{|Dv|^2}\Big) D_\gamma \bar{e} D_\alpha(\bar{e}^a \eta^2 \sigma) dM dt + \\
&\quad + \tilde{\gamma}\int_{Q_{r,\tau}} \sigma\eta^2 \bar{e}^a F(V)\Big(2F(V)g^{\alpha\beta}g^{\gamma\bar{\gamma}}D_\gamma D_\beta v\cdot D_{\bar{\gamma}}D_\alpha v + \\
&\qquad\qquad + \frac{1}{2}H(V)\frac{g^{\gamma\bar{\gamma}}D_\gamma|Dv|^2 D_{\bar{\gamma}}|Dv|^2}{|Dv|^2}\Big) dM dt + \\
&\quad + \widetilde{C}\tilde{k}^2\int_{Q_{r,\tau}} \sigma\eta^2\bar{e}^a\Big|\frac{d\chi}{du}\Big|^2 dM dt \leq \\
&\leq \gamma\int_{Q_{r,\tau}} \sigma\eta^2 \bar{e}^a (\min\{V^{p-2}, 1\})^2 |Dv|^2\Big(\frac{1}{\tilde{e}_0} + |Dv|^2\Big) dM dt\,.
\end{aligned} \tag{6.1}$$

We make estimation of each term in both hand side of (6.1). First we deal with the left hand side. By (2.7) with $P = Du$, $Q = D\bar{e}$ and Schwarz's and Young's inequality, the integrated function in the third term is bounded below by

$$\frac{\gamma}{2}\min\{V^{p-2}, 1\}\bar{e}^{a-1}|D\bar{e}|^2\eta^2 + 8\gamma\min\{V^{p-2}, 1\}\bar{e}^{a+1}|D\eta|^2\,. \tag{6.2}$$

Similarly, the fourth term is estimated below by

$$\gamma(\min\{V^{p-2},1\})^2|D^2v|^2\bar{e}^a\,. \tag{6.3}$$

Since, by our assumption $r_0 > 1$,

$$\frac{1}{\sqrt{\tilde{e}_0}} < \rho_0 < 1\,, \tag{6.4}$$

we have, by (2.2) and (4.31),

$$\bar{e}^a(\min\{V^{p-2},1\})^2|Dv|^2\Big(\frac{1}{\tilde{e}_0}+|Dv|^2\Big) \le p\,4^{1-\frac{2}{p}}(1+4p)\bar{e}^{\,a+\frac{2}{p}}\,. \tag{6.5}$$

A substitution of (6.2)-(6.5) into (6.1) gives

$$\begin{aligned}\frac{1}{a+1}&\int_{B_r\times\{t=0\}}\bar{e}^{a+1}\eta^2 dM+\\ &+\frac{\gamma}{2}\int_{Q_{r,\tau}}\min\{V^{p-2},1\}\bar{e}^{a-1}|D\bar{e}|^2\eta^2\sigma dMdt+\\ &+\tilde{\gamma}\int_{Q_{r,\tau}}\sigma\eta^2\bar{e}^a\Big(\gamma(\min\{V^{p-2},1\})^2|D^2v|^2+\widetilde{C}\tilde{k}^2\Big|\frac{d\chi}{du}\Big|^2\Big)dMdt\le\\ \le&\frac{1}{a+1}\int_{Q_{r,\tau}}\bar{e}^{a+1}\eta^2\partial_t\sigma dMdt+\gamma\int_{Q_{r,\tau}}\sigma\big(|D\eta|^2\bar{e}^{a+1}+\eta^2\bar{e}^{a+\frac{2}{p}}\big)dMdt\,.\end{aligned} \tag{6.6}$$

Here choose the cutoff function σ with $\operatorname{supp}\sigma\subset(-\tau,\bar{t}\,]$ for each $\bar{t}$, $-(1-\sigma_2)\tau\le\bar{t}\le 0$, to have the inequality (6.6) with replacing 0 by $\bar{t}$. We set $w=\bar{e}(v)$. By Hölder's and Sobolev's inequality (see [21], (3.4), p. 75; also refer to [7], [8], pp. 232-233), we have, for any $\delta>0$,

$$\begin{aligned}&\int_{Q_{r,(1-\sigma_2)\tau}}\eta^{2\left(1+\frac{2}{m}\right)}w^{\delta+a+2-\frac{2}{p}}dMdt\le\\ &\le\int_{-(1-\sigma_2)\tau}^{0}\Big(\int_{\{t\}\times B_r}(\eta^2w^{a+2-\frac{2}{p}})^{\frac{m}{m-2}}dM\Big)^{\frac{m-2}{m}}\times\\ &\quad\times\Big(\int_{\{t\}\times B_r}\eta^2w^{\frac{\delta m}{2}}dM\Big)^{\frac{2}{m}}dt\le\\ &\le\gamma(m)\int_{Q_{r,(1-\sigma_2)\tau}}\Big|D\big(\eta w^{\frac{a}{2}+1-\frac{1}{p}}\big)\Big|^2dMdt\times\\ &\quad\times\Big(\sup_{-(1-\sigma_2)\tau\le t\le 0}\int_{\{t\}\times B_r}\eta^2w^{\frac{\delta m}{2}}dM\Big)^{\frac{2}{m}}\,.\end{aligned} \tag{6.7}$$

Noting that, by (2.2), (4.31) and (6.4),

$$|Dw^{\frac{a}{2}+1-\frac{1}{p}}|^2 \le 2^{3-\frac{4}{p}} p\Big(\frac{a}{2}+1-\frac{1}{p}\Big)^2 w^a\Big(\big(\min\{V^{p-2},1\}\big)^2 \big|D^2 v\big|^2 + \tilde{k}^2\Big|\frac{d\chi}{dv}\Big|^2\Big)$$

and applying (6.6) with $a+1 = \delta m/2$ for the resulting inequality, we have

$$\begin{aligned}&\int_{Q_{(1-\sigma_1)r,(1-\sigma_2)\tau}} w^{\delta+a+2-\frac{2}{p}} dM dt \le \\ &\le \gamma(p)(1+a)^2\Big(\int_{Q_{r,\tau}} w^{a+\frac{2}{p}}(1+|D\eta|^2+\partial_t\sigma) dM dt\Big)^{1+\frac{2}{m}}.\end{aligned} \tag{6.8}$$

Let us define the sequence $\{a_i\}$, $i = 0,1,2,\cdots$, by the relation

$$\begin{cases} a_{i+1} = \delta + a + 2 - \dfrac{2}{p} \\ a_i = a + \dfrac{2}{p}. \end{cases}$$

Then, for initial value $a_0 > 2/p$,

$$a_i + A = \Big(1+\frac{2}{m}\Big)^i (a_0 + A), \quad A = (m+1)\Big(1-\frac{2}{p}\Big)$$

and we also define sequences $\{r_i\}$ and $\{\tau_i\}$, $i = 0,1,2,\cdots$, by

$$r_i = \frac{1}{2}(1+2^{-i}), \quad \tau_i = r_i^2. \tag{6.9}$$

In (6.8), we choose $Q_{(1-\sigma_1)r,(1-\sigma_2)\tau} = Q_{r_{i+1},\tau_{i+1}}$ and $Q_{r,\tau} = Q_{r_i,\tau_i}$ and make routine calculation to have

$$\begin{aligned}&\frac{1}{|Q_{r_{i+1},\tau_{i+1}}|}\int_{Q_{r_{i+1},\tau_{i+1}}} w^{a_{i+1}} dM dt \le \\ &\le \gamma(m,p) a_i^2 2^{i\gamma(m)}\Big(\frac{1}{|Q_{r_i,\tau_i}|}\int_{Q_{r_i,\tau_i}} w^{a_i} dM dt\Big)^{1+\frac{2}{m}}.\end{aligned} \tag{6.10}$$

By Moser's iteration, we obtain, from (6.10) and $a_0 > 2/p$,

$$\sup_{Q_{1/2}} w \le \gamma(m,p)\Big(\frac{1}{|Q_1|}\int_{Q_1} w^{a_0} dM dt\Big)^{\frac{1}{a_0+A}}. \tag{6.11}$$

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