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L^p cohomology for locally polycylindrical domains in $\mathbb{C}^n$

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RIASSUNTO: *Si determinano delle stime in L^p per l'operatore $\bar{\partial}$ su un dominio localmente policilindrico e si applicano queste stime per risolvere il problema di Corona ed il problema di Gleason mediante una coomologia limitata.*

ABSTRACT: *L^p estimates are obtained for the $\bar{\partial}$-operator on a generalization of a polycylinder called a locally polycylindrical domain and the estimates are applied to the solution of the Corona and the Gleason problems through cohomology with bounds.*

A polycylinder in $\mathbb{C}^n$ is a set of the form $\Omega = D_1 \times D_2 \times \ldots \times D_n$, with each D_j a non-empty bounded open set in $\mathbb{C}$. Clearly this is a generalization of a polydisc. Normally a further generalization of a polydisc after the polycylinder is an analytic polyhedron, but the analytic polyhedron quickly becomes unwieldy to work with and a more amenable generalization of the polycylinder which extends to manifolds is the following: A bounded open set in $\mathbb{C}^n$ is called a locally polycylindrical domain, if for every point x_0 in the boundary $\partial\Omega$ of Ω, there is a neighborhood U_{x_0} of x_0 such that $\Omega_\cap U_{x_0}$ is a polycylinder. Clearly a domain such as Ω is Stein.

KEY WORDS AND PHRASES: *Polycyliders - $\bar{\partial}$-operator - Cohomology with bounds*
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Most of the “hard analysis” problems which were solved on strictly pseudoconvex domains after the introduction of the Ramirez-Henkin kernel, can be solved on polycylinders, therefore, since the locally polycylindrical domain is special enough, it is natural to ask whether these problems can be solved on a locally polycylindrical domain. Here we select two related problems which we solve on a locally polycylindrical domain. These are the Gleason problem and the Weak Corona problem. Let Ω be a bounded set in $\mathbb{C}^n$ containing 0 and let $A(\Omega) = H^\infty(\Omega) \cap C(\bar{\Omega})$ be the set of bounded holomorphic functions in Ω continuous on $\bar{\Omega}$. The Gleason problem for Ω is “if $f \in A(\Omega)$ and $f(0) = 0$, are there $f_1, \ldots, f_n \in A(\Omega)$ such that $f(z_1, \ldots, z_n) = \sum\limits_{j=1}^{n} z_j f_j(z_1, \ldots, z_n)$?”.

The Weak Corona problem is formulated in [2]: Let X be a relatively compact domain of a topological space Y. Let $f_0, \ldots, f_N$ be complex-valued continuous functions on X; $f_0, \ldots, f_N$ satisfy the Weak Corona assumption (on X) if the following two conditions hold:

(a) $f_0, \ldots, f_N$ have no common zeros on X;

(b) a positive number $\delta > 0$ exists so that for each $z \in \partial X$ (= boundary of X in Y), an index $i \in \{0, \ldots, N\}$, $i = i(z)$ and an open neighborhood V_z of z in Y are given such that $|f_i(W)| \geq \delta$ on $V_z \cap X$.

Let A be a function algebra on X. The Weak Corona problem is solvable in A (on X) when for $f_0, \ldots, f_N \in A$ which satisfy the Weak Corona assumption, $f_0, \ldots, f_N$ represent 1 in A (that is, there are $g_0, \ldots, g_N$ in A such that $f_0 g_0 + \ldots + f_N g_N = 1$ on X).

What we do in this paper is to reduce the solutions of the Gleason and the Weak Corona problem to the works in [2] and [10], by showing that certain cohomology groups vanish. Actually our vanishing theorems are more than what we need to solve the Gleason and the Weak Corona problems on locally polycylindrical domains.

1 – Preliminaries

If $U \subset \mathbb{C}^n$ is an open set and $f \in C^\infty(U)$ and $1 \leq p \leq \infty$, we define

$$\|f\|^{(0)}_{L^p(U)} = \|f\|^{(0,0)}_{L^p(U)} = \|f\|_{L^p(U)},$$

$$\|f\|^{(0)}_{L^p(\bar{U})} = \|f\|^{(0,0)}_{L^p(\bar{U})} \quad \text{if } f \text{ is continuous}$$

to the boundary, $\|f\|_{L^p(U)}$, $\|f_{L^p(\bar{U})}$ being the L^p-norms on U and $\bar{U}$ respectively:

$$\|f\|^{(0,r)}_{L^p(U)} = \max_{i_1<..<i_r} \left\| \frac{\partial^r f}{\partial \bar{z}_{i_1} \dots \partial \bar{z}_{i_r}} \right\|^{(0)}_{L^p(U)} \quad \text{for } 1 \le r \le n,$$

$$\|f\|^{(0,r)}_{L^p(\bar{U})} = \max_{i_1<..<i_r} \left\| \frac{\partial^r f}{\partial \bar{z}_{i_1} \dots \partial \bar{z}_{i_r}} \right\|_{L^p(\bar{U})} \quad \text{for } 1 \le r \le n,$$

if the relevant partial derivatives of f are continuous to the boundary, and

$$\|f\|^{(n)}_{L^p(U)} = \max_{0\le r\le n} \|f\|^{(0,r)}_{L^p(U)}; \ \|f\|^{(n)}_{L^p(\bar{U})} = \max_{0\le r\le n} \|f\|^{(0,r)}_{L^p(\bar{U})}.$$

If $f = \sum_{(i_1,\dots,i_q)}{}' f_{i_1\dots i_q} d\bar{z}_{i_1}\wedge \dots \wedge d\bar{z}_{i_q}$ is a $C^\infty(0,q)$-form on U, where $\sum'$ means the summation is over increasing multi-indices we write f as $\sum_I{}' f_I d\bar{z}^I$ for short, $I = (i_1, \dots, i_q)$, and set

$$\|f\|^{(n)}_{L^p_{(0,q)}(U)} = \max_I \|f_I\|^{(n)}_{L^p(\bar{U})},$$

and where the relevant derivatives of the coefficients f_I are continuous to the boundary, we set

$$\|f\|^{(n)}_{L^p_{(0,q)}(\bar{U})} = \max_I \|f_I\|^{(n)}_{L^p(\bar{U})}.$$

From [5] we have the following

THEOREM 1. *Let Ω be a polycylinder in $\mathbb{C}^n$ and $1 \le p \le \infty$. There is a $K > 0$ such that if f is a smooth $\bar{\partial}$-closed $(0, q+1)$-form on Ω with $\|f\|^{(n)}_{L^p_{(0,q+1)}(\Omega)} < \infty$, then there is a smooth $(0,q)$-form u on Ω with $\bar{\partial}u = f$ and*

$$\|u\|^{(n)}_{L^p_{(0,q)}(\bar{\Omega})} \le K\|f\|^{(n)}_{L^p_{(0,q+1)}(\Omega)}.$$

Modifying the proof of Theorem 1 in [5] we get

THEOREM 2. *Let Ω be a polycylinder in $\mathbb{C}^n$ and $1 \leq p \leq \infty$. There is a $K > 0$ such that if f is a smooth $\bar{\partial}$-closed $(0, q+1)$-form on Ω and $\|f\|^{(n)}_{L^p_{(0,q+1)}(\bar{\Omega})}$ is defined and finite, then there is a smooth $(0, q)$-form u on Ω with $\bar{\partial} u = f$ and $\|u\|^{(n)}_{L^p_{(0,q)}(\bar{\Omega})}$ is defined and*

$$\|u\|^{(n)}_{L^p_{(0,q)}(\bar{\Omega})} \leq K \|f\|^{(n)}_{L^p_{(0,q+1)}(\bar{\Omega})}.$$

Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions in $\mathbb{C}^n$. If $U \subset \mathbb{C}^n$ is open and $r > 0$ is an integer, let $\Gamma(U, \mathcal{O}^r)$ be the sections of $\mathcal{O}^r$on U, then $\Gamma_p(U, \mathcal{O}^r) := \{ f = (f_1, \ldots, f_r) \in \Gamma(U, \mathcal{O}^r) : \|f_1\|_{L^p(U)} + \ldots + \|f_r\|_{L^p(U)} < \infty \}$.

If F is a coherent analytic sheaf on a neighborhood of the closure $\bar{U}$ of the polycylinder U, then by Cartan's theorem A there is an exact sequence

$$\mathcal{O}^m \xrightarrow{\lambda} F \longrightarrow 0$$

of $\mathcal{O}$-homomorphisms in a neighborhood of $\bar{U}$, where m is a positive integer. The L^p-bounded sections of F over $U, \Gamma_p(U, F)$ is defined by

$$\Gamma_p(U, F) = \lambda(\Gamma_p(U, \mathcal{O}^m)).$$

It can be shown that the definition of $\Gamma_p(U, F)$ does not depend on λ and m [6], p. 260.

Now, let Ω be a locally polycylindrical domain and let F be a coherent analytic sheaf in a neighborhood of the closure of Ω. Then Ω is expressible as the union of a finite number of polycylinders, so let $\mathcal{U} = \{U_j\}_{j \in I}$ be a finite set of polycylinders such that $\Omega = \bigcup_{j \in I} U_j$. We define the L^p-bounded alternate q-cochain group $C^q_p(\mathcal{U}, F)$ of the covering $\mathcal{U}$ with values in F by

$$C^q_p(\mathcal{U}, F) := \{ c = (c_\alpha) \in C^q(\mathcal{U}, F) : c_a \in \Gamma_p(U_\alpha, F) \quad \forall \alpha = (\alpha_0, \ldots, \alpha_q) \in I^{q+1} \}, \text{ where } U_\alpha = U_{\alpha 0} \cap \ldots \cap U_{\alpha_q}$$

and $C^q(\mathcal{U}, F)$, the alternate q-cochain group of the cover $\mathcal{U}$ with values in F.

The coboundary operator

$$\delta : \; C^q(\mathcal{U}, F) \longrightarrow C^{q+1}(\mathcal{U}, F)$$

maps $C_p^q(\mathcal{U},F)$ into $C_p^{q+1}(\mathcal{U},F)$, hence we have a complex

$$C_p^0(\mathcal{U},F)\xrightarrow{\delta}C_p^1(\mathcal{U},F)\xrightarrow{\delta}\cdots\longrightarrow C_p^q(\mathcal{U},F)\xrightarrow{\delta}C_p^{q+1}(\mathcal{U},F)\longrightarrow\cdots$$

and $H_p^q(\mathcal{U},F)$ is the qth cohomology group of this complex.

Now, using the formalism in the proof of Theorem 5 we have the following.

THEOREM 3. *The natural map*

$$H_p^q(\mathcal{U},F)\longrightarrow H^q(\Omega,F)$$

is an isomorphism for $q\ge 0$ and $1\le p\le\infty$.

We can now replace $\Gamma_p(U,\mathcal{O}^r)$ by

$$\Gamma_p(\bar U,\mathcal{O}^r):=\{f=(f_1,\dots,f_r)\in\Gamma(U,\mathcal{O}^r):\ f_j\in C(\bar U),\ 1\le j\le r$$
$$\text{and } \|f_1\|_{L^p(\bar U)}+\cdots+\|f_r\|_{L^p(\bar U)}<\infty\}$$

in the definition of $H_p^q(\mathcal{U},F)$ to get $\bar H_p^q(\mathcal{U},F)$, where $\Gamma_p(\mathcal{U},F)$ corresponds to $\Gamma_p(U,F)$, $\bar C_p^q(\mathcal{U},F)$ corresponds to $\bar C_p^q(\mathcal{U},F)$ and $\bar H_p^q(\mathcal{U},F)$ is the qth cohomology group of the complex

$$\bar C_p^0(\mathcal{U},F)\xrightarrow{\delta}\bar C_p^1(\mathcal{U},F)\longrightarrow\cdots\xrightarrow{\delta}\bar C_p^q(\mathcal{U},F)\xrightarrow{\delta}\bar C_p^{q+1}(\mathcal{U},F)\xrightarrow{\delta}\cdots$$

We then have, with the same proof.

THEOREM 4. *The natural map*

$$\bar H_p^q(\mathcal{U},F)\longrightarrow H_p^q(\Omega,F)$$

is an isomorphism for $q\ge 0$ and $1\le p\le\infty$.

Let Ω be a locally polycylindrical domain and $\mathcal{U} = \{U_j\}_{j\in I}$ be as above, so that each U_j is a polycylinder and $\Omega = \bigcup_{j\in I} U_j$. Let $\mathcal{E}^{0,q}$ be the sheaf of germs of C^∞ forms of type $(0,q)$ on $\mathbb{C}^n$, and $F^{0,q}$ the sheaf of germs of $\bar\partial$-closed C^∞ forms of type $(0,q)$ on $\mathbb{C}^n$. Define $\Gamma_p(\Omega,\mathcal{E}^{0,q})$ and $\Gamma(\Omega,F^{0,q})$ and $\hat\Gamma_p(\Omega,\mathcal{E}^{0,q})$ by

$$\Gamma_p(\Omega,\mathcal{E}^{0,q}) := \left\{ f \in \Gamma(\Omega,\mathcal{E}^{0,q}) :\ \|f\|^{(n)}_{L^p_{0,q)}(\Omega)} < \infty \right\},$$

$$\Gamma_p(\Omega,F^{0,q}) := \left\{ f \in \Gamma(\Omega,F^{0,q}) : \|f\|^{(n)}_{L^p_{(0,q)}(\Omega)} < \infty \right\}$$

$$\hat\Gamma_p(\Omega,\mathcal{E}^{0,q}) := \{ f \in \Gamma_p(\Omega,\mathcal{E}^{0,q}) :\ \bar\partial f \in \Gamma_p(\Omega,\mathcal{F}^{0,q+1})\}\,.$$

Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions in $\mathbb{C}^n$ as before. Then we use Theorem 1 to prove the following:

THEOREM 5. *$H^q_p(\mathcal{U},\mathcal{O})$ is for $q>0$ and $1\le p\le\infty$ isomorphic to the quotient space*

$$\Gamma_p(\Omega,F^{p,q})/(\bar\partial\Gamma_p(\Omega,\mathcal{E}^{0,q-1}))\bigcap\Gamma_p(\Omega,\mathcal{E}^{0,q})$$

Finally, we have the theorem to correspond to the above theorem, when we use Theorem 2:

THEOREM 6. *$\bar H^q_p(\mathcal{U},\mathcal{O})$ is for $q>0$ and $1\le p\le\infty$ isomorphic to the quotient space*

$$\{f \in \Gamma_p(\Omega,F^{0,q}) : \|f\|^{(n)}_{L^p_{(0,q)}(\bar\Omega)} \textit{ is defined and finite}\}/$$

$$\{\bar\partial g \in \Gamma_p(\Omega,F^{0,q}) : \|g\|^{(n)}_{L^p_{(0,q-1)}(\bar\Omega)} \textit{ and } \|\bar\partial g\|^{(n)}_{L^p_{(0,q)}(\bar\Omega)} \textit{ are defined and finite}\}$$

Let Ω be a locally polycylindrical domain. Using the fact that Ω is Stein and therefore Cartan's theorem B holds on (Ω), Theorem 4 and Theorem 5, we get

THEOREM 7. *If f is a smooth $\bar\partial$-closed $(0,q+1)$-form on Ω with $\|f\|^{(n)}_{L^p_{(0,q+1)}(\Omega)} < \infty$, then there is a smooth $(0,q)$-form u on Ω with $\bar\partial u = f$ and $\|u\|^{(n)}_{L^p_{(0,q)}(\Omega)} < \infty$.*

Corresponding to Theorem 7, using Theorem 6 instead of Theorem 5, we have

THEOREM 8. *If f is smooth $\bar{\partial}$-closed $(0, q+1)$-form on Ω with $\|f\|^n_{L^p_{(0,q+1)}(\bar{\Omega})}$ defined and finite, then there is a smooth $(0, q)$-form u on Ω with $\bar{\partial}u = f$ and $\|u\|^{(n)}_{(L^p(\bar{\Omega})}$ is defined and finite.*

Let Ω be a locally polycylindrical domain and $U \neq \phi$ a set open in $\bar{\Omega}$, then $B^p_\Omega(U)$ is the Banach space of holomorphic functions f on $\Omega \cap U$ such that $\|f\|_{L^p(\Omega\cap U)} < \infty$, $1 \leq p \leq \infty$. If V is open $\bar{\Omega}$ with $\phi \neq V \subset U$, the restriction map $r^U_V : B^p_\Omega(U) \longrightarrow B^p_\Omega(V)$ is defined. Then $B^p_0 = \{B^p\Omega(U); r^U_V\}$ is then the canonical presheaf of L^p-holomorphic functions on $\bar{\Omega}$. The associated sheaf $\mathcal{B}_p$ is the sheaf of germs of L^p-holomorphic functions on Ω.

From Theorem 7, using arguments similar to those in [5] section 2, we get the following:

THEOREM 9. *Let Ω be a locally polycylindrical domain and $\mathcal{B}_p$ the sheaf of germs of L^p-holomorphic functions on $\bar{\Omega}$. Then*

$$H^q(\bar{\Omega}, \mathcal{B}_p) = 0 \; for \; q \geq 1 \; and \; 1 \leq p \leq \infty.$$

Similar to the definition of the sheaves $\mathcal{B}_p$, $1 \leq p \leq \infty$, we can define $\mathcal{A}$ the sheaf of germs of holomorphic functions in Ω which are continuous on $\bar{\Omega}$.

Then using Theorem 8, we have

THEOREM 9. *Let Ω be a locally polycylindrical domain and A the sheaf of germs of holomorphic functions in Ω that are continuous on $\bar{\Omega}$. Then*

$$H^q(\bar{\Omega}, \mathcal{A}) = 0 \; for \; q \geq 1.$$

2 – Dolbeault's Isomorphism Theorem with Bounds

All the theorems in Section 1 except Theorems 5 and 6 have proofs which are similar to the corresponding theorems in [5], so we do not prove them in this paper.

To prove Theorem 5, with $U_\alpha = U_{\alpha_0} \cap \ldots \cap U_{\alpha_q}$ *for a multi-index* $\alpha = (\alpha_0, \ldots, \alpha_q) \in I^{q+1}$*, let us consider the exact sequence*

$$0 \longrightarrow \Gamma_p(U_\alpha, F^{0,q}) \longrightarrow \hat{\Gamma}_p(U_\alpha, \mathcal{E}^{0,q}) \xrightarrow{\bar\partial} \Gamma_p(U_\alpha, F^{0,q+1}) \longrightarrow 0.$$

The exactness follows from Theorem 1. With the cochain groups $C^r_p(\Omega, F^{0,q})$ *and* $\hat{C}^r_p(\mathcal{U}, \mathcal{E}^{0,q})$ *defined from the* $\Gamma_p(U_\alpha, F^{0,p})$ *and the* $\hat{\Gamma}_p(U_\alpha, \mathcal{E}^{0,q})$ *in the obvious way, we have the short exact sequence*

$$0 \longrightarrow C^r_p(\mathcal{U}, F^{0,q}) \longrightarrow \hat{C}^r_p(\mathcal{U}, \mathcal{E}^{0,q}) \xrightarrow{\bar\partial} C^r_p(\mathcal{U}, F^{0,q+1}) \longrightarrow 0.$$

From which we get the long exact sequence

$$0 \longrightarrow \Gamma_p(\Omega, F^{0,q}) \longrightarrow \hat{\Gamma}_p(\Omega, \mathcal{E}^{0,q}) \xrightarrow{\bar\partial} \Gamma_p(\Omega, F^{0,q+1})$$

$$\longrightarrow H^1_p(\mathcal{U}, F^{0,q}) \longrightarrow \hat{H}^1_p(\mathcal{U}, \mathcal{E}^{0,q}) \longrightarrow H^1_p(\mathcal{U}, F^{0,q+1}) \longrightarrow H^2_p(\mathcal{U}, F^{0,p}) \longrightarrow \cdots$$

where $H^r_p(\mathcal{U}, F^{0,q})$ *is the* r*th cohomology group of the complex*

$$C^0_p(\mathcal{U}, F^{0,q}) \longrightarrow C^1_p(\mathcal{U}, F^{0,q}) \longrightarrow \cdots$$

and $\hat{H}^r_p(\mathcal{U}, \mathcal{E}^{0,q})$ *that of the complex*

$$\hat{C}^0_p(\mathcal{U}, \mathcal{E}^{0,q}) \longrightarrow \hat{C}^1_p(\mathcal{U}, \mathcal{E}^{0,q}) \longrightarrow \cdots$$

It is easy to show that $\hat{H}^r_p(\mathcal{U}, \mathcal{E}^{0,q}) = 0$ *for* $r > 0$*. Therefore*

$$H^r_p(\mathcal{U}, F^{0,q+1}) \approx H^{r+1}_p(\mathcal{U}, F^{0,q}),\ r \geq 1,$$

$$H^1_p(\mathcal{U}, F^{0,q-1}) \approx \Gamma_p(\Omega, F^{0,q}) / \{\bar\partial \Gamma_p(\Omega, \mathcal{E}^{0,q-1}) \cap \Gamma_p(\Omega, \mathcal{E}^{0,q}).$$

Hence when $q > 0$

$$H_p^q(\mathcal{U}, \mathcal{O}) = H_p^q(\mathcal{U}, F^{0,0}) \approx H_p^{q-1}(\mathcal{U}, F^{0,1}) \approx$$

$$\approx H_p^1(\mathcal{U}, F^{0,q-1}) \approx \Gamma_p(\Omega, F^{0,q})/\bar{\partial}\Gamma_p(\Omega, \mathcal{E}^{0,q-1}) \cap \Gamma_p(\Omega, \mathcal{E}^{0,q})\,.$$

This proves Theorem 5, and the proof of Theorem 6 is similar.

3 – Conclusion

To show that the Weak Corona problem is solvable in $H^\infty(\Omega)$*, where* Ω *is a locally polycylindrical domain and* $H^\infty(\Omega)$ *is the algebra of bounded holomorphic functions on* Ω*, we use the reduction to the* L^∞ *version of Theorem 9 contained in [2], and to solve the Gleason problem on* Ω *we use the reduction to Theorem 10 as contained in [10].*

Therefore the Weak Corona problem and the Gleason problem are solvable on locally polycylindrical domains.

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