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The skew derivative problem for the Helmholtz equation outside cuts in a plane

P.A. KRUTITSKII

RIASSUNTO: *Si studia l'equazione di Helmholtz nel piano con tagli usando la teoria del potenziale ed il metodo delle equazioni integrali al contorno. Il problema è riportato ad una equazione integrale singolare di Cauchy e quindi ad una equazione di Fredholm di seconda specie. Sono discusse le applicazioni alla diffrazione delle onde lunghe oceaniche.*

ABSTRACT: *The skew derivative problem for the propagative Helmholtz equation outside cuts in a plane is studied by potential theory and the boundary integral equation method. The problem is reduced to the Cauchy singular integral equation and then to the Fredholm integral equation of the second kind, which is uniquely solvable. Applications to the diffraction of long ocean waves are discussed.*

1 – Introduction

In the present paper we study the skew derivative problem for the 2-D Helmholtz equation outside cuts in a plane. The skew derivative problems for an open boundary are too complicated to be effectively studied by a classical approach, while Dirichlet and Neumann problems were actively studied [4], [5], [8], [16], [17], [18], [24], [25], [28], [29]. To solve the 2-D skew derivative problem outside cuts by a classical approach, we

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must look for a solution of this problem by a linear combination of single and double layer potentials, because the problem can not be solved by only one of them. In this way we arrive at a very complicated system of boundary integro-differential equations. The system contains hyper-singular integrals, Cauchy singular integrals, compact operators and the derivative of the densitiy of the double-layer potential. Clearly, the system is too complicated to be studied by standard methods. The basic lack of the classical approach is so, that single and double layer potentials have different orders of singularities at the boundary. In the present paper we suggest to solve the 2-D skew derivative problem outside cuts in another way, namely, with the help of the nonclassical angular potential, which has the same order of singularity as a single layer potential [6], [10], [11], [12], [13], [14]. Looking for a solution of the problem as a sum of angular and single layer potentials, we reduce the problem to a Cauchy singular integral equation with some additional conditions. By inversion of Cauchy singular operator, we obtain the uniquely solvable Fredholm equation of the second kind. Therefore the solution of the problem can be computed by standard codes.

The skew derivative problem for the 2-D Helmholtz equation models scattering long ocean waves by islands. The corresponding mathematical model takes into account daily rotation of the earth [15], [20], [22], [23]. Without rotation the skew derivative boundary condition transforms to the Neumann boundary condition. The model of long ocean waves on the rotating earth is appropriate for the description of the dynamics of any waves, which period can not be neglected in comparison with the period of daily rotation of the earth. The most important type of such waves are tidal waves. So, traditionally any long ocean waves on the rotating earth are called tidal waves, even if they are not caused by the moon [15], [23].

Free tidal waves in the elliptic lake and scattering tidal waves by an elliptic island are studied in [9], [19], [21]. Scattering tidal waves by a wavebreaker in the form of a half-line or two half-lines was treated in [2], [3]. General properties of tidal waves are discussed in [15], [20], [22], [23].

The skew derivative problem outside cuts in a plane, studied in the present paper, can be used to model diffraction of tidal waves by spits and reefs of an arbitrary shape.

Our results can be used for mathematical modeling in marine engineering, ecology and environmental sciences.

The skew derivative problem for an open boundary is different from the skew derivative problem for a closed boundary. In many cases the skew derivative problem for a closed boundary can be reduced to the Cauchy singular integral equation by means of only one single layer potential. In view of this, the skew derivative problem for a closed boundary was thoroughly studied unlike open boundary, where the classical approach leads to the system of hypersingular integro-differential equations as noted above.

2 – Formulation of the problem

By a simple open curve we mean a smooth nonclosed arc without self-intersections [17].

In the plane $x = (x_1, x_2) \in R^2$ we consider simple open curves $\Gamma_1, \ldots, \Gamma_N \in C^{2,\lambda}$, $\lambda \in (0,1]$, so that they do not have points in common. We put

$$\Gamma = \bigcup_{n=1}^{N} \Gamma_n \,.$$

We assume that each curve Γ_n is parametrized by the arc length s:

$$\Gamma_n = \{x : \; x = x(s) = (x_1(s), x_2(s)), \; s \in [a_n, b_n]\}, \qquad n = 1, \ldots, N\,,$$

so that $a_1 < b_1 < \ldots < a_N < b_N$. Therefore points $x \in \Gamma$ and values of the parameter s are in one-to-one correspondence. Below the set of the intervals on the Os axis

$$\bigcup_{n=1}^{N} [a_n, b_n]$$

will be denoted by Γ also.

The tangent vector to Γ at the point $x(s)$ we denote by $\tau_x = (\cos\alpha(s), \sin\alpha(s))$, where $\cos\alpha(s) = x_1'(s)$, $\sin\alpha(s) = x_2'(s)$. Let $\mathbf{n}_x = (\sin\alpha(s), -\cos\alpha(s))$ be a normal vector to Γ at $x(s)$. The direction of $\mathbf{n}_x$ is chosen such that it will coincide with the direction of τ_x if $\mathbf{n}_x$ is rotated anticlockwise through an angle of $\pi/2$.

We consider Γ as a set of cuts. The side of Γ which is on the left, when the parameter s increases, will be denoted by Γ^+, and the opposite side will be denoted by Γ^-.

We say, that the function $u(x)$ belongs to the smoothness class $\mathbf{K}$ if

1) $u \in C^0(\overline{R^2\backslash\Gamma}) \cap C^2(R^2\backslash\Gamma)$, and u is continuous at the ends of Γ,

2) $\nabla u \in C^0(\overline{R^2\backslash\Gamma}\backslash X)$, where X is a point-set, consisting of the end-points of Γ :

$$X = \bigcup_{n=1}^{N} (x(a_n) \cup x(b_n)) \,,$$

3) in the neighbourhood of any point $x(d) \in X$ for some constants $\mathcal{C} > 0$, $\epsilon > -1$ the inequality holds

$$|\nabla u| \leq \mathcal{C}|x - x(d)|^{\epsilon} \,, \tag{1}$$

where $x \to x(d)$ and $d = a_n$ or $d = b_n$, $n = 1, \ldots, N$.

Remark. In the definition of the class $\mathbf{K}$ we consider Γ as a set of cuts. In particular, by $C^0(\overline{R^2\backslash\Gamma})$ we denote a class of functions, which are continuously extended on Γ from the left and right, but their values on Γ from the left and right can be different, so that the functions may have a jump across Γ.

Let us formulate the oblique derivative problem for the Helmholtz equation in $R^2\backslash\Gamma$.

Problem $\mathbf{U}$. To find a function $u(x)$, which belongs to the class $\mathbf{K}$, satisfies the Helmholtz equation

$$\Delta u(x) + k^2 u(x) = 0 \,, \ x \in R^2\backslash\Gamma \,; \ \Delta = \partial_{x_1}^2 + \partial_{x_2}^2 \,, \ k = \operatorname{Re} k > 0 \,, \tag{2a}$$

satisfies the boundary condition

$$\Big(\frac{\partial}{\partial \mathbf{n}_x} u(x(s)) + \beta \frac{\partial}{\partial \tau_x} u(x(s))\Big)\Big|_{x(s)\in\Gamma^{\pm}} = F^{\pm}(s) \tag{2b}$$

and meets the conditions at infinity. With respect to the constant β we suppose that $\operatorname{Re}\beta = 0$ and $\beta = i\,c$, where c is a real number. In addition, we require the radiation conditions at infinity

$$u = O(|x|^{-1/2}), \ \ \frac{\partial u}{\partial |x|} - iku = o(|x|^{-1/2}), \ \ |x| = \sqrt{x_1^2 + x_2^2} \to \infty \,. \tag{2c}$$

All conditions of the problem $\mathbf{U}$ must be satisfied in the classical sense.

The Neumann problem for the Helmholtz equation in the exterior of cuts [11] is a particular case of our problem when $\beta = 0$.

The problem $\mathbf{U}$ describes scattering long ocean waves, in particular, tidal waves by spits and reefs. In this model [15], [20], [22], [23]

$$k^2 = \frac{\sigma^2 - 4\omega}{gh}, \qquad \beta = i\,\frac{2\omega}{\sigma},$$

where g is the gravity acceleration, h is the depth of the fluid layer (for example, ocean) supposed uniform, σ is the frequency of forced oscillations, ω is the earth's angular velocity of daily rotation. The function $u(x)$ enables to find basic parameters of the fluid motion

$$\zeta = \mathrm{Re}\,(ue^{-i\sigma t}),$$
$$\mathbf{v} = \frac{g}{\sigma^2 - 4\omega^2}\mathrm{Re}\left[e^{-i\sigma t}\Big(-i\sigma\frac{\partial u}{\partial x_1} + 2\omega\frac{\partial u}{\partial x_2}\;\;,\;\; -i\sigma\frac{\partial u}{\partial x_2} - 2\omega\frac{\partial u}{\partial x_1}\Big)\right],$$

where ζ is the elevation of the surface above its equilibrium point, $\mathbf{v}$ is the velocity vector.

The natural restriction in this model is $\sigma > 2\omega \geq 0$, otherwise we do not observe a wave motion. If rotation is absent ($\omega = 0$), then the oblique derivative problem transforms to the Neumann problem. The derivation of the equations and detailed description of the model is given in [15], [20], [22], [23].

Let us return to the mathematical analysis of the problem $\mathbf{U}$.

On the basis of the energy equalities [26] and the Rellich Lemma [1], [27] we can easily prove the following assertion.

THEOREM 1. *If* $\Gamma \in C^{2,\lambda}$, $\lambda \in (0,1]$, *then the problem* $\mathbf{U}$ *has at most one solution.*

PROOF. Let $u_0(x)$ be an arbitrary solution of the homogeneous problem $\mathbf{U}$. Let us show, that $u_0(x) \equiv 0$. To prove this with the help of the energy equalities, we envelope open curves by closed contours, let tend contours to the curves and use the smoothness of the solution of the problem $\mathbf{U}$. In this way we obtain the identity

$$\begin{aligned}&\lim_{r\to\infty}(\|\nabla u_0\|^2_{L_2(C_r\setminus\Gamma)} - k^2\|u_0\|^2_{L_2(C_r\setminus\Gamma)}) = \\ &= \int_\Gamma \bar{u}_0^+\Big(\frac{\partial u_0}{\partial \mathbf{n}_x}\Big)^+ ds - \int_\Gamma \bar{u}_0^-\Big(\frac{\partial u_0}{\partial \mathbf{n}_x}\Big)^- ds + i\,k\lim_{r\to\infty}\int_{\partial C_r}|u_0|^2 dl,\end{aligned} \tag{3}$$

where the conditions (1), (2c) are taken into account and C_r is the circle of the radius r with the center in the origin. By $\bar{u}_0(x)$ we denote a function, which is complex conjugate to $u_0(x)$. Clearly, $\bar{u}_0(x)$ belongs to the class $\mathbf{K}$. By $\int_\Gamma \ldots\ ds$ we mean $\sum_{n=1}^N \int_{a_n}^{b_n} \ldots ds$.

It follows from Corollary 3.7 and Section 3.10 of the monograph [1], that any solution of equation (2a) satisfying the radiation condition (2c) has the following asymptotic behaviour at infinity

$$u_0(x) = \frac{e^{ikr}}{\sqrt{r}} \mathcal{F}_0(\phi) + O\Big(\frac{1}{r^{3/2}}\Big), \qquad r = |x| \to \infty\,,$$

where ϕ is a polar angle and $\mathcal{F}_0(\phi)$ is a continuous function. This asymptotic formula proves the existence of the limit

$$\lim_{r\to\infty} \int_{\partial C_r} |u_0|^2 dl = \int_0^{2\pi} |\mathcal{F}_0(\phi)|^2 d\phi$$

in the right side of (3) in case condition (2c). Consequently, the limit in the left side of (3) also exists.

Let $u_0(x) = U_1(x) + i\,U_2(x)$ and $\bar{u}_0(x) = U_1(x) - i\,U_2(x)$, where $U_1(x)$, $U_2(x)$ are real functions. Using the homogeneous boundary condition (2b), we obtain

$$\begin{aligned}
&\int_\Gamma \bar{u}_0^+ \Big(\frac{\partial u_0}{\partial \mathbf{n}_x}\Big)^+ ds - \int_\Gamma \bar{u}_0^- \Big(\frac{\partial u_0}{\partial \mathbf{n}_x}\Big)^- ds = \\
&= -\beta \int_\Gamma \Big[\bar{u}_0^+ \Big(\frac{\partial u_0}{\partial \tau_x}\Big)^+ - \bar{u}_0^- \Big(\frac{\partial u_0}{\partial \tau_x}\Big)^-\Big] ds = \\
&= -\beta \int_\Gamma \Big[U_1^+\Big(\frac{\partial U_1}{\partial \tau_x}\Big)^+ + U_2^+\Big(\frac{\partial U_2}{\partial \tau_x}\Big)^+ + i\Big\{U_1^+\Big(\frac{\partial U_2}{\partial \tau_x}\Big)^+ - U_2^+\Big(\frac{\partial U_1}{\partial \tau_x}\Big)^+\Big\} + \\
&\quad - U_1^-\Big(\frac{\partial U_1}{\partial \tau_x}\Big)^- - U_2^-\Big(\frac{\partial U_2}{\partial \tau_x}\Big)^- - i\Big\{U_1^-\Big(\frac{\partial U_2}{\partial \tau_x}\Big)^- - U_2^-\Big(\frac{\partial U_1}{\partial \tau_x}\Big)^-\Big\}\Big] ds = \\
&= -\frac{\beta}{2} \sum_{m=1}^N \Big\{[U_1^+(x(b_m))]^2 - [U_1^+(x(a_m))]^2 + [U_2^+(x(b_m))]^2 - [U_2^+(x(a_m))]^2 + \\
&\quad - \Big([U_1^-(x(b_m))]^2 - [U_1^-(x(a_m))]^2 + [U_2^-(x(b_m))]^2 - [U_2^-(x(a_m))]^2\Big)\Big\} + \\
&\quad - \beta i \int_\Gamma \Big[U_1^+\Big(\frac{\partial U_2}{\partial \tau_x}\Big)^+ - U_2^+\Big(\frac{\partial U_1}{\partial \tau_x}\Big)^+ - U_1^-\Big(\frac{\partial U_2}{\partial \tau_x}\Big)^- + U_2^-\Big(\frac{\partial U_1}{\partial \tau_x}\Big)^-\Big] ds\,,
\end{aligned} \tag{4}$$

In accordance with the smoothness properties of the function u_0, which belongs to the class $\mathbf{K}$, we have

$$u_0^+(x(b_m)) = u_0^-(x(b_m))\,, \quad u_0^+(x(a_m)) = u_0^-(x(a_m))\,, \quad m = 1, \dots , N\,.$$

Consequently,

$$\tag{5} \begin{gathered} U_j^+(x(b_m)) = U_j^-(x(b_m)),\ U_j^+(x(a_m)) = U_j^-(x(a_m))\,, \\ m = 1, \dots , N\,; \qquad j = 1, 2\,. \end{gathered}$$

Putting (4) into (3) and using (5), we arrive at the identity

$$\begin{aligned} &\lim_{r\to\infty} (\|\nabla u_0\|^2_{L_2(C_r\setminus\Gamma)} - k^2\|u_0\|^2_{L_2(C_r\setminus\Gamma)}) = \\ &= -i\beta \int_\Gamma \Big[U_1^+\Big(\frac{\partial U_2}{\partial \tau_x}\Big)^+ - U_2^+\Big(\frac{\partial U_1}{\partial \tau_x}\Big)^+ - U_1^-\Big(\frac{\partial U_2}{\partial \tau_x}\Big)^- + U_2^-\Big(\frac{\partial U_1}{\partial \tau_x}\Big)^-\Big] ds + \\ &\quad + i\,k \lim_{r\to\infty} \int_{\partial C_r} |u_0|^2 dl\,. \end{aligned}$$

We recall that $\operatorname{Re}\beta = 0$, $\beta = i\operatorname{Im}\beta$ and take the imaginary part in the latter identity, then we obtain

$$\lim_{r\to\infty} \int_{\partial C_r} |u_0|^2 dl = 0\,,$$

where we take into account, that $k = \operatorname{Re} k > 0$.

Since $k = \operatorname{Re} k > 0$ and condition (2c) holds, then $u_0(x) \equiv 0$ on the basis of the Rellich Lemma [1], [27]. Thus, $u_0(x)$ is the trivial solution of the homogeneous problem $\mathbf{U}$. Hence the homogeneous problem $\mathbf{U}$ has only the trivial solution, and the theorem is proved due to the linearity of the problem $\mathbf{U}$. ∎

3 – Integral equations at the boundary

By $\mathcal{H}_0^{(1)}(z)$ we denote the Hankel function of the first kind

$$\mathcal{H}_0^{(1)}(z) = \frac{\sqrt{2}\exp(iz - i\pi/4)}{\pi\sqrt{z}} \int_0^\infty \exp(-t)t^{-1/2}\Big(1 + \frac{it}{2z}\Big)^{-1/2} dt\,.$$

Let us construct the solution of the problem $\mathbf{U}$, assuming that $F^+(s)$, $F^-(s)$ from (2b) are arbitrary functions from the Banach space $C^{0,\lambda}(\Gamma)$, where the Hölder index $\lambda \in (0,1]$.

The solution of the problem $\mathbf{U}$ can be obtained with the help of potential theory for the equation (2a). We seek a solution of the problem in the following form

$$u[\mu,\nu](x) = v[\mu+\beta\nu](x) + w[\nu-\beta\mu](x)\,, \tag{6}$$

where

$$w[\nu-\beta\mu](x) = \frac{i}{4}\int_\Gamma (\nu(\sigma)-\beta\mu(\sigma))\mathcal{H}_0^{(1)}(k|x-y(\sigma)|)d\sigma$$

is a single layer potential for the equation (2a), and

$$v[\mu+\beta\nu](x) = \frac{i}{4}\int_\Gamma (\mu(\sigma)+\beta\nu(\sigma))V(x,\sigma)d\sigma$$

is an angular potential [10] for the equation (2a). The kernel $V(x,\sigma)$ is defined on each curve Γ_n $(n=1,\dots,N)$ by the formula

$$V(x,\sigma) = \int_{a_n}^{\sigma} \frac{\partial}{\partial \mathbf{n}_y}\mathcal{H}_0^{(1)}(k|x-y(\xi)|)d\xi\,,\ \ \sigma\in[a_n,b_n]\,,$$

where

$$y = y(\xi) = (y_1(\xi), y_2(\xi)),\ \ |x-y(\xi)| = \sqrt{(x_1-y_1(\xi))^2+(x_2-y_2(\xi))^2}\,.$$

Below we suppose that the density of the angular potential satisfies the following additional conditions [10], [11]

$$\int_{a_n}^{b_n} (\mu(\sigma)+\beta\nu(\sigma))\,d\sigma = 0,\ \ n=1,\dots,N\,. \tag{7}$$

Integrating $v[\mu+\beta\nu](x)$ by parts and using (7), we express the angular potential in terms of a double layer potential

$$v[\mu+\beta\nu](x) = -\frac{i}{4}\int_\Gamma \rho(\sigma)\frac{\partial}{\partial \mathbf{n}_y}\mathcal{H}_0^{(1)}(k|x-y(\sigma)|)d\sigma\,,$$

with the density

$$\rho(\sigma) = \int_{a_n}^{\sigma} (\mu(\xi) + \beta\nu(\xi))\, d\xi, \quad \sigma \in [a_n, b_n], \quad n = 1, \dots, N\,.$$

Consequently, $v[\mu+\beta\nu](x)$ satisfies both equation (2a) outside Γ and the conditions at infinity (2c).

In addition to (7) we require that $\mu(s)$, $\nu(s)$ belong to the following Banach spaces: $\nu\ (s) \in C^{0,\lambda}(\Gamma)$, $\mu(s) \in C_q^{\omega}(\Gamma)$, $\omega \in (0,1]$, $q \in [0,1)$.

We say, that $\mu(s) \in C_q^{\omega}(\Gamma)$ if

$$\mu(s) \prod_{n=1}^{N} |s-a_n|^q |s-b_n|^q \in C^{0,\omega}(\Gamma)\,,$$

where $C^{0,\omega}(\Gamma)$ is the Hölder space with the index ω and

$$\|\mu(s)\|_{C_q^{\omega}(\Gamma)} = \left\|\mu(s) \prod_{n=1}^{N} |s-a_n|^q |s-b_n|^q \right\|_{C^{0,\omega}(\Gamma)}.$$

As shown in [10], for such $\mu(s)$, $\nu(s)$ the potentials $v[\mu+\beta\nu](x)$, $w[\nu-\beta\mu](x)$ belong to the class $\mathbf{K}$. In particular, the inequality (1) holds with $\epsilon = -q$, if $q \in (0,1)$.

Therefore, the function (6) belongs to the class $\mathbf{K}$ and satisfies all conditions of the problem $\mathbf{U}$ except the boundary condition (2b).

To satisfy the boundary condition, we put (6) in (2b), use the limit formulas for the angular potential from [10] and arrive at the integral equation for the densities $\mu(s)$, $\nu(s)$:

$$\begin{aligned}
&-\frac{1+\beta^2}{2\pi}\int_{\Gamma}\mu(\sigma)\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}d\sigma \pm \frac{1}{2}(1+\beta^2)\nu(s)+\\
&+\frac{i}{4}\int_{\Gamma}(\mu(\sigma)+\beta\nu(\sigma))\frac{\partial}{\partial\mathbf{n}_x}V_0(x(s),\sigma)d\sigma+\\
&+\beta\frac{i}{4}\int_{\Gamma}(\mu(\sigma)+\beta\nu(\sigma))\frac{\partial}{\partial s}V(x(s),\sigma)d\sigma+\\
&+\frac{i}{4}\int_{\Gamma}(\nu(\sigma)-\beta\mu(\sigma))\frac{\partial}{\partial\mathbf{n}_x}\mathcal{H}_0^{(1)}(k|x(s)-y(\sigma)|)d\sigma+\\
&+\beta\frac{i}{4}\int_{\Gamma}(\nu(\sigma)-\beta\mu(\sigma))\frac{\partial}{\partial s}h(k|x(s)-y(\sigma)|)d\sigma = F^{\pm}(s)\,,\quad s\in\Gamma\,,
\end{aligned} \tag{8}$$

where

$$V_0(x,\sigma)=\int_{a_n}^{\sigma}\frac{\partial}{\partial\mathbf{n}_y}h(k|x-y(\xi)|)d\xi\,,\quad \sigma\in[a_n,b_n]\,,\quad n=1,2,\ldots,N\,,$$
$$h(z)=\mathcal{H}_0^{(1)}(z)-\frac{2i}{\pi}\ln\frac{z}{k}\,.$$

By $\varphi_0(x,y)$ we denote the angle between the vector $\overrightarrow{xy}$ and the direction of the normal $\mathbf{n}_x$. The angle $\varphi_0(x,y)$ is taken to be positive if it is measured anticlockwise from $\mathbf{n}_x$ and negative if it is measured clockwise from $\mathbf{n}_x$. Besides, $\varphi_0(x,y)$ is continuous in $x,y\in\Gamma$ if $x\neq y$.

The first term in (8) is a Cauchy singular integral [17].

Equation (8) is obtained as $x\to x(s)\in\Gamma^{\pm}$ and comprises 2 integral equations. The upper sign denotes the integral equation on Γ^+, the lower sign denotes the integral equation on Γ^-. In addition to the integral equation (8) we have the conditions (7).

Subtracting the integral equation (8), we find

$$\nu(s)=\frac{1}{1+\beta^2}(F^+(s)-F^-(s))\in C^{0,\lambda}(\Gamma)\,. \tag{9}$$

We note, that $\nu(s)$ is found completely and belongs to the required class of smoothness.

We introduce the function $f(s)$ on Γ by the formula

$$\begin{aligned} f(s)=\frac{1}{2}(F^+(s)+F^-(s))-\beta\frac{i}{4}\int_{\Gamma}\nu(\sigma)\frac{\partial}{\partial\mathbf{n}_x}V_0(x(s),\sigma)d\sigma+\\ -\beta^2\frac{i}{4}\int_{\Gamma}\nu(\sigma)\frac{\partial}{\partial s}V(x(s),\sigma)d\sigma+\\ -\frac{i}{4}\int_{\Gamma}\nu(\sigma)\frac{\partial}{\partial\mathbf{n}_x}\mathcal{H}_0^{(1)}(k|x(s)-y(\sigma)|)d\sigma+\\ -\beta\frac{i}{4}\int_{\Gamma}\nu(\sigma)\frac{\partial}{\partial s}h(k|x(s)-y(\sigma)|)d\sigma \end{aligned} \tag{10}$$

where $\nu(s)$ is given in (9). As shown in [10], Lemma 3, Theorem 6 and in [11], Lemmas 3, 4, $f(s)$ belongs to $C^{0,p_0}(\Gamma)$, where $p_0=\lambda$ if $0<\lambda<1$ and $p_0=1-\epsilon_0$ for any $\epsilon_0\in(0,1)$ if $\lambda=1$.

Adding the integral equation (8), we obtain the singular integral equation [17] for $\mu(s)$ on Γ:

$$
\begin{aligned}
&-\frac{1+\beta^2}{2\pi}\int_\Gamma \mu(\sigma)\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}d\sigma+\\
&+\frac{i}{4}\int_\Gamma \mu(\sigma)\frac{\partial}{\partial \mathbf{n}_x}V_0(x(s),\sigma)d\sigma+\beta\frac{i}{4}\int_\Gamma \mu(\sigma)\frac{\partial}{\partial s}V(x(s),\sigma)d\sigma+\\
&-\beta\frac{i}{4}\int_\Gamma \mu(\sigma)\frac{\partial}{\partial \mathbf{n}_x}\mathcal{H}_0^{(1)}(k|x(s)-y(\sigma)|)d\sigma+\\
&-\beta^2\frac{i}{4}\int_\Gamma \mu(\sigma)\frac{\partial}{\partial s}h(k|x(s)-y(\sigma)|)d\sigma=f(s)\,,\quad s\in\Gamma\,,
\end{aligned}
\tag{11a}
$$

where $f(s)$ is defined by (10); $V_0(x,\sigma)$, $h(z)$ are introduced in (8), and $V(x,\sigma)$ is the kernel of the angular potential.

Due to (9), conditions (7) take the form

$$
\int_{a_n}^{b_n}\mu(\sigma)\,d\sigma=\frac{-\beta}{1+\beta^2}\int_{a_n}^{b_n}(F^+(\sigma)-F^-(\sigma))\,d\sigma,\quad n=1,\ldots,N\,. \tag{11b}
$$

Thus, if $\mu(s)$ is a solution of equations (11) from the space $C_q^\omega(\Gamma)$, $\omega\in(0,1]$, $q\in[0,1)$, then the potential (6) meets all requirements of the problem **U**. The following theorem holds.

THEOREM 2. *If $\Gamma\in C^{2,\lambda}$, $F^\pm(s)\in C^{0,\lambda}(\Gamma)$, $\lambda\in(0,1]$, and if equations* (11) *have a solution $\mu(s)$ from the Banach space $C_q^\omega(\Gamma)$, $\omega\in(0,1]$, $q\in[0,1)$, then the solution of the problem* **U** *is given by* (6), *where $\nu(s)$ is taken from* (9).

Our further treatment will be aimed to the proof of the solvability of the system (11) in the Banach space $C_q^\omega(\Gamma)$. Moreover, we reduce the system (11) to a Fredholm equation of the second kind, which can be easily computed by classical methods.

It can be easily proved that

$$
\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}-\frac{1}{\sigma-s}\in C^{0,\lambda}(\Gamma\times\Gamma)
$$

(see [10, Lemma 3] for details). Therefore we can rewrite (11a) in the form

$$
\frac{1}{\pi}\int_\Gamma \mu(\sigma)\frac{d\sigma}{\sigma-s}+\int_\Gamma \mu(\sigma)Y(s,\sigma)d\sigma=-\frac{2}{1+\beta^2}f(s),\quad s\in\Gamma\,, \tag{12}
$$

where $f(s)$ is defined by (10) and

$$Y(s,\sigma)=\Big\{\frac{1}{\pi}\Big(\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}-\frac{1}{\sigma-s}\Big)+ \\ -\frac{1}{1+\beta^2}\frac{i}{2}\Big[\frac{\partial}{\partial\mathbf{n}_x}V_0(x(s),\sigma)+\beta\frac{\partial}{\partial s}V(x(s),\sigma)+ \\ -\beta\frac{\partial}{\partial\mathbf{n}_x}\mathcal{H}_0^{(1)}(k|x(s)-y(\sigma)|)-\beta^2\frac{\partial}{\partial s}h(k|x(s)-y(\sigma)|)\Big]\Big\}.$$

It follows from [10], Lemma 3, Theorem 6 and from [11], Lemmas 3, 4 that $Y(s,\sigma)\in C^{0,p_0}(\Gamma\times\Gamma)$, $p_0=\lambda$ if $0<\lambda<1$ and $p_0=1-\epsilon_0$ for any $\epsilon_0\in(0,1)$ if $\lambda=1$.

4 – The Fredholm integral equation and the solution of the problem

Inverting the singular integral operator in (12), we arrive at the following integral equation of the second kind [17]:

$$(13)\quad \mu(s)+\frac{1}{Q(s)}\int_\Gamma\mu(\sigma)A_0(s,\sigma)d\sigma+\frac{1}{Q(s)}\sum_{n=0}^{N-1}G_n s^n=\frac{1}{Q(s)}\Phi_0(s),\quad s\in\Gamma,$$

where

$$A_0(s,\sigma)=-\frac{1}{\pi}\int_\Gamma\frac{Y(\xi,\sigma)}{\xi-s}Q(\xi)d\xi\,,$$

$$Q(s)=\prod_{n=1}^{N}|\sqrt{s-a_n}\sqrt{b_n-s}|\mathrm{sign}(s-a_n)\,,$$

$$\Phi_0(s)=\frac{1}{1+\beta^2}\frac{1}{\pi}\int_\Gamma\frac{2Q(\sigma)f(\sigma)}{\sigma-s}d\sigma\,,$$

and $G_0,\dots,G_{N-1}$ are arbitrary constants.

To derive equations for $G_0,\dots,G_{N-1}$, we substitute $\mu(s)$ from (13) in the conditions (11b), then we obtain

$$(14)\qquad \int_\Gamma\mu(\sigma)l_n(\sigma)d\sigma+\sum_{m=0}^{N-1}B_{nm}G_m=H_n,\quad n=1,\dots,N\,,$$

where

$$l_n(\sigma) = -\int_{a_n}^{b_n} Q^{-1}(s)A_0(s,\sigma)ds\,,$$

$$\text{(15)}\qquad H_n = -\int_{a_n}^{b_n} Q^{-1}(s)\Phi_0(s)ds - \frac{\beta}{1+\beta^2}\int_{a_n}^{b_n}(F^+(s)-F^-(s))ds\,,$$

$$B_{nm} = -\int_{a_n}^{b_n} Q^{-1}(s)s^m ds\,.$$

By B we denote the $N\times N$ matrix with the elements B_{nm} from (15). As shown in [11, Lemma 7], the matrix B is invertible. The elements of the inverse matrix will be called $(B^{-1})_{nm}$. Inverting the matrix B in (14), we express the constants $G_0,...,G_{N-1}$ in terms of $\mu(s)$

$$G_n = \sum_{m=1}^{N}(B^{-1})_{nm}\Big[H_m - \int_\Gamma \mu(\sigma)l_m(\sigma)d\sigma\Big]\,.$$

We substitute G_n in (13) and obtain the following integral equation for $\mu(s)$ on Γ

$$\text{(16)}\qquad \mu(s) + \frac{1}{Q(s)}\int_\Gamma \mu(\sigma)A(s,\sigma)d\sigma = \frac{1}{Q(s)}\Phi(s)\,,\quad s\in\Gamma\,,$$

where

$$A(s,\sigma) = A_0(s,\sigma) - \sum_{n=0}^{N-1} s^n \sum_{m=1}^{N}(B^{-1})_{nm}l_m(\sigma)\,,$$

$$\Phi(s) = \Phi_0(s) - \sum_{n=0}^{N-1} s^n \sum_{m=1}^{N}(B^{-1})_{nm}H_m\,.$$

It can be shown using the properties of singular integrals [7], [17], that $\Phi_0(s)$, $A_0(s,\sigma)$ are Hölder functions if $s\in\Gamma$, $\sigma\in\Gamma$. Therefore, $\Phi(s)$, $A(s,\sigma)$ are also Hölder functions if $s\in\Gamma$, $\sigma\in\Gamma$. Consequently, any solution of (16) belongs to $C^\omega_{1/2}(\Gamma)$, and below we look for $\mu(s)$ on Γ in this space.

Instead of $\mu(s)\in C^\omega_{1/2}(\Gamma)$ we introduce the new unknown function $\mu_*(s) = \mu(s)Q(s)\in C^{0,\omega}(\Gamma)$ and rewrite (16) in the form

$$\text{(17)}\qquad \mu_*(s) + \int_\Gamma \mu_*(\sigma)Q^{-1}(\sigma)A(s,\sigma)d\sigma = \Phi(s)\,,\quad s\in\Gamma\,.$$

Thus, the system of equations (11) for $\mu(s)$ has been reduced to the equation (17) for the function $\mu_*(s)$. It is clear from our consideration that any solution of (17) gives a solution of system (11).

As noted above, $\Phi(s)$ and $A(s,\sigma)$ are Hölder functions if $s \in \Gamma$, $\sigma \in \Gamma$. More precisely (see [7], [17]), $\Phi(s) \in C^{0,p}(\Gamma)$, $p = \min\{1/2, \lambda\}$, and $A(s,\sigma)$ belongs to $C^{0,p}(\Gamma)$ in s uniformly with respect to $\sigma \in \Gamma$.

We arrive at the following assertion.

LEMMA 1. *If* $\Gamma \in C^{2,\lambda}$, $\lambda \in (0,1]$, $\Phi(s) \in C^{0,p}(\Gamma)$, $p = \min\{\lambda, 1/2\}$, *and* $\mu_*(s)$ *from* $C^0(\Gamma)$ *satisfies the equation* (17), *then* $\mu_*(s) \in C^{0,p}(\Gamma)$.

The condition $\Phi(s) \in C^{0,p}(\Gamma)$ holds if $F^{\pm}(s) \in C^{0,\lambda}(\Gamma)$.

Hence below we will seek $\mu_*(s)$ from $C^0(\Gamma)$.

Since $A(s,\sigma) \in C^0(\Gamma \times \Gamma)$, the integral operator from (17):

$$\mathbf{A}\mu_*(s) = \int_\Gamma \mu_*(\sigma) Q^{-1}(\sigma) A(s,\sigma)\, d\sigma$$

is a compact operator mapping $C^0(\Gamma)$ into itself. Therefore, (17) is a Fredholm equation of the second kind in the Banach space $C^0(\Gamma)$.

Let us show that homogeneous equation (17) has only a trivial solution. Then, according to Fredholm's theorems, the inhomogeneous equation (17) has a unique solution for any right-hand side. We will prove this by a contradiction. Let $\mu_*^0(s) \in C^0(\Gamma)$ be a non-trivial solution of the homogeneous equation (17). According to the Lemma 1, $\mu_*^0(s) \in C^{0,p}(\Gamma)$, $p = \min\{\lambda, 1/2\}$. Therefore the function $\mu^0(s) = \mu_*^0(s) Q^{-1}(s) \in C^p_{1/2}(\Gamma)$ converts the homogeneous equation (16) into identity. Using the homogeneous identity (16), we check, that $\mu^0(s)$ satisfies conditions (11b). Besides, acting on the homogeneous identity (16) with a singular operator with the kernel $(s-t)^{-1}$, we find that $\mu^0(s)$ satisfies the homogeneous equation (12). Consequently, $\mu^0(s)$ satisfies the homogeneous equation (11a). On the basis of Theorem 2, $u[\mu^0, 0](x)$ is a solution of the homogeneous problem $\mathbf{U}$. According to Theorem 1: $u[\mu^0, 0](x) \equiv 0$, $x \in R^2 \backslash \Gamma$. Using the limit formulas for tangent and normal derivatives of potentials [10], we obtain

$$\begin{aligned}
&\lim_{x \to x(s) \in \Gamma^+} \left\{ \beta \frac{\partial}{\partial \mathbf{n}_x} u[\mu^0, 0](x) - \frac{\partial}{\partial \tau_x} u[\mu^0, 0](x) \right\} + \\
&- \lim_{x \to x(s) \in \Gamma^-} \left\{ \beta \frac{\partial}{\partial \mathbf{n}_x} u[\mu^0, 0](x) - \frac{\partial}{\partial \tau_x} u[\mu^0, 0](x) \right\} = \\
&= -(1+\beta^2)\mu^0(s) \equiv 0, \quad s \in \Gamma.
\end{aligned}$$

Consequently, if $s \in \Gamma$, then $\mu^0(s) \equiv 0$, $\mu^0_*(s) = \mu^0(s)Q^{-1}(s) \equiv 0$, and we arrive at the contradiction to the assumption that $\mu^0_*(s)$ is a non-trivial solution of the homogeneous equation (17). Thus, the homogeneous Fredholm equation (17) has only a trivial solution in $C^0(\Gamma)$.

We have proved the following assertion.

Theorem 3. *If $\Gamma \in C^{2,\lambda}$, $\lambda \in (0,1]$, then* (17) *is a Fredholm equation of the second kind in the space $C^0(\Gamma)$. Moreover, equation* (17) *has a unique solution $\mu_*(s) \in C^0(\Gamma)$ for any $\Phi(s) \in C^0(\Gamma)$.*

As a consequence of the Theorem 3 and the Lemma 1 we obtain the corollary.

Corollary. *If $\Gamma \in C^{2,\lambda}$, $\lambda \in (0,1]$ and $\Phi(s) \in C^{0,p}(\Gamma)$, where $p = \min\{\lambda, 1/2\}$, then the unique solution of* (17) *in $C^0(\Gamma)$, ensured by Theorem 3, belongs to $C^{0,p}(\Gamma)$.*

We recall that $\Phi(s)$ belongs to the class of smoothness required in the corollary if $F^{\pm}(s) \in C^{0,\lambda}(\Gamma)$. As mentioned above, if $\mu_*(s) \in C^{0,p}(\Gamma)$ is a solution of (17), then $\mu(s) = \mu_*(s)Q^{-1}(s) \in C^p_{1/2}(\Gamma)$ is a solution of system (11). We obtain the statement.

Proposition. *If $\Gamma \in C^{2,\lambda}$, $F^{\pm}(s) \in C^{0,\lambda}(\Gamma)$, $\lambda \in (0,1]$, then the system of equations* (11) *has a solution $\mu(s) \in C^p_{1/2}(\Gamma)$, $p = \min\{1/2, \lambda\}$, which is expressed by the formula $\mu(s) = \mu_*(s)Q^{-1}(s)$, where $\mu_*(s) \in C^{0,p}(\Gamma)$ is the unique solution of the Fredholm equation* (17) *in $C^0(\Gamma)$.*

Remark. The system of linear integral equations (11) has no more than one solution $\mu(s) \in C^{\omega}_q(\Gamma)$, $\omega \in (0,1]$, $q \in [0,1)$, because the homogeneous system (11) has only the trivial solution. The proof of this fact by a contradiction almost coincides with the proof of the Theorem 3. Together with the proposition this means, that the system (11) is uniquely solvable, and the numerical solution of integral equations can be found directly from (11) by numerical inversion of the integral operator in (11). In doing so, Hölder functions can be approximated by piecewise linear functions, which also obey Hölder inequality.

On the basis of the Theorem 2 and the proposition we arrive at the following result.

Theorem 4. *If* $\Gamma \in C^{2,\lambda}$, $F^{\pm}(s) \in C^{0,\lambda}(\Gamma)$, $\lambda \in (0,1]$, *then the solution of the problem* $\mathbf{U}$ *exists and is given by* (6)*, where* $\nu(s)$ *is defined in* (9) *and* $\mu(s)$ *is a solution of equations* (11) *from* $C^{p}_{1/2}(\Gamma)$, $p = \min\{1/2, \lambda\}$, *ensured by the proposition.*

It can be checked directly that the solution of the problem $\mathbf{U}$ satisfies condition (1) with $\epsilon = -1/2$. Explicit expressions for singularities of the solution gradient at the end-points of the open curves can be easily obtained with the help of formulas presented in [10], [11].

Theorem 4 ensures existence of a classical solution of the problem $\mathbf{U}$ when $\Gamma \in C^{2,\lambda}$, $F^{\pm}(s) \in C^{0,\lambda}(\Gamma)$. On the basis of our consideration we suggest the following scheme for solving the problem $\mathbf{U}$. First, we find the unique solution $\mu_*(s)$ of the Fredholm equation (17) from $C^0(\Gamma)$. This solution automatically belongs to $C^{0,p}(\Gamma)$, $p = \min\{\lambda, 1/2\}$. Second, we construct the solution of equations (11) from $C^{p}_{1/2}(\Gamma)$ by the formula $\mu(s) = \mu_*(s)Q^{-1}(s)$. This solution automatically belongs to $C^{p}_{1/2}(\Gamma)$. Finally, substituting $\nu(s)$ from (9) and $\mu(s)$ in (6), we obtain the solution of the problem $\mathbf{U}$.

If $\beta = 0$, then the problem $\mathbf{U}$ transforms to the Neumann problem for the Helmholtz equation in the exterior of cuts in a plane. The Neumann problem has been studied in [17], and its solution coincides with the solution of the problem $\mathbf{U}$ from the Theorem 4 in case $\beta = 0$.

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Indirizzo dell'Autore:

P.A. Krutitskii – Dept. of Mathematics – Faculty of Physics – Moscow State University – Moscow 119899 – Russia
E-mail: krutitsk@math.phys.msu.su

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