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Weak solution to evolution problems of harmonic maps from noncompact manifolds

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RIASSUNTO: *In questo lavoro, si costruisce una soluzione debole per la equazione di tipo termico delle mappe armoniche da una varietà non-compatta fino a una sfera, usando il metodo dei movimenti minimizzanti con l'aiuto della energia relativa.*

ABSTRACT: *In this paper we construct a weak solution for the heat-type equation of harmonic maps from a noncompact manifold into a sphere using the so-called minimizing movement method with the aid of the relative energy.*

1 – Introduction

This paper is concerned with weak solutions to the evolution problems of harmonic maps from noncompact Riemannian manifolds into spheres.

To construct solutions for differential equations with time-variable, Rothe's time-discretization method has been used in various context. In 1971, REKTORYS [18] combined the time-discretization method and the direct method of calculus of variations to construct solutions of parabolic equations. Roughly speaking, their method is summarized as follows. For

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the equation

$$\frac{\partial u}{\partial t} - \Big(\text{the Euler-Lagrange equation of } \int_\Omega f(x,u,Du)d\mu\Big) = 0\,, \tag{1.1}$$

they consider the auxiliary variational functionals

$$G_n(u) = \int_\Omega \Big\{\frac{|u-u_{n-1}|^2}{h} + f(x,u,Du)\Big\}d\mu\,, \tag{1.2}$$

and define u_n successively as a minimizer of $G_n(u)$. Using the sequence $\{u_n\}$, they construct approximate solutions and prove that the approximate solutions converge to a solution of (1.1) as $h \to 0$. In [18] existence of weak solutions of linear parabolic equations was proved.

The method mentioned above was rediscovered by Kikuchi [12]. In [12] parabolic systems associated to the certain variational functionals are studied. Recently, such methods are used for many problems: evolution problems of harmonic maps, Navier-Stokes equations ([14]), semilinear hyperbolic systems ([15], [16], [17], [21]), etc. For evolution problems of harmonic maps into spheres, weak solutions were constructed by Bethuel-Coron-Ghidaglia-Soyeur [2] on cylindrical domains, and by the author [22] on non-cylindrical domains.

On the other hand, in [7], De Giorgi introduced a new concept of *minimizing movement*. Let us see his definition in [7].

Definition 1.1 (De Giorgi). Let S be a topological space. Let $F : ((1,+\infty)\times\mathbb{Z}\times S\times S) \to \overline{\mathbb{R}}$ and $u : \mathbb{R} \to S$; we say that u is a generalized minimizing movement associated to F, S, and we write $u \in GMM(F,S)$, if there exists a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ such that $\lim \lambda_i = +\infty$ and a sequence $\{w_i\}_{i\in\mathbb{N}}$ of functions $w_i : \mathbb{Z} \to S$ such that for any $t \in \mathbb{R}$

$$\lim_{i\to+\infty} w_i([\lambda_i t]) = u(t)$$

and for any $i \in \mathbb{N}, k \in \mathbb{Z}$

$$F(\lambda_i, k, w_i(k+1), w_i(k)) = \min_{s\in S} F(\lambda_i, k, s, w_i(k))\,.$$

If we set

$$F(\lambda, k, u, v) = \int_\Omega \{\lambda|u-v|^2 + f(x,u,Du)\}\,dx\,,$$

we can see that G_n of (1.2) corresponds to F in Definition 1.1. Moreover, for a suitable choice of S, $u \in GMM(F,S)$ will be a solution of (1.1) in some sense.

De Giorgi's minimizing movement provides new fields of vision for calculus of variations and differential equations. Especially, in [1] and [10], we can find remarkable use of minimizing movements combined with the geometric measure theory. See also [11] in which abstract evolution equations on non-cylindrical domain are treated.

Now, let us consider evolution problems of harmonic maps.

Let $M = (M^m, g)$ be a smooth Riemannian m-manifold and S^{n-1} a sphere $\{u : u \in \mathbb{R}^n, |u| = 1\} \subset \mathbb{R}^n$. Throughout this paper, $x = (x^1, \dots, x^m)$ denotes a local coordinate system on M, $(g_{\alpha\beta}(x))$ a metric tensor with respect to the coordinates $(x^\alpha)_{1\le\alpha\le m}$ and $(g^{\alpha\beta}(x)) = (g_{\alpha\beta}(x))^{-1}$. For C^1-map $u = (u^1(x), \dots, u^n(x)) : \Omega \to S^{n-1} \subset \mathbb{R}^n$, the *energy density* $e(u)(x)$ of u at $x \in M$ is defined as

$$e(u)(x) = \frac{1}{2}\|Du(x)\|^2 = \frac{1}{2} g^{\alpha\beta}(x) D_\alpha u(x) \cdot D_\beta u(x)\,,$$

where "$\cdot$" denotes the inner product of $\mathbb{R}^n$. For a bounded domain Ω, the *energy* of u on Ω is given by

$$\mathcal{E}(u;\Omega) = \int_\Omega e(u)\,d\mu\,, \tag{1.3}$$

where $d\mu = \sqrt{\det(g_{\alpha\beta})(x)}dx$ stands for the volume element of M The Euler-Lagrange equation of the energy functional $\mathcal{E}$ is given by

$$\Delta_M u^i + u^i\|Du\|^2 = 0 \quad \text{in} \quad \Omega \quad \text{for} \quad i = 1, \dots, n\,. \tag{1.4}$$

Here and in the sequel, Δ_M denotes the Laplace-Beltrami operator of M. Moreover, we denote by $\tau(u)$ the left-hand side of the above equation, and call it the *tension field* of u. A (weak) solution of (1.4) is called a (*weakly*) *harmonic map*.

We consider the following equation on a noncompact Riemannian manifold M:

$$\frac{\partial u}{\partial t} - \Delta_M u - u\|Du\|^2 = 0 \tag{1.5}$$

which is associated to evolution problems of harmonic maps into spheres. For the case that the source manifold M is a compact manifold without boundary, CHEN [4] constructed weak solutions of (1.5) using penalty method and Galerkin's method. The result of [4] was generalized by CHEN-STRUWE [5] to arbitrary compact target manifold. Moreover, in [5], partial regularity of the weak solutions is also shown. For the case that M has nonempty boundary, see [6]. On the other hand, as mentioned before, in [2] and in [22] weak solutions of (1.5) are constructed by the method which we will employ in this paper.

For the case that the sectional curvatures of the target manifold are nonpositive, the evolution problem of harmonic map is firstly considered by EELLS-SAMPSON [9]. Therefore the evolution equation of harmonic maps are called as *Eells-Sampson equations*, and (1.5) is a special case of Eells-Sampson equations. For a target manifold with nonpositive sectional curvatures, LI-TAM [13] proved the existence of the unique smooth solution for Eells-Sampson equation on a noncompact complete manifold whose Ricci curvature are bounded below by a negative constant.

In this paper, $H^{k,l}(M,\mathbb{R}^n)$ and $H^{k,l}_{\text{loc}}(M,\mathbb{R}^n)$ denote the standard Sobolev spaces of $\mathbb{R}^n$-valued maps defined on M. $H^{k,l}_0(M,\mathbb{R}^n)$ stands for the closure of $C^\infty_0(M,\mathbb{R}^n)$ in $H^{k,l}(M,\mathbb{R}^n)$.

In order to construct weak solutions of (1.5) on the noncompact manifold M using the minimizing movement method, we have to consider variational problems on noncompact domains. Therefore, in general, we should seek locally minimizing maps which have infinite total energies. To find locally energy minimizing weakly harmonic maps on noncompact manifolds, DING [8] introduced a new concept of *relative energy* E_w with respect to a given map $w \in C^2$. The relative energy is defined for $u \in Y_w = \{v \in H^{1,2}_{\text{loc}}(M,S^{n-1}); v-w \in H^{1,2}_0(M,\mathbb{R}^n)\}$ as follows:

$$E_w(u) = \frac{1}{2}\int_M \|D(u-w)\|^2\,d\mu - \int \Delta_{{}_M} w\cdot u d\mu\,.$$

For the case that w has finite total energy, $u \in Y_w$ also has finite energy and $E_w(u)$ is nothing other than the difference of the energies of u and w, $\mathcal{E}(u;M)-\mathcal{E}(w;M)$. Therefore a E_w-minimizer minimizes $\mathcal{E}$ also, and so it will be weakly harmonic. On the other hand, $E_w(u)$ is well-defined for some class of w with infinite total energy. Thus, we can find weakly harmonic maps with infinite total energy.

Now, let us state our main result.

Theorem 1.2. *Let $M = (M^m, g)$ be a complete Riemannian manifold. Assume that the spectrum of the Laplace-Beltrami operator Δ_M of M has a positive infimum $\lambda(M) > 0$, and that $w \in H^{2,2}_{\mathrm{loc}}(M, S^{n-1})$ satisfies*

$$\lim_{R\to\infty} \|Dw\|_{L^\infty(M\setminus B_R(p_0))} \to 0 \quad \text{for some fixed } p_0 \in M \tag{1.6}$$

$$|\tau(w)| \in L^2(M)\,. \tag{1.7}$$

Then, for any $u_0 \in \{u \in H^{1,2}_{\mathrm{loc}}(M, S^{n-1}); u - w \in H^{1,2}_0(M, \mathbb{R}^n)\}$, there exist a weak solution $u(x,t)$ of (1.5) *which satisfies the conditions*

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in M, \tag{1.8}$$

$$u(x,t) - w(x) \in H^{1,2}_0(M, \mathbb{R}^n) \quad \text{for all} \quad t > 0\,. \tag{1.9}$$

Remark. If M is a simply connected complete manifold whose sectional curvatures are bounded below and above by two negative constants, then the assumption $\lambda(M) > 0$ is satisfied. Moreover, for such a case, we can define the geometric boundary $S(\infty)$ of M, and the condition (1.9) can be considered as a boundary condition at infinity, $u(x,t) = w(x)$ on $S(\infty)$.

2 – Construction of weak solutions

Assume that $w \in H^{1,2}_{\mathrm{loc}}(M, S^{n-1})$ satisfies the conditions (1.6) and (1.7). Let us define the following subsets of $H^{1,2}_0(M, \mathbb{R}^n)$:

$$\begin{aligned}
H^{1,2}_{\mathrm{loc}}(M, S^{n-1}) &= \{u \in H^{1,2}_{\mathrm{loc}}(M, \mathbb{R}^n)\,;\, |u(x)| = 1 \text{ a.e. } x \in M\},\\
X_w &= \{f \in H^{1,2}_0(M, \mathbb{R}^n)\,;\, f + w \in H^{1,2}_{\mathrm{loc}}(M, S^{n-1})\},\\
Y_w &= \{u \in H^{1,2}_{\mathrm{loc}}(M, S^{n-1})\,;\, u - w \in H^{1,2}_0(M, \mathbb{R}^n)\}\,.
\end{aligned}$$

We consider a functional $F_w : (0,1) \times X_w \times X_w \to \mathbb{R}$ defined as

$$F_w(h,f,g) = \int_M \Big\{\frac{|f-g|^2}{2h} + \frac{1}{2}\|Df\|^2 - \Delta_M w \cdot f\Big\} d\mu\,, \tag{2.1}$$

where $d\mu$ stand for the volume element on M. Given $g \in X_w$, because of Lemma 2.2, $\widetilde{F}_w$ is well-defined and coercive for $f \in X_w$.

For $u \in Y_w$, we have $u-w \in X_w$. Therefore we can define a functional $\widetilde{F}_w : (0,1) \times Y_w \times Y_w \to \mathbb{R}$ by

$$\widetilde{F}_w(h,u,v) = F_w(h, u-w, v-w)\,. \tag{2.2}$$

Moreover, for $v, u \in Y_w$ and $\Omega \subset\subset M$, put

$$\mathcal{F}_w(h,u,v,\Omega) = \int_\Omega \Big\{ \frac{|u-v|^2}{2h} + \frac{1}{2}\|Du\|^2 \Big\} d\mu\,.$$

Then, it is easy to see that for $v, u \in Y_w$ and $\psi \in C_0^\infty(M, \mathbb{R}^n)$ with $\operatorname{spt}\psi \subset \Omega$

$$\begin{aligned} &\widetilde{F}_w(h, u+\psi, v) - \widetilde{F}_w(h,u,v) = \\ &\quad = \mathcal{F}_w(h, u+\psi, v, \Omega) - \mathcal{F}_w(h,u,v,\Omega)\,. \end{aligned} \tag{2.4}$$

On the other hand, given $v \in Y_w$, $u \in Y_w$ minimizes $\widetilde{F}_w(h,\cdot,v)$ if and only if $u-w$ minimizes $F_w(h,\cdot,v-w)$. Therefore, if $u-w$ minimizes $F_w(h,\cdot,v)$, then u is a local minimizer of $\mathcal{F}_w$ and satisfies the Euler-Lagrange equation of $\mathcal{F}_w$. Thus we have the following lemma.

Lemma 2.1. *Let $v \in Y_w$, $h > 0$ be given. If $f \in X_w$ is a minimizer of $F_w(h,\cdot,v-w)$, then $u = f + w$ satisfies*

$$\begin{aligned} \int_{\operatorname{spt}\psi} \Big\{ &\frac{u-v}{h}\cdot\psi - \frac{1}{2h}|u-v|^2 u\cdot\psi + \\ &+ g^{\alpha\beta} D_\alpha u \cdot D_\beta \psi - \|Du\|^2 u\cdot\psi \Big\} d\mu = 0\,, \end{aligned} \tag{2.5}$$

for all $\psi \in C_0^\infty(M, \mathbb{R}^n)$.

Proceeding as in [8], we can prove the coercivity of F_w.

Lemma 2.2. *Assume that the spectrum of the Laplace-Beltrami operator of M has a positive infimum $\lambda(M)$, and that $w \in H^{2,2}_{\mathrm{loc}}(M, S^{n-1})$*

satisfies (1.6) *and* (1.7). *Then there exist positive constants* C_0, C_1, C_2 *and* C_3 *depending only on* $\lambda(M)$ *and* w *such that*

$$\begin{aligned}(2.6)\qquad &\int_M \Big(\frac{|f-g|^2}{2h} + C_0\|Df\|^2 + C_1\Big)d\mu \geq \\ &\geq F_w(h,f,g) \geq \int_M \Big(\frac{|f-g|^2}{2h} + C_2\|Df\|^2 - C_3\Big)d\mu\end{aligned}$$

for all $g, f \in X_w$.

Proof. It is enough to estimate the term $\int_M \Delta_M w \cdot f d\mu$. We have

$$(2.7)\qquad \int_M \Delta_M w \cdot f d\mu = \int_M \tau(w)\cdot f d\mu - \int_M w\|Dw\|^2 \cdot f d\mu\,.$$

Since the assumption on the spectrum of Δ_M means that

$$(2.8)\qquad \lambda(M)\|f\|^2_{L^2(M)} \leq \|Df\|^2_{L^2(M)} \quad \text{for all} \quad f\in C_0^\infty(M)\,,$$

we can estimate the first term of the right hand side of (2.7) as

$$\begin{aligned}(2.9)\qquad &\int_M \tau(w)\cdot f d\mu \leq \|\tau(w)\|_{L^2(M)}\|f\|_{L^2(M)} \leq \\ &\leq \lambda^{-\frac{1}{2}}\|\tau(w)\|_{L^2(M)}\|Df\|_{L^2(M)}\,.\end{aligned}$$

About the second term of (2.7), using the relation $|w|^2 = |w+f|^2 = 1$, we have

$$(2.10)\qquad |w\cdot f| = \frac{1}{2}\,\big||w+f|^2 - |w|^2 - |f|^2\big| = \frac{1}{2}|f|^2 \leq 1\,.$$

Therefore, we obtain

$$\begin{aligned}(2.11)\qquad &\int_M |w\|Dw\|^2\cdot f|\, d\mu = \frac{1}{2}\int_M \|Dw\|^2|f|^2 d\mu \leq \\ &\leq \frac{1}{2}\int_{M\setminus B_R} \|Dw\|^2|f|^2 d\mu + \frac{1}{2}\int_{B_R}\|Dw\|^2|f|^2 d\mu \leq \\ &\leq \frac{1}{2}\varepsilon(R)\int_{M\setminus B_R}|f|^2 d\mu + \frac{1}{2}\int_{B_R}\|Dw\|^2|f|^2 d\mu \leq \\ &\leq \frac{1}{2}\varepsilon(R)\int_{M\setminus B_R}|f|^2 d\mu + C(R)\,,\end{aligned}$$

where $\varepsilon(R) = \|Dw\|^2_{L^\infty(M\setminus B_R)}$ and $C(R) = \int_{B_R} \|Dw\|^2 d\mu$. From (2.7), (2.8), (2.9) and (2.11) we get

$$\begin{aligned}\int_M |\Delta_M w \cdot f|\, d\mu \le \frac{1}{2}\varepsilon(R)\lambda^{-1}(M)\|Df\|^2_{L^2} + \\ + \lambda^{-\frac{1}{2}}(M)\|Df\|_{L^2} + C(R)\,.\end{aligned} \tag{2.12}$$

Now, because of the assumption (1.6) on w, we can choose $R > 0$ so that $\varepsilon(R)$ is sufficiently small in (2.12). Thus we get (2.6). □

As mentioned in [8], the estimates for $\int \Delta_M w \cdot f d\mu$ imply the lower semi-continuity of F_w.

LEMMA 2.3. *Let w be as in Lemma 2.2. Then for any $g \in X_w$, $F_w(h, \cdot, g)$ is lower semi-continuous with respect to the weak topology in X_w.*

PROOF. For a given $f \in X_w$, take a sequence $\{f_k\} \subset H_0^{1,2}(M, \mathbb{R}^n)$ such that

$$f(x) + w(x) + f_k(x) \in S^{n-1} \quad \text{for a.e. } x \in M \tag{2.13}$$

$$f_k \to 0 \quad \text{weakly in} \quad H^{1,2}(M, \mathbb{R}^n)\,. \tag{2.14}$$

It is enough to show that

$$\int |\Delta_M w \cdot f_k| d\mu \to 0 \text{ as } k \to \infty\,. \tag{2.15}$$

Putting $\sup_k \|f_k\|_{H^{1,2}} = K$ and proceeding as in (2.11), we can see that

$$\int_M |w\|Dw\|^2 \cdot f_k|\, d\mu \le \frac{1}{2}\varepsilon(R)K + \frac{1}{2}\int_{B_R} \|Dw\|^2 |f_k|^2 d\mu\,. \tag{2.16}$$

By taking R to be large enough, we can take the first term to be arbitrarily small. Let us see that the second term tends to 0. Using Rellich's compactness theorem on B_R, we can see that $f_k \to 0$ strongly in $L^2(B_R, \mathbb{R}^n)$, and hence $f_k(x) \to 0$ for almost every $x \in B_R$. On the other hand, (2.13) implies that $|f| < 2$. Therefore, we can use Lebesgue's convergence theorem to see that $\int_{B_R} \|Dw\|^2 |f_k|^2 d\mu \to 0$ as $n \to \infty$ for any $R > 0$. Thus, we obtain

$$\int_M |w\|Dw\|^2 \cdot f_k|\, d\mu \to 0 \text{ as } k \to \infty\,. \tag{2.17}$$

Using (2.7), (2.17) and the assumption (1.7), we get (2.15). ∎

Lemmas 2.2 and 2.3 guarantee the existence of minimizers of $F_w(h,\cdot,g)$ in the class X_w. Thus, given $f_0^h = f_0 \in X_w$, we can define $f_k^h \in X_w$ for $k \geq 1$ successively by

$$F_w(h, f_k^h, f_{k-1}^h) = \min_{f \in X_w} F_w(h, f, f_{k-1}^h)\,. \tag{2.18}$$

Proposition 2.4. *Let M and w be as in Theorem* 1.2*, and let f_0 be in the class X_w. Then, for $\{f_k^h\}$ defined by* (2.18)*, we have the following energy-type estimate:*

$$\begin{aligned}\sum_{k=1}^{N} \int_M \frac{1}{2h} \left|f_k^h - f_{k-1}^h\right|^2 d\mu + \int_M \frac{1}{2} \left\|Df_N^h\right\|^2 d\mu \leq \\ \leq C(M, w, f_0) \text{ for all } N \in \mathbb{N}\,.\end{aligned} \tag{2.19}$$

Proof. Since f_k^h is a minimizer, we have $F_w(h, f_k^h, f_{k-1}^h) \leq F_w(h, f_{k-1}^h, f_{k-1}^h)$ and so

$$\begin{aligned}\int_M \Big[\frac{1}{2h} \left|f_k^h - f_{k-1}^h\right|^2 + \frac{1}{2} \left\|Df_k^h\right\|^2 - \Delta_M w \cdot f_k^h\Big] d\mu \leq \\ \leq \int_M \Big[\frac{1}{2} \left\|Df_{k-1}^h\right\|^2 - \Delta_M w \cdot f_{k-1}^h\Big] d\mu\,.\end{aligned} \tag{2.20}$$

Summing up the above estimate from $k = 1$ to N, we get

$$\begin{aligned}\sum_{k=1}^{N} \int_M \frac{|f_k^h - f_{k-1}^h|^2}{2h} d\mu + \int_M \Big\{\frac{1}{2}\|Df_k^h\|^2 - \Delta_M w \cdot f_k^h\Big\} d\mu \leq \\ \leq \int_M \Big\{\frac{1}{2}\|Df_0\|^2 - \Delta_M w \cdot f_0\Big\} d\mu\,.\end{aligned} \tag{2.21}$$

Now, taking into account (2.12), we can deduce (2.19) from (2.21). ∎

Now, using Proposition 2.4, we can prove our main theorem.

Proof of Theorem 1.2. Let M, w and u_0 as in Theorem 1.2. Put $f_0 = u_0 - w$, and define f_k^h by (2.18) successively.

Let us define

$$\begin{cases} \bar{f}_h(x,t) = \begin{cases} f_0(x) & \text{for}\, t = 0\,, \\ f_k^h(x) & \text{for}(k-1)h < t \le kh,\ k \ge 1\,, \end{cases} \\ f_h(x,t) = \dfrac{t-(k-1)h}{h} f_k^h(x) + \dfrac{kh-t}{h} f_{k-1}^h(x) \\ \qquad\qquad\qquad \text{for}\ (k-1)h < t \le kh,\ k \ge 1. \end{cases} \tag{2.22}$$

Then Proposition 2.4 gives us the following energy estimate.

$$\int_0^T \int_M \frac{1}{2} \|D_t f_h\|^2 \, d\mu\, dt \le C(w, u_0, M) \quad \text{for all} \quad T > 0\,. \tag{2.23}$$

From the above estimate we can deduce that

$$\begin{aligned} & f_h \to f \text{ weakly in } H^{1,2}(M \times (0,T), \mathbb{R}^n)\,, \\ & \bar{f}_h \to \bar{f} \text{ weakly in } L^2(M \times (0,T), \mathbb{R}^n) \text{ and } L^\infty((0,T); H_0^{1,2}(M, \mathbb{R}^n)), \end{aligned} \tag{2.24}$$

for some $f \in H^{1,2}((0,T) \times M, \mathbb{R}^n)$ and $\bar{f} \in L^\infty((0,T) \times H_0^{1,2}(M, \mathbb{R}^n)$. Moreover, as in [22], the estimate (2.23) implies that $f = \bar{f}$. In fact

$$\int_0^T \int_M |f_h - \bar{f}_h|^2 d\mu\, dt \le \int_0^T \int_M h^2 \|D_t f_h\|^2 \, d\mu\, dt \le h^2 TC \to 0 \text{ as } h \to 0. \tag{2.25}$$

Using Rellich's compactness theorem for f_h on every $\Omega \subset\subset M$ and taking into account (2.25), we can see

$$f_h,\, \bar{f}_h \to f \text{ strongly in } L^2(\Omega \times (0,T)) \ \forall\, \Omega \subset\subset M\,. \tag{2.26}$$

Now, put

$$\begin{aligned} u_k^h &= f_k^h + w, \quad u_h = f_h + w\,, \\ \bar{u}_h &= \bar{f}_h + w \qquad u = f + w\,. \end{aligned}$$

Then, (2.24) and (2.26) imply that

$$\begin{aligned} u_h \to u \quad & \text{weakly in} \quad H^{1,2}(M \times (0,T), \mathbb{R}^n)\,, \\ \bar{u}_h \to u \quad & \text{weakly in} \quad L^\infty((0,T) \times H^{1,2}(M, \mathbb{R}^n)) \\ u_h,\, \bar{u}_h \to u \quad & \text{strongly in} \quad L^2(\Omega \times (0,T)) \ \forall\, \Omega \subset\subset M\,. \end{aligned} \tag{2.27}$$

We can also see that since $f \in L^\infty((0,T) \times H_0^{1,2}(M, \mathbb{R}^n))$,

$$u \in L^\infty((0,T) \times Y_w)\,.$$

On the other hand, by Lemma 2.1 we have

$$\int_{\operatorname{spt}\psi} \Big[\frac{u_k^h - u_{k-1}^h}{h} \cdot \psi - \frac{1}{2h}|u_k^h - u_{k-1}^h|^2 u_k^h \cdot \psi + \\ + g^{\alpha\beta} D_\alpha u_k^h \cdot D_\beta \psi - \|Du_k^h\|^2 u \cdot \psi\Big] d\mu = 0$$

for all $\psi \in C_0^\infty(M, \mathbb{R}^n)$, and therefore

$$\int_0^T \int_{\operatorname{spt}\psi} \Big[D_t u_h(x,t) \cdot \psi(x) - \frac{h}{2}\|D_t u_h(x,t)\|^2 \bar{u}_h(x,t)\psi(x) + \\ + g^{\alpha\beta}(x) D_\alpha \bar{u}_h(x,t) \cdot D_\beta \psi(x) + \\ - \|D\bar{u}_h(x,t)\|^2 \bar{u}_h(x,t) \cdot \psi(x)\Big] \eta(t) d\mu\, dt = 0$$

for all $\psi \in C_0^\infty(M, \mathbb{R}^n)$ and $\eta \in C_0^\infty(0,T)$. Since the set of all finite linear combinations of maps of the form $\psi(x)\eta(t)$ is dense in $H_0^{1,2}(M \times (0,T))$, we get

$$\int_0^T \int_M \Big[D_t u_h \cdot \varphi(x,t) - \frac{h}{2}\|D_t u_h\|^2 \bar{u}_h \varphi(x,t) + \\ + g^{\alpha\beta} D_\alpha \bar{u}_h \cdot D_\beta \varphi(x,t) - \|D\bar{u}_h\|^2 \bar{u}_h \cdot \varphi(x,t)\Big] d\mu\, dt = 0 \tag{2.29}$$

for all $\varphi \in C_0^\infty(M \times (0,T))$. By completion, (2.29) holds for any $\varphi \in H_0^{1,2} \cap L^\infty(\Omega \times (0,T))$ for every $\Omega \subset\subset M$. Taking $\varphi^i = \bar{u}_h^j \Phi_{ij}$ for $(\Phi_{ij}) \in C_0^\infty(M \times (0,T), \mathbb{R}^{n^2})$ with $\Phi_{ij} = -\Phi_{ji}$, we obtain

$$\int_0^T \int_M [D_t u_h^i \bar{u}_h^j \Phi_{ij} + D_\alpha \bar{u}_h^i D_\beta \Phi_{ij} \bar{u}_h^j] d\mu\, dt = 0\,. \tag{2.30}$$

Now, letting $h \to 0$ in (2.30) and taking subsequence if necessary, we get

$$\int_0^T \int_M [D_t u^i u^j \Phi_{ij} + g^{\alpha\beta} D_\alpha u^i u^j D_\beta \Phi_{ij}] d\mu\, dt = 0 \tag{2.31}$$

for all $(\Phi_{ij}) \in C_0^\infty(M \times (0,T), \mathbb{R}^{n^2})$ with $\Phi_{ij} = -\Phi_{ji}$ (2.31) tells us that $u(x,t)$ satisfies

$$D_t u \wedge u - \frac{1}{\sqrt{g}} D_\beta \left(\sqrt{g} g^{\alpha\beta} D_\alpha u \wedge u\right) = 0$$

weakly. Consequently, using the argument in [4] or [20] which is familiar to us now, we conclude that u is a weak solution of the equation

$$D_t u - \Delta_M u - u\|Du\|^2 = 0\,.$$

Since $u_h(x,0) = u_0(x)$ for all $h > 0$ and $x \in M$, we have $u(x,0) = u_0(x)$ by the continuity of trace operator. Moreover, (2.23) implies that

$$\bar{f}_h(\cdot,t) \to \tilde{f}_t(\cdot) \quad \text{weakly in} \quad H^{1,2}(M,\mathbb{R}^n) \quad \text{for every } t\,, \tag{2.32}$$

for some $\tilde{f}_t \in H_0^{1,2}(M,\mathbb{R}^n)$. Using Rellich's theorem, we can see that $\bar{f}_h$ converges to $\tilde{f}_t$ strongly in $L^2(\Omega,\mathbb{R}^n)$ for any $\Omega \subset\subset M$. On the other hand, (2.26) implies that $\tilde{f}_h(\cdot,t)$ converges to $f(\cdot,t)$ strongly in $L^2(\Omega)$ for almost all t. Therefore, in the Sobolev spaces, it is not necessary to distinguish $f(x,t)$ from $\tilde{f}_t(x)$. Thus, we can say that $u(\cdot,x) - w(\cdot) = f(\cdot,t) = \tilde{f}_t(\cdot) \in H_0^{1,2}(M,\mathbb{R}^n)$ for every t. □

Remark. The last part of the above proof shows also that

$$f^h_{[t/h]+1}(x) = \bar{f}_h(\cdot,t) \to f(\cdot,t) \quad \text{weakly in} \quad H^{1,2}(M,\mathbb{R}^n) \quad \text{for every} \quad t\,.$$

Therefore, if we extend $f(x,t)$ for $t < 0$ by f_0, $f(x,t)$ is a generalized minimizing movement associated to

$$F(\lambda,k,f,g) = \begin{cases} F_w\Big(\dfrac{1}{\lambda}, f, g\Big) & \text{for} \quad k \ge 1, \\ \displaystyle\int_M |f - f_0|^2 d\mu & \text{for} \quad k < 0\,, \end{cases}$$

and $S = X_w$ with the weak $H^{1,2}$-topology.

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