Rendiconti di Matematica, Serie VII
Volume 19, Roma (1999), 427-447

# Complex Finsler structures on CR-holomorphic vector bundles

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RIASSUNTO: *Per ogni fibrato vettoriale CR-olomorfo* (*nel senso di* N. TANAKA, [16]) $E$, *su una CR varietà fortemente pseudoconvessa* $M$, *dotato di una struttura di Finsler complessa* $F$, *si costruisce una connessione* $D$ *nel pullback* $\pi^*E$ *di* $E$ *via* $\pi : E \setminus \{0\} \longrightarrow M$, *simile alla connessione di Rund* (*cfr.* M. ABATE – G. PATRIZIO, [1]) *nella geometria di Finsler complessa.*

ABSTRACT: *For any CR-holomorphic vector bundle* (*in the sense of N. Tanaka,* [16]) $E$, *over a strictly pseudoconvex CR manifold* $M$, *equipped with a convex complex Finsler structure* $F$, *we build a connection* $D$ *in the pullback bundle* $\pi^*E$ *of* $E$ *by* $\pi : E \setminus \{0\} \longrightarrow M$, *similar to the Rund connection* (*cf.* M. ABATE – G. PATRIZIO, [1]) *in complex Finsler geometry.*

## 1 – Introduction

Let $M$ be a CR manifold and $E \longrightarrow M$ a complex vector bundle over $M$. Let $j : M \longrightarrow E$ be the zero section, i.e. $j(x) = 0_x$, $x \in M$, and set $E_0 = E \setminus j(M)$. A function $F : E \longrightarrow \mathbf{R}$ is a *complex Finsler structure* on $E$ if i) $F \in C^\infty(E_0)$, ii) $F(v) \geq 0$ and $F(v) = 0 \Longleftrightarrow v \in j(M)$, and iii) $F(\lambda v) = |\lambda|^2 F(v)$ for any $\lambda \in \mathbf{C}$, $v \in E$. Let $\pi : E_0 \longrightarrow M$ be the projection. Let $\pi^* E \longrightarrow E_0$ be the pullback of $E \longrightarrow M$ by $\pi$. Any *convex*

KEY WORDS AND PHRASES: *CR-holomorphic bundle - Complex Finsler structure - Tanaka-Webster connection.*
A.M.S. CLASSIFICATION: 53C60 - 32C16

complex Finsler structure $F$ on $E$ induces a Hermitian metric $H$ on $\pi^*E$. When $M$ is a complex manifold and $E \longrightarrow M$ a holomorphic vector bundle over $M$ there is a natural connection $D$ in $\pi^*E$ (the *Rund connection*, cf. [10]) i.e. the Hermitian connection of $(\pi^*E, H)$ (cf. e.g. [19], p. 79). Indeed, if this is the case then $\pi^*E \longrightarrow E_0$ is a Hermitian vector bundle over $E_0$. See also M. Abate – G. Patrizio, [1], p. 88. As known, the CR analogue of the canonical Hermitian connection of a Hermitian vector bundle is the *Tanaka connection* (cf. [16], p. 39) of a CR-holomorphic vector bundle (over a strictly pseudoconvex CR manifold) endowed with a Hermitian metric. Then, is there any natural choice of connection $D$ in $\pi^*E$ (so that $DH = 0$)? It is to be observed that, as opposed to the holomorphic category (where $E_0$ is a complex manifold), given a CR-holomorphic vector bundle $E \longrightarrow M$ over a CR manifold $M$, $E_0$ is not a CR manifold. On the other hand, Tanaka's construction (cf. [16], p. 40-41) relies heavily on the existence of the *Tanaka-Webster connection* (cf. [16], p. 29-30, and [18]) of the base CR manifold. We circumvent this difficulty by observing that, while $E_0$ carries no CR structure, the vertical foliation $\mathcal{F}$ of $E_0$ (i.e. the foliation $\mathcal{F}$ tangent to $\mathrm{Ker}(d\pi)$) possesses a natural transverse CR structure, and therefore one may use the theory of transversally CR foliations as developed in [4]-[5] (and the *transverse Webster connection* there). The canonical connection $D = D(F, N)$ we build (cf. Theorem 1) depends on $H$, on the complex structure of $T(\mathcal{F})$, respectively on the CR-holomorphic structure of $E$ (expressed by the occurence of two differential operators $\overline{\partial}_{\mathcal{F}} : \Gamma^\infty(\pi^*E) \longrightarrow \Gamma^\infty(T_{0,1}(\mathcal{F})^* \otimes \pi^*E)$ and $\overline{\partial}_{\mathcal{H}} : \Gamma^\infty_B(\pi^*E) \longrightarrow \Gamma^\infty(\overline{\mathcal{H}}^* \otimes \pi^*E)$, cf. Section 3), and on a fixed choice of complement $N$ of $T(\mathcal{F})$ in $T(E_0)$. While the choice of $N$ is arbitrary, $D(F, N)$ is shown to be independent of the choice of contact form $\theta_M$ on $M$, used to build it (hence $D(F, N)$ is a CR invariant). The construction in Theorem 1 only works for strictly pseudoconvex CR manifolds $M$, in particular $M$ should have CR codimension 1, and it is an interesting question, raised by the referee, whether any useful generalization, to the higher codimension case, may be produced.

## 2 – CR geometry

We briefly recall the notions of CR and pseudohermitian geometry we need, such as the existence (and axiomatic description) of the Tanaka-

Webster connection (consequently $T_{1,0}(M)$ may be organized as a CR-holomorphic bundle). The main reference is [16]. Cf. also [9]. As to (transversally) CR foliations, we rely upon recent work in [4]-[5].

## 2.1 – CR manifolds

Let $M$ be a real $(2n+k)$-dimensional $C^\infty$ manifold and $T_{1,0}(M)$ a *CR structure* on $M$, i.e. a rank $n$ complex subbundle of the complexified tangent bundle $T(M) \otimes \mathbf{C}$ so that

$$T_{1,0}(M) \cap T_{0,1}(M) = (0)$$

$$[\Gamma^\infty(T_{1,0}(M)), \Gamma^\infty(T_{1,0}(M))] \subseteq \Gamma^\infty(T_{1,0}(M))$$

where $T_{0,1}(M) = \overline{T_{1,0}(M)}$ and an overbar denotes complex conjugation. The pair $(M, T_{1,0}(M))$ is a *CR manifold* (of *type* $(n,k)$). The integers $n$ and $k$ are the *CR dimension* and *CR codimension*, respectively. A CR manifold of CR codimension $k=0$ is a complex manifold. When $k=1$ we refer to $M$ as a CR manifold of *hypersurface type*. It is with this type of CR manifolds that the present paper is mainly concerned.

Let $(M, T_{1,0}(M))$ be a CR manifold and $F : T_{1,0}(M) \longrightarrow [0,\infty)$ a complex Finsler structure on $T_{1,0}(M)$. Then $(M,F)$ is a *Finslerian CR manifold*. To give a class of examples, we recall that a *complex Minkowski norm* on a complex linear space $V$ is a map $v \mapsto \|v\|$, $v \in V$, so that i) $\|v\| \geq 0$ and $\|v\| = 0 \iff v = 0$, ii) $\|\lambda v\| = |\lambda| \, \|v\|$, and iii) for any linear basis $\{e_1, \cdots, e_n\}$ of $V$ the map $f(z^1, \cdots, z^n) = \|z^i e_i\|$ is at least of class $C^4$ at $z \neq 0$. A pair $(V, \|\cdot\|)$ is a *complex Minkowski space*. Two complex Minkowski spaces $V, W$ are *congruent* if there is a $\mathbf{C}$-linear isomorphism $\varphi : V \longrightarrow W$ so that $\|\varphi(v)\| = \|v\|$ for any $v \in V$. Let $M$ be a CR manifold and $F : T_{1,0}(M) \longrightarrow [0,\infty)$ a complex Finsler structure. Then $T_{1,0}(M)_x$ is a complex Minkowski space in a natural way, for any $x \in M$. The Finslerian CR manifold $M$ is *modelled* on $(V, \|\cdot\|)$ if $T_{1,0}(M)_x \approx V$ (congruent complex Minkowski spaces) for any $x \in M$. Let $(M, T_{1,0}(M))$ be a CR manifold of CR dimension $n$ and let $B(M) \longrightarrow M$ be the principal $GL(n,\mathbf{C})$-bundle whose associated bundle of standard fibre $\mathbf{C}^n$ is $T_{1,0}(M) \longrightarrow M$, i.e.

$$(B(M) \times \mathbf{C}^n)/\, GL(n,\mathbf{C}) \approx T_{1,0}(M)$$

(a vector bundle isomorphism). Let $(V, \|\cdot\|)$ be a complex Minkowski space and $\{e_1, \cdots, e_n\}$ a fixed linear basis of $V$. Let

$$G = \{[U^\alpha_\beta] \in GL(n, \mathbf{C}) : f(U^1_\alpha z^\alpha, \cdots, U^n_\alpha z^\alpha) = f(z^1, \cdots, z^n),$$

$$\forall\, (z^1, \cdots, z^n) \in \mathbf{C}^n\}\,.$$

Then $G$ is a closed subgroup of $GL(n, \mathbf{C})$. Let $H \subset G$ be a Lie subgroup and $B \longrightarrow M$ a principal $H$-subbundle of $B(M) \longrightarrow M$. A pair $(M, B)$ is referred to as a *CR* $\{V, H\}$*-manifold* (by analogy with [12]). Let $v \in T_{1,0}(M)_x$ and $\{T_\alpha\}$ a (local) frame of $T_{1,0}(M)$ defined on an open neighborhood $U$ of $x$, adapted to $B$. Then $v = v^\alpha T_\alpha(x)$ for some $v^\alpha \in \mathbf{C}$. Set

$$F(v) = \|v^\alpha e_\alpha\|^2\,.$$

Therefore, any CR $\{V, H\}$-manifold is a Finslerian CR manifold modelled on $(V, \|\cdot\|)$.

Let $(M, T_{1,0}(M))$ be a CR manifold. Its *Levi distribution*

$$H(M) = Re\{T_{1,0}(M) \oplus T_{0,1}(M)\}$$

carries the complex structure

$$J : H(M) \longrightarrow H(M), \;\; J(Z + \overline{Z}) = i(Z - \overline{Z}), \;\; Z \in T_{1,0}(M)\,.$$

Here $i = \sqrt{-1}$. Let $H(M)^\perp \subset T^*(M)$ be the conormal bundle of $H(M)$ i.e.

$$H(M)^\perp_x = \{\omega \in T^*_x(M) : \mathrm{Ker}(\omega) \supseteq H(M)_x\}, \;\; x \in M\,.$$

Assume $M$ to be an orientable CR manifold of hypersurface type, of CR dimension $n$. Then $H(M)^\perp$ is a trivial real line bundle over $M$. A nowhere zero global section $\theta \in \Gamma^\infty(H(M)^\perp)$ is a *pseudohermitian structure* on $M$. The CR manifold $M$ is *nondegenerate* if the *Levi form*

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}), \;\; Z, W \in T_{1,0}(M)$$

is nondegenerate for some pseudohermitian structure $\theta$ (and thus for all). If this is the case then $\theta$ is a *contact form*, i.e. $\theta \wedge (d\theta)^n$ is a volume form on $M$. The CR manifold $M$ is *strictly pseudoconvex* if $L_\theta$ is positive definite for some $\theta$.

Let $M$ be a nondegenerate CR manifold and $\theta$ a contact form. There is a unique nowhere zero tangent vector field $T$ on $M$ (the *characteristic direction* of $(M, \theta)$) so that

$$\theta(T) = 1 \,, \; T \rfloor d\theta = 0 \,.$$

The *Webster metric* $g_\theta$ is the semi-Riemannian metric on $M$ given by

$$g_\theta(T, T) = 1 \,, \; g_\theta(T, X) = 0$$

$$g_\theta(X, Y) = (d\theta)(X, JY)$$

for any $X, Y \in H(M)$. For any nondegenerate CR manifold $M$ on which a contact form $\theta$ has been fixed, there is a unique linear connection $\nabla$ on $M$ (the *Tanaka-Webster connection* of $(M, \theta)$) so that i) $H(M)$ is parallel with respect to $\nabla$, ii) $\nabla J = 0$, $\nabla g_\theta = 0$, iii) the torsion $T_\nabla$ of $\nabla$ is *pure*, i.e.

$$T_\nabla(Z, W) = 0$$

$$T_\nabla(Z, \overline{W}) = 2iL_\theta(Z, \overline{W})T$$

$$\tau \circ J + J \circ \tau = 0$$

for any $Z, W \in T_{1,0}(M)$, where $\tau X = T_\nabla(T, X)$, $X \in T(M)$, is the *pseudohermitian torsion*. Cf. [16] and [18]. See also [8], p. 173-174.

### 2.2 – CR-holomorphic bundles

Let $E \longrightarrow M$ be a complex vector bundle, of standard fibre $\mathbf{C}^r$, over a CR manifold $M$. It is *CR-holomorphic* if it is endowed with a differential operator

$$\overline{\partial}_E : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(T_{0,1}(M)^* \otimes E)$$

so that

$$\overline{\partial_E}(fu) = f\overline{\partial}_E u + (\overline{\partial}_M f) \otimes u$$

$$[\overline{Z}, \overline{W}]u = \overline{Z}\,\overline{W}u - \overline{W}\,\overline{Z}u$$

for any $f \in C^\infty(M), u \in \Gamma^\infty(E)$, and $Z, W \in \Gamma^\infty(T_{1,0}(M))$. Here

$$\overline{\partial}_M : C^\infty(M) \longrightarrow \Gamma^\infty(T_{0,1}(M)^*) \,, \; (\overline{\partial}_M f)\overline{Z} = \overline{Z}(f)$$

is the *tangential Cauchy-Riemann operator* (on functions). Also one adopts the notation $\overline{Z}u = (\overline{\partial}_E u)\overline{Z}$.

EXAMPLES. i) Let $V$ be a complex $N$-dimensional manifold and $M \subset V$ a real hypersurface endowed with the CR structure

$$T_{1,0}(M) = [T(M) \otimes \mathbf{C}] \cap T^{1,0}(V)$$

where $T^{1,0}(V)$ is the holomorphic tangent bundle over $V$ (if $(U, z^j)$ is a local system of complex coordinates on $V$ then the portion of $T^{1,0}(V)$ over $U$ is the span of $\{\partial/\partial z^j : 1 \leq j \leq N\}$). Then the portion over $M$ of any holomorphic vector bundle over $V$ is a CR-holomorphic vector bundle over $M$.

ii) Let $(M, T_{1,0}(M))$ be a CR manifold and set

$$\hat{T}(M) = \frac{T(M) \otimes \mathbf{C}}{T_{0,1}(M)} .$$

Let $\rho : T(M) \otimes \mathbf{C} \longrightarrow \hat{T}(M)$ the projection. Then $\hat{T}(M)$ is a CR-holomorphic vector bundle with the differential operator

$$\overline{\partial}_{\hat{T}(M)} : \Gamma^\infty(\hat{T}(M)) \longrightarrow \Gamma^\infty(T_{0,1}(M)^* \otimes \hat{T}(M))$$

$$(\overline{\partial}_{\hat{T}(M)} u)\overline{Z} = \rho \left[\overline{Z}, W\right]$$

for any $u \in \Gamma^\infty(\hat{T}(M))$, $Z \in \Gamma^\infty(T_{1,0}(M))$, and some $W \in \Gamma^\infty(T(M) \otimes \mathbf{C})$ so that $\rho W = u$.

iii) Let $M$ be a nondegenerate CR manifold, $\theta$ a contact form on $M$, and $\nabla$ the Tanaka-Webster connection of $(M, \theta)$. Then $T_{1,0}(M)$ is a CR-holomorphic vector bundle with the differential operator

$$\overline{\partial}_{T_{1,0}(M)} : \Gamma^\infty(T_{1,0}(M)) \longrightarrow \Gamma^\infty(T^*_{0,1} \otimes T_{1,0}(M))$$

$$(\overline{\partial}_{T_{1,0}(M)} Z)\overline{W} = \nabla_{\overline{W}} Z \ , \ Z, W \in \Gamma^\infty(T_{1,0}(M)) \, .$$

Cf. e.g. [17], p. 569.

## 2.3 – CR foliations

Let $M$ be a $C^\infty$ manifold and $\mathcal{F}$ a codimension $2n+1$ foliation of $M$. Let $T(\mathcal{F})$ be the tangent bundle of the foliation and $\nu(\mathcal{F}) = T(M)/T(\mathcal{F})$ its normal (or transverse) bundle. Let $\pi_{\mathcal{F}} : T(M) \longrightarrow \nu(\mathcal{F})$ be the projection. Let $\mathring{\nabla}$ be the *Bott connection* of $(M, \mathcal{F})$, i.e.

$$\mathring{\nabla}_X s = \pi_{\mathcal{F}}[X, Y]$$

for any $X \in \Gamma^\infty(T(\mathcal{F}))$, $s \in \Gamma^\infty(\nu(\mathcal{F}))$, and some $Y \in \mathcal{X}(M)$ so that $\pi_{\mathcal{F}} Y = s$. Let $\mathcal{H} \subset \nu(\mathcal{F}) \otimes \mathbf{C}$ be a complex subbundle, of complex rank $n$. Set $H = Re\{\mathcal{H} \oplus \overline{\mathcal{H}}\} \subset \nu(\mathcal{F})$. Then $H$ carries the complex structure $J : H \longrightarrow H$, $J(\alpha + \overline{\alpha}) = i(\alpha - \overline{\alpha})$, $\alpha \in \mathcal{H}$. Then $\mathcal{H}$ is a *transverse almost CR structure* (of *transverse CR dimension* $n$) if 1) $\mathcal{H} \cap \overline{\mathcal{H}} = (0)$, 2) $H$ is parallel with respect to the Bott connection of $\mathcal{F}$, and 3) $\mathcal{L}_X J = 0$ for any $X \in \Gamma^\infty(T(\mathcal{F}))$. Lie derivatives are defined with respect to $\mathring{\nabla}$.

Let $L(\mathcal{F}) = L(M, \mathcal{F}) \subset \mathcal{X}(M)$ be the Lie subalgebra of all *foliate vector fields* (or infinitesimal automorphisms of $\mathcal{F}$), cf. e.g. [15], p. 35. Let $\ell(\mathcal{F}) = \ell(M, \mathcal{F}) \subset \Gamma^\infty(\nu(\mathcal{F}))$ be the Lie algebra of all *transverse vector fields* (i.e. $s \in \ell(\mathcal{F}) \Longleftrightarrow s = \pi_{\mathcal{F}} Y$ for some $Y \in L(\mathcal{F})$). Let $\Gamma^\infty_B(\nu(\mathcal{F}))$ consist of all $s \in \Gamma^\infty(\nu(\mathcal{F}))$ with $\mathcal{L}_X s = 0$ for any $X \in \Gamma^\infty(T(\mathcal{F}))$. Note that $\Gamma^\infty_B(\mathcal{F}) = \ell(\mathcal{F})$ (so that the Lie bracket $[s, r]$ of any $s, r \in \Gamma^\infty_B(\nu(\mathcal{F}))$ is well defined).

A transverse almost CR structure $\mathcal{H} \subset \nu(\mathcal{F}) \otimes \mathbf{C}$ is *integrable* if for any $x \in M$ there is an open neighborhood $U \subseteq M$, $x \in U$, and there is a frame $\{\zeta_1, \cdots, \zeta_n\}$ of $\mathcal{H}$ on $U$ so that $\zeta_\alpha \in \Gamma^\infty_B(\nu(\mathcal{F}) \otimes \mathbf{C})$ and $[\zeta_\alpha, \zeta_\beta] \in \Gamma^\infty(\mathcal{H})$ for any $1 \leq \alpha, \beta \leq n$. Such a local frame of $\mathcal{H}$ is termed *admissible*. An integrable transverse almost CR structure is a *transverse CR structure* on $(M, \mathcal{F})$. When $\mathcal{F}$ is the trivial foliation by points a transverse CR structure is an ordinary CR structure.

Let $(N, T_{1,0}(N))$ be a CR manifold of hypersurface type, of CR dimension $n$. A *CR automorphism* $f : N \longrightarrow N$ is a $C^\infty$ diffeomorphism and a CR map (i.e. $(d_x f) T_{1,0}(N)_x \subseteq T_{1,0}(N)_{f(x)}$, $x \in N$). Let $\Gamma^\infty_{CR}(N)$ be the pseudogroup of all (local) CR automorphisms of $(N, T_{1,0}(N))$ (of class $C^\infty$). Let $\mathcal{F}$ be a $\Gamma^\infty_{CR}(N)$-foliation of $M$ (in the sense of [11]). Then $\mathcal{F}$ is a (*transversally*) *CR foliation* (of *transverse CR dimension* $n$). As such (by Theor. 1 in [5], p. 55) $\mathcal{F}$ carries a transverse CR structure $\mathcal{H}$. For instance, let $f : M \longrightarrow N$ be a $C^\infty$ submersion from a $C^\infty$ manifold onto a CR manifold $N$. The vertical distribution $\mathrm{Ker}(df)$ is integrable, thus giving rise to a CR foliation of $M$ (whose leaves are the connected components of the fibres of $f$) which is transversally nondegenerate (strictly pseudoconvex) if $N$ is a nondegenerate (strictly pseudoconvex) CR manifold (cf. Theor. 4 in [5], p. 60). CR foliations also arise on certain degenerate CR manifolds (cf. [5], p. 64-68). See [3] for an application of the CR foliation theory to the Beltrami equations on degenerate CR manifolds.

## 3 – The canonical connection

Let $(E, \overline{\partial}_E)$ be a CR-holomorphic vector bundle (of standard fibre $\mathbf{C}^r$) over the CR manifold $(M, T_{1,0}(M))$. Let $\pi^* E \longrightarrow E_0$ be the pullback of $E$ by $\pi$. Given a section $s : M \longrightarrow E$ its *natural lift* is the section

$$\tilde{s} : E_0 \longrightarrow \pi^* E \ , \ \tilde{s}(u) = (u, s(\pi(u))) \ , \ u \in E_0 \, .$$

Given a local frame $\{s_1, \cdots, s_r\}$ of $E$ on the open set $U \subseteq M$, $\{\tilde{s}_1, \cdots, \tilde{s}_r\}$ is a local frame of $\pi^* E$ on $\pi^{-1}(U) \subseteq E_0$.

Let $\mathcal{F}$ be the *vertical foliation* on $E_0$, i.e. the simple foliation defined by the $C^\infty$ submersion $\pi : E_0 \longrightarrow M$. Let $p^* \mathcal{F}$ be the pullback of $\mathcal{F}$ to $\pi^* E$ (cf. [15], p. 30), where $p : \pi^* E \longrightarrow E_0$ is the projection (a $C^\infty$ submersion). A section $\sigma : E_0 \longrightarrow \pi^* E$ is *foliate* if

$$(d_u \sigma) T(\mathcal{F})_u \subseteq T(p^* \mathcal{F})_{\sigma(u)} \ , \ u \in E_0 \, .$$

Note that the foliate sections in $\pi^* E$ are precisely the natural lifts of the sections in $E$. Indeed, let $\zeta^j : \pi^{-1}(U) \longrightarrow \mathbf{C}$ be complex fibre coordinates, i.e. $u = \zeta^j(u) s_j(\pi(u))$, for any $u \in \pi^{-1}(U)$. A section $\sigma : \pi^{-1}(U) \longrightarrow \pi^* E$ is locally represented as

$$\sigma(x, \zeta) = (x, \zeta, f^j(x, \zeta) e_j)$$

for some $C^\infty$ functions $f^j : \pi^{-1}(U) \longrightarrow \mathbf{C}$. Here $\{e_j\} \subset \mathbf{C}^r$ is the canonical linear basis. Then $\sigma$ is foliate if and only if $(d\sigma) X \in T(p^* \mathcal{F})$ for any $X = X^j \partial / \partial \zeta^j + \overline{X^j} \partial / \partial \overline{\zeta}^j \in T(\mathcal{F})$. This is equivalent to $X(f^j) = 0$, i.e.

$f^j \in \Omega^0_B(\mathcal{F})$. Hence $f^j$ descends to a function on $U$ (denoted by the same symbol) and $\sigma$ is the natural lift of $f^j s_j$.

Let $\Gamma^\infty_B(\pi^* E)$ be the space of all foliate sections in $(\pi^* E, p^* \mathcal{F}) \longrightarrow (E_0, \mathcal{F})$. Let $\mathcal{H}$ be the transverse CR structure of $\mathcal{F}$. We shall need the differential operator

$$\overline{\partial}_{\mathcal{H}} : \Gamma^\infty_B(\pi^* E) \longrightarrow \Gamma^\infty(\overline{\mathcal{H}}^* \otimes \pi^* E)$$

defined by

$$(\overline{\partial}_{\mathcal{H}} \tilde{s})_u \overline{\alpha} = (\overline{\partial}_E s)_{\pi(u)} (d_T \pi)_u \overline{\alpha}$$

for any $\alpha \in \mathcal{H}_u, u \in E_0$. The map $(d_T \pi)_u : \nu(\mathcal{F})_u \longrightarrow T_{\pi(u)}(M)$ is naturally induced by $d\pi$ (a $\mathbf{R}$-linear isomorphism, because of $T(\mathcal{F}) = \mathrm{Ker}(d\pi)$).

Let $F : E \longrightarrow [0, \infty)$ be a complex Finsler structure and set

$$H_u(Z, W) = H_{j\bar{k}}(u) Z^j \overline{W^k}$$

$$H_{j\bar{k}} = \frac{1}{2} \frac{\partial^2 F^2}{\partial \zeta^j \partial \bar{\zeta}^k}$$

$$Z = Z^j \tilde{s}_j(u), W = W^j \tilde{s}_j(u) \in (\pi^* E)_u, u \in \pi^{-1}(U)\,.$$

Then $H$ is globally defined. We say $F$ is *convex* if $H$ is positive definite. If $F$ is convex $H$ is a Hermitian metric in $\pi^* E \longrightarrow E_0$. There are, however, interesting examples of degenerate complex Finsler structures. For instance, let $(M, T_{1,0}(M))$ be a *parallelizable CR manifold*, i.e. $T_{1,0}(M)$ admits a global frame $\{T_1, \cdots, T_n\}$. Let $F : T_{1,0}(M) \longrightarrow [0, \infty)$ be given by

$$F(u) = \left| u^1 \cdots u^n \right|^{2/n}$$

(a CR analogue of the real Finsler metric in [2]) where $u = u^\alpha T_\alpha(x), u \in T_{1,0}(M)_x, x \in M$. Then

$$H_{j\bar{k}} = \frac{1}{2n^2} \frac{F}{\zeta^j \bar{\zeta}^k}$$

hence $F$ is not convex ($\det[H_{j\bar{k}}] = 0$).

As remarked before, $E_0$ is not a CR manifold hence Tanaka's result does not apply. However $E_0$ carries the CR foliation $\mathcal{F}$ and the synthetic object $(\pi^* E, \overline{\partial}_{\mathcal{H}})$ is analogous to a CR-holomorphic vector bundle. We shall additionally need the differential operator

$$\overline{\partial}_{\mathcal{F}} : \Gamma^\infty(\pi^* E) \longrightarrow \Gamma^\infty(T_{0,1}(\mathcal{F})^* \otimes \pi^* E)$$

$$(\overline{\partial}_{\mathcal{F}} s) \frac{\partial}{\partial \overline{\zeta}^j} = \frac{\partial f^k}{\partial \overline{\zeta}^j} \tilde{s}_k$$

$$s = f^k \tilde{s}_k \ , \ f^k : \pi^{-1}(U) \longrightarrow \mathbf{C}$$

where $T_{0,1}(\mathcal{F}) \subset T(\mathcal{F}) \otimes \mathbf{C}$ is locally the span of $\{\partial/\partial\overline{\zeta}^j : 1 \leq j \leq r\}$. Note that $\overline{\partial}_{\mathcal{F}} s$ is globally defined (and $\overline{\partial}_{\mathcal{F}} \tilde{s}_j = 0$).

Let $M$ be a nondegenerate CR manifold, of signature $(r, s)$ (the signature of the Levi form). Then $\mathcal{F}$ is a (transversally) nondegenerate CR foliation, of the same signature. We shall need the *basic complex* of $(E_0, \mathcal{F})$

$$\Omega^0_B(\mathcal{F}) \xrightarrow{d_B} \Gamma^\infty_B(\nu(\mathcal{F})^*) \xrightarrow{d_B} \Gamma^\infty_B(\Lambda^2 \nu(\mathcal{F})^*) \xrightarrow{d_B} \cdots \xrightarrow{d_B} \Gamma^\infty_B(\Lambda^{2n+1} \nu(\mathcal{F})^*) \longrightarrow 0.$$

Let $\theta_M$ be a contact form on $M$ and $\theta \in \Gamma^\infty_B(\nu(\mathcal{F})^*)$ be the naturally induced *transverse pseudohermitian structure* (i.e. $\theta_u = (\theta_M)_{\pi(u)} \circ (d_T \pi)_u \ , \ u \in E_0$) on $(E_0, \mathcal{F})$. Let $L_\theta(\alpha, \overline{\beta}) = -(d_B \theta)(\alpha, \overline{\beta}), \alpha, \beta, \in \mathcal{H}$, be the *transverse Levi form*. The *trace* $\Lambda_\theta \varphi$ of a bilinear form $\varphi$ on $\mathcal{H} \otimes \overline{\mathcal{H}}$ is given by

$$i \Lambda_\theta \varphi = \sum_{\alpha=1}^{n} \epsilon_\alpha \varphi(\zeta_\alpha, \zeta_{\bar{\alpha}})$$

where $\{\zeta_\alpha\}$ is an orthonormal (i.e. $L_\theta(\zeta_\alpha, \zeta_{\bar{\beta}}) = \epsilon_\alpha \delta_{\alpha\beta}, \epsilon_1 = \cdots = \epsilon_r = -\epsilon_{r+1} = \cdots = -\epsilon_{r+s} = 1$) admissible local frame of $\mathcal{H}$.

In the sequel, we also fix a complement to $T(\mathcal{F})$ in $T(E_0)$, i.e. a vector bundle $N \longrightarrow E_0$ so that

$$T(E_0) = T(\mathcal{F}) \oplus N \tag{1}$$

(for instance, let $h$ be a Riemannian metric on $E_0$ and $N = T(\mathcal{F})^\perp$ the $h$-orthogonal complement of $T(\mathcal{F})$ in $T(E_0)$). Let $\sigma : \nu(\mathcal{F}) \longrightarrow N$ be the natural bundle isomorphism (associated with the direct sum decomposition (1)). We establish the following

THEOREM 1. *Let $F : E \longrightarrow [0, \infty)$ be a convex complex Finsler structure on a CR holomorphic vector bundle $(E, \overline{\partial}_E)$ over a strictly pseudoconvex CR manifold $(M, T_{1,0}(M))$. There exists a unique connection $D = D(F, N)$ in $\pi^* E \longrightarrow E_0$, depending on the data $(F, N)$, so that*

1) $D_{\overline{Z}} s = (\overline{\partial}_{\mathcal{F}} s)\overline{Z}$,

2) $D_{\sigma(\overline{\alpha})} \tilde{v} = (\overline{\partial}_{\mathcal{H}} \tilde{v})\overline{\alpha}$,

3) $DH = 0$,

4) $\Lambda_\theta R^D = 0$,

*for some contact form $\theta_M$ on $M$, and for any $Z \in \Gamma^\infty(T_{1,0}(\mathcal{F})), s \in \Gamma^\infty(\pi^* E)$, $\alpha \in \Gamma^\infty(\mathcal{H})$, and $v \in \Gamma^\infty(E)$. In particular, $D$ is a CR invariant (and axiom 4 holds for any contact form on $M$).*

Here $R^D$ is the curvature tensor field of $D$. In axiom 4, $R^D$ is thought of as the $End(E)$-valued bilinear form

$$(\alpha, \overline{\beta}) \mapsto R^D(\sigma(\alpha), \sigma(\overline{\beta})) \ , \ \alpha, \beta \in \mathcal{H} \, .$$

Before proving Theorem 1, we wish to look at the analogy with real Finsler geometry (cf. e.g. M. Matsumoto, [13]). A *nonlinear connection* on $M$ is a $C^\infty$ distribution $N$ on $V(M) = T(M) \setminus 0$ so that $T_u(V(M)) = \mathrm{Ker}(d_u \pi) \oplus N_u, \ u \in V(M)$ (cf. N. Barthel, [6]) where $\pi : V(M) \longrightarrow M$ is the projection. A *Finsler connection* on $M$ is a pair $(\nabla, N)$ consisting of a connection $\nabla$ in $\pi^* T(M)$ and a nonlinear connection $N$ on $M$ (cf. [13]). The *vertical lift* is the bundle isomorphism $\gamma : \pi^* T(M) \longrightarrow \mathrm{Ker}(d\pi)$ given by $\gamma_u(u, X) = \frac{dC}{dt}(0)$, where $C(t) = u + tX, \ |t| < \epsilon$. Given a nonlinear connection $N$ on $M$, the *horizontal lift* is the bundle isomorphism $\beta : \pi^* T(M) \longrightarrow N, \ \beta_u = \left( L_u|_{N_u} \right)^{-1}$, where

$$L_u Y = (u, (d_u \pi) Y) \tag{2}$$

for any $Y \in T_u(V(M)), \ u \in V(M)$. With any Finsler connection $(\nabla, N)$ one may associate two concepts of torsion, namely $T_L(X, Y) = \nabla_X LY - \nabla_Y LX - L[X, Y]$ and $T_K(X, Y) = \nabla_X KY - \nabla_Y KX - K[X, Y], \ X, Y \in \mathcal{X}(V(M))$, where $K = \gamma^{-1} \circ \pi^\perp$ is the *Dombrowski map* (here $\pi^\perp$ :

$T(V(M)) \longrightarrow \mathrm{Ker}(d\pi)$ is the projection). Given a real Finsler metric $F : T(M) \longrightarrow [0, +\infty)$, there is a naturally associated Riemannian bundle metric $g$ in $\pi^* T(M) \longrightarrow V(M)$, and one may consider the family of Finsler connections satisfying $\nabla g = 0$. Then a canonical connection (the *Cartan connection*, cf. [7]) may be chosen from this set, by additionally requiring that $T_F(\beta X, \beta Y) = 0$ and $T_K(\gamma X, \gamma Y) = 0$, for any $X, Y \in \Gamma^\infty(\pi^* T(M))$. Moreover, if $(\nabla, N)$ is the Cartan connection, then $N$ is also uniquely determined in terms of $F$. Other canonical connections (e.g. the *Berwald*, or *Rund connection*, cf. [13]) are of current use in real Finsler geometry. Now, given a complex vector bundle $E$ over a CR manifold $M$, there is an analogous notion of nonlinear connection, i.e. a $C^\infty$ distribution $N$ on $E_0$ so that (1) holds, and it is only natural that there should be a freedom of choice of $N$, just as in the case of Finsler connections. The (globally defined) bundle isomorphism

$$\gamma : \pi^* E \longrightarrow T_{1,0}(\mathcal{F}), \;\; \gamma(\tilde{s}_j) = \frac{\partial}{\partial \zeta^j}, \;\; 1 \leq j \leq r$$

may play the role of the vertical lift, yet the bundle morphism (2) is $\pi^* T(M)$-valued, rather than $\pi^* E$-valued, hence $T_L$ is not well defined (for a connection $D$ in $\pi^* E$). Therefore, there is no obvious 'torsion-free' requirement, and one may not expect that axioms 1-4 in Theorem 1 should influence upon the choice of $N \subset T(E_0)$.

Let $M$ be a CR manifold. Geometric objects depending only on the CR structure of $M$ are usually referred to as *CR invariants*. For instance, the signature of the Levi form (of a nondegenerate CR manifold) is a CR invariant. In CR geometry, several objects are built in terms of the given CR structure and a fixed pseudohermitian structure $\theta_M$. An example is the Tanaka-Webster connection (of $(M, \theta_M)$). Such an object is a CR invariant if it is invariant under a transformation $\hat{\theta}_M = e^{2f}\theta_M$, $f \in C^\infty(M)$ (and in this respect, CR geometry is, of course, analogous to conformal geometry). The Tanaka-Webster connection is not a CR invariant. While, as argued above, there is an apparent freedom in the choice of complement $N$ to $T(\mathcal{F})$ in $T(E_0)$ (which, as suggested by the referee, might be useful in applications), once $F$ and $N$ are fixed, the connection $D = D(F, N)$ furnished by Theorem 1 may be shown to be a CR invariant.

We firstly establish uniqueness. Let $D$ be a connection obeying to 1-4, where $\theta$ is the transverse pseudohermitian structure associated with a fixed contact form $\theta_M$ on $M$. Axiom 3 yields

$$Z(H(u,v)) = H(D_Z u, v) + H(u, D_{\overline{Z}} v) \tag{3}$$

for any $Z \in \Gamma^\infty(T_{1,0}(\mathcal{F}))$ and $u, v \in \Gamma^\infty(\pi^* E)$. Set

$$D_{\partial/\partial\zeta^j} \tilde{s}_k = C^i_{jk} \tilde{s}_i \,. \tag{4}$$

By axiom 1

$$D_{\partial/\partial\overline{\zeta}^j} \tilde{s}_k = 0 \,. \tag{5}$$

Hence (by (3) and (5))

$$C^i_{jk} = H^{i\bar{\ell}} \frac{\partial H_{k\bar{\ell}}}{\partial \zeta^j} \,. \tag{6}$$

where $[H^{i\bar{j}}] = [H_{i\bar{j}}]^{-1}$. Next, let $T$ be the characteristic direction of $(M, \theta_M)$ and $\xi \in \Gamma^\infty_B(\nu(\mathcal{F}))$ the corresponding *transverse characteristic direction* on $(E_0, \mathcal{F})$ (i.e. $(d_T\pi)_u \xi_u = T_{\pi(u)}, u \in E_0$). By axioms 2-3

$$D_{\sigma(\bar{\alpha})} \tilde{s}_j = (\overline{\partial}_{\mathcal{H}} \tilde{s}_j)\overline{\alpha} \tag{7}$$

$$H(D_{\sigma(\alpha)} \tilde{s}_j, \tilde{s}_k) = (\sigma\alpha)(H_{j\bar{k}}) - H(\tilde{s}_j, (\overline{\partial}_{\mathcal{H}} \tilde{s}_k)\overline{\alpha}) \,. \tag{8}$$

Taking into account the direct sum decompositions

$$T(E_0) = T(\mathcal{F}) \oplus \sigma\, \nu(\mathcal{F})$$

$$\nu(\mathcal{F}) \otimes \mathbf{C} = \mathcal{H} \oplus \overline{\mathcal{H}} \oplus \mathbf{C}\xi$$

we are left with the computation of $D_{\sigma(\xi)} u$ for $u \in \Gamma^\infty(\pi^* E)$. To this end, define $D^2 u$ by setting

$$(D^2 u)(X, Y) = D_X D_Y u - D_{\sigma(\nabla_X \pi_{\mathcal{F}} Y)} u$$

for any $X, Y \in \mathcal{X}(E_0)$ and $u \in \Gamma^\infty(\pi^* E)$. Here

$$\nabla : \Gamma^\infty(\nu(\mathcal{F})) \longrightarrow \Gamma^\infty(T^*(E_0) \otimes \nu(\mathcal{F}))$$

is the *transverse Webster connection* of $(E_0, \theta)$ (cf. Theorem 10 in [5], p. 73). Next, define $B$ by setting

(9) $$B(X,Y)u = (D^2 u)(X,Y) - (D^2 u)(Y,X)\,.$$

This may be also written as

(10) $$B(X,Y)u = R^D(X,Y)u - D_{\sigma\, T_\nabla(X,Y)} u + D_{\pi^\perp [X,Y]} u\,.$$

Here $T_\nabla$ is given by

$$T_\nabla(X,Y) = \nabla_X \pi_{\mathcal{F}} Y - \nabla_Y \pi_{\mathcal{F}} X - \pi_{\mathcal{F}}[X,Y]$$

and $\pi^\perp : T(E_0) \longrightarrow T(\mathcal{F})$ is the projection. Let $\{\zeta_\alpha\}$ be a local orthonormal admissible frame of $\mathcal{H}$. Define $S_U \in \Gamma^\infty(U, T(\mathcal{F}))$ by setting

$$S_U = i \sum_{\alpha=1}^{n} \pi^\perp [\sigma\, \zeta_\alpha, \sigma\, \zeta_{\bar\alpha}]\,.$$

If $\{\zeta'_\alpha\}$ is another orthonormal admissible frame of $\mathcal{H}$, defined on the open set $U', U \cap U' \neq \emptyset$, then (as $\pi^\perp \sigma = 0$)

$$\zeta'_\alpha = U^\beta_\alpha \zeta_\beta$$

$$\pi^\perp [\sigma\, \zeta'_\alpha, \sigma\, \zeta'_{\bar\alpha}] = U^\beta_\alpha U^{\bar\gamma}_{\bar\alpha} \pi^\perp [\sigma\, \zeta_\beta, \sigma\, \zeta_{\bar\gamma}]$$

$$\sum_{\alpha=1}^{n} U^\beta_\alpha U^{\bar\gamma}_{\bar\alpha} = \delta^{\beta\gamma}$$

hence the local sections $S_U$ glue up to a (globally defined) section $S \in \Gamma^\infty(T(\mathcal{F}))$. Set $X = \sigma(\zeta_\alpha), Y = \sigma(\zeta_{\bar\alpha})$ in (10) and take traces. As

$$T_\nabla(\sigma\, \alpha, \sigma\, \bar\beta) = 2i L_\theta(\alpha, \bar\beta)\xi$$

for any $\alpha, \beta \in \mathcal{H}$ (cf. [5], p. 73), it follows that (by axiom 4)

(11) $$2n D_{\sigma\, \xi} u = -(\Lambda_\theta B)u - D_S u$$

for any $u \in \Gamma^\infty(\pi^* E)$. The formulae (4)-(8) and (11) show that $D$ (obeying to the axioms 1-4) is unique. To prove existence, let

$$D : \Gamma^\infty(\pi^* E) \longrightarrow \Gamma^\infty(T(E_0)^* \otimes \pi^* E)$$

be defined by (4)-(8) and (11). Then $D$ is a connection in the vector bundle $\pi^* E$. For instance, the property

$$D_{\sigma\,\xi} f u = f D_{\sigma\,\xi} u + (\sigma\,\xi)(f) u \tag{12}$$

may be checked as follows. Firstly, note that

$$B(X,Y) f u = f B(X,Y) u + \left(\pi^\perp [X,Y]\right)(f) u - \left(\sigma\, T_\nabla(X,Y)\right)(f) u \tag{13}$$

for any $X, Y \in \Gamma^\infty(\sigma\, H)$, provided that

$$D_X f u = f D_X u + X(f) u \tag{14}$$

for any $X \in \Gamma^\infty(\sigma\, H)$. To check (14) note that (7)-(8) prescribe $D_{\sigma\,\bar\alpha}$, respectively $D_{\sigma\,\alpha}$, on foliate sections (natural lifts of sections in $E$) only. Then we extend $D_X, X \in \Gamma^\infty(\sigma\, H)$, as a derivation, to the whole of $\Gamma^\infty(\pi^* E)$. We still must check that (14) holds for $f \in C^\infty(M)$ and $u = \tilde s, s \in \Gamma^\infty(E)$. Here, we do not distinguish notationally between $f$ and its vertical lift $f \circ \pi \in \Omega^0_B(\mathcal F)$. We have

$$\begin{aligned} D_{\sigma\,\bar\alpha} f \tilde s &= (\overline\partial_{\mathcal H} f \tilde s)\bar\alpha = (\overline\partial_E f s)(d_T \pi)\bar\alpha = \\ &= \left(f\,\overline\partial_E s + (\overline\partial_M f) \otimes s\right)(d_T\pi)\bar\alpha = \\ &= f(\overline\partial_{\mathcal H} \tilde s)\bar\alpha + ((d_T\pi)\bar\alpha)(f)\tilde s = \\ &= f\, D_{\sigma\,\bar\alpha} \tilde s + (\sigma\,\bar\alpha)(f)\,\tilde s \end{aligned}$$

by $\sigma \circ \pi_{\mathcal F} : T(\mathcal F)^\perp \subset T(E_0)$ and $(d_T\pi) \circ \pi_{\mathcal F} = d\pi$. Finally (13) leads to

$$(\Lambda_\theta B)(f u) = f(\Lambda_\theta B) u - S(f) u - 2n(\sigma\,\xi)(f) u$$

for any $f \in C^\infty(E_0)$ and $u \in \Gamma^\infty(\pi^* E)$. Hence (by (11)) one gets (12).

It remains to be checked that $D$ satisfies the axioms 1-4. By (5) and (7) the connection $D$ obeys to axioms 1-2. Also $\Lambda_\theta R^D = 0$ as a consequence of (10)-(11). It remains that we check axiom 3. Note firstly that

$$X(H(u,v)) = H(D_X u, v) + H(u, D_{\overline{X}} v) \tag{15}$$

for any $X \in \Gamma^\infty([(\sigma\ H) \oplus T(\mathcal{F})] \otimes \mathbf{C})$. Indeed (4)-(6) lead to (3), and (3) and (8) (and their complex conjugates) lead to (15). A calculation shows that (by (15))

$$H(B(X,Y)u,v) + H(u, B(\overline{X},\overline{Y})v) = \\ = \left(\pi^\perp[X,Y]\right)(H(u,v)) - \left(\sigma\, T_\nabla(X,Y)\right)(H(u,v))$$

for any $X, Y \in \Gamma^\infty((\sigma\ H) \otimes \mathbf{C})$. Set $X = \sigma\,\zeta_\alpha$ and $Y = \overline{X}$ and take traces. We obtain

$$H((\Lambda_\theta B)u,v) + H(u,(\Lambda_\theta B)v) = -S(H(u,v)) - 2n(\sigma\,\xi)(H(u,v))\,.$$

At this point, substitute $\Lambda_\theta B$ from (11) and use (by (15), as $S \in T(\mathcal{F})$) $D_S H = 0$. This procedure gives $D_{\sigma(\xi)} H = 0$. ∎

To prove the last statement in Theorem 1, let $\hat{\theta}$ be the transverse pseudohermitian structure associated with the contact form $\hat{\theta}_M = e^{2f}\theta_M$, $f \in C^\infty(M)$. Then $\hat{\theta} = e^{2f\circ\pi}\theta$. Next, let $\hat{D}$ be the connection determined by axioms 1-4 (where $\theta$ is replaced by $\hat{\theta}$). Then (by (4)-(5) and (7)-(8))

$$\hat{D}_X s = D_X s\ ,\ \hat{D}_{\sigma(z)} s = D_{\sigma(z)} s$$

for any $X \in T(\mathcal{F})$, $s \in \Gamma^\infty(\pi^* E)$ and $z \in \Gamma^\infty(H)$. To see how $D_{\sigma(\xi)}$ changes under a transformation $\hat{\theta} = e^{2\,f\circ\pi}\theta$, note first that

$$d_B\hat{\theta} = e^{2(f\circ\pi)}\left\{d_B\theta + 2d_B(f\circ\pi)\wedge\theta\right\}$$

hence

$$e^{2(f\circ\pi)}\hat{\xi} = \xi - ih^{\alpha\bar\beta}(\sigma\zeta_{\bar\beta})(f\circ\pi)\zeta_\alpha + ih^{\bar\alpha\beta}(\sigma\zeta_\beta)(f\circ\pi)\zeta_{\bar\alpha} \tag{16}$$

where $h_{\alpha\bar\beta} = L_\theta(\zeta_\alpha,\zeta_{\bar\beta})$ and $[h^{\alpha\bar\beta}] = [h_{\alpha\bar\beta}]^{-1}$. Note that $h_{\alpha\bar\beta}$ are basic functions. We need to derive the transformation law (under $\hat{\theta} = e^{2(f\circ\pi)}\theta$)

for the transverse Webster connection $\nabla$. We recall (cf. [5]) that $\nabla$ is given by

$$\begin{cases} \nabla_{\sigma(\overline{\alpha})}\beta = \rho_+\pi_{\mathcal{F}}\left[\sigma\overline{\alpha}, \sigma\beta\right] \\ \nabla_{\sigma(\alpha)}\beta = U_{\alpha\beta} \\ \nabla_{\sigma(\xi)}\beta = \mathcal{L}_{\sigma(\xi)}\beta + T_\xi\beta \\ \nabla\xi = 0 \end{cases} \tag{17}$$

$$\omega(U_{\alpha\beta}, \overline{\gamma}) = (\sigma\alpha)(\omega(\beta, \overline{\gamma})) - \omega\left(\beta, \rho_-\pi_{\mathcal{F}}\left[\sigma\alpha, \sigma\overline{\gamma}\right]\right)$$

$$T_\xi = -\frac{1}{2}J \circ \left(\mathcal{L}_{\sigma(\xi)}J\right)$$

together with

$$\begin{cases} \nabla_{\sigma\alpha}\bar{\beta} = \overline{\nabla_{\sigma\bar{\alpha}}\beta} \\ \nabla_{\sigma\bar{\alpha}}\bar{\beta} = \overline{\nabla_{\sigma\alpha}\beta} \\ \nabla_{\sigma\xi}\bar{\beta} = \overline{\nabla_{\sigma\xi}\beta} \\ \nabla_X = \mathring{\nabla}_X \end{cases}$$

for any $\alpha, \beta \in \mathcal{H}$ and $X \in T(\mathcal{F})$. Here $\omega = -d_B\theta$. In the original construction (cf. [5]) of $\nabla$ one chose $N$ to be the orthogonal complement to $T(\mathcal{F})$ (rather than an arbitrary nonlinear connection on $E_0$), with respect to a bundle-like Riemannian metric on $E_0$ whose associated transverse metric is the *transverse Webster metric* $g_\theta$

$$g_\theta(z, w) = (d_B\theta)(z, Jw)\ ,\ g_\theta(z, \xi) = 0\ ,\ g_\theta(\xi, \xi) = 1$$

for any $z, w \in H$. However, a slight modification of the proof of Theorem 10 in [5] shows that $T_\xi$, and $\nabla$ itself, do not depend upon the choice of $N$ entering their explicit expressions. Finally, $\rho_+ : \nu(\mathcal{F}) \otimes \mathbf{C} \longrightarrow \mathcal{H}$ and $\rho_- : \nu(\mathcal{F})\otimes\mathbf{C} \longrightarrow \overline{\mathcal{H}}$ are the projections associated with the decomposition $\nu(\mathcal{F}) \otimes \mathbf{C} = \mathcal{H} \oplus \overline{\mathcal{H}} \oplus \mathbf{C}\xi$. Let $\{\theta^\alpha\}$, respectively $\{\hat{\theta}^\alpha\}$, be the (local) basic 1-forms determined by

$$\theta^\alpha(\zeta_\beta) = \delta^\alpha_\beta\ ,\ \theta^\alpha(\zeta_{\bar{\beta}}) = 0\ ,\ \theta^\alpha(\xi) = 0$$

respectively

$$\hat{\theta}^\alpha(\zeta_\beta) = \delta^\alpha_\beta\ ,\ \hat{\theta}^\alpha(\zeta_{\bar{\beta}}) = 0\ ,\ \hat{\theta}^\alpha(\hat{\xi}) = 0\,.$$

Then

$$\hat{\theta}^\alpha = \theta^\alpha + ih^{\alpha\bar{\beta}}(\sigma\zeta_{\bar{\beta}})(f\circ\pi)\theta$$

hence

$$\hat{\rho}_+ = \rho_+ + ih^{\alpha\bar{\beta}}(\sigma\zeta_{\bar{\beta}})(f\circ\pi)\theta\otimes\zeta_\alpha\,.$$

Let $\hat{\nabla}$ be the transverse Webster connection of $(M,\mathcal{F},\hat{\theta})$. Then (by (17))

$$\hat{\nabla}_{\sigma(\overline{\alpha})}\beta = \nabla_{\sigma(\overline{\alpha})}\beta + ih^{\lambda\bar{\mu}}(\sigma\zeta_{\bar{\mu}})(f\circ\pi)(\pi^*\theta_M)([\sigma\bar{\alpha},\sigma\beta])\zeta_\lambda\,.$$

Consequently

$$(\hat{D}^2u)(\zeta_\alpha,\zeta_{\bar{\alpha}}) = (D^2u)(\zeta_\alpha,\zeta_{\bar{\alpha}}) + ih^{\lambda\bar{\mu}}(\sigma\zeta_\lambda)(f\circ\pi)(\pi^*\theta_M)([\sigma\zeta_\alpha,\sigma\zeta_{\bar{\alpha}}])D_{\sigma\zeta_{\bar{\mu}}}u$$

where from

$$\begin{aligned}&\hat{B}(\zeta_\alpha,\zeta_{\bar{\alpha}})u = B(\zeta_\alpha,\zeta_{\bar{\alpha}})u +\\ &+ ih^{\lambda\bar{\mu}}(\pi^*\theta_M)([\sigma\zeta_\alpha,\sigma\zeta_{\bar{\alpha}}])\{(\sigma\zeta_\lambda)(f\circ\pi)D_{\sigma\zeta_{\bar{\mu}}}u - (\sigma\zeta_{\bar{\mu}})(f\circ\pi)D_{\sigma\zeta_\lambda}u\}\,.\end{aligned}$$

Next, if $\{\zeta_\alpha\}$ is $L_\theta$-orthonormal, then $\{e^{-(f\circ\pi)}\zeta_\alpha\}$ is $L_{\hat{\theta}}$-orthonormal, hence

$$\begin{aligned}&ie^{2(f\circ\pi)}(\Lambda_{\hat{\theta}}\hat{B})u = i(\Lambda_\theta B)u +\\ &+ h^{\lambda\bar{\mu}}(\pi^*\theta_M)\Big(i\sum_{\alpha=1}^n[\sigma\zeta_\alpha,\sigma\zeta_{\bar{\alpha}}]\Big)\{(\sigma\zeta_\lambda)(f\circ\pi)D_{\sigma\zeta_{\bar{\mu}}}u - (\sigma\zeta_{\bar{\mu}})(f\circ\pi)D_{\sigma\zeta_\lambda}u\}\,.\end{aligned}$$

As $(\pi^*\theta_M)T(\mathcal{F}) = 0$ and $\sigma\pi_{\mathcal{F}}Y = Y_N$ (the projection of $Y$ on $N$) and

$$2ih_{\alpha\bar{\beta}}\xi = \nabla_{\sigma\zeta_\alpha}\zeta_{\bar{\beta}} - \nabla_{\sigma\zeta_{\bar{\beta}}}\zeta_\alpha - \pi_{\mathcal{F}}[\sigma\zeta_\alpha,\sigma\zeta_{\bar{\beta}}]$$

it follows that

$$2in\sigma(\xi) = \sum_{\alpha=1}^n \sigma\left(\nabla_{\sigma\zeta_\alpha}\zeta_{\bar{\alpha}} - \nabla_{\sigma\zeta_{\bar{\alpha}}\zeta_\alpha}\right) - \sum_{\alpha=1}^n[\sigma\zeta_\alpha,\sigma\zeta_{\bar{\alpha}}] - iS$$

for any $L_\theta$-orthonormal (admissible) frame $\{\zeta_\alpha\}$. We may conclude that

$$(\pi^*\theta_M)\Big(i\sum_{\alpha=1}^n[\sigma\zeta_\alpha,\sigma\zeta_{\bar{\alpha}}]\Big) = 2n$$

hence $\Lambda_\theta B$ transforms as

$$ie^{2(f\circ\pi)}\left(\Lambda_{\hat\theta}\hat B\right)u = i\left(\Lambda_\theta B\right)u +$$
$$+\,2nh^{\lambda\bar\mu}\{(\sigma\zeta_\lambda)(f\circ\pi)D_{\sigma\zeta_{\bar\mu}}u - (\sigma\zeta_{\bar\mu})(f\circ\pi)D_{\sigma\zeta_\lambda}u\}\,.$$

Next, taking into account that $\hat S = e^{-2(f\circ\pi)}S$ we get

$$e^{2(f\circ\pi)}\hat D_{\sigma(\hat\xi)}u = D_{\sigma(\xi)}u + ih^{\alpha\bar\beta}\{(\sigma\zeta_\alpha)(f\circ\pi)D_{\sigma\zeta_{\bar\beta}}u - (\sigma\zeta_{\bar\beta})(f\circ\pi)D_{\sigma\zeta_\alpha}u\}$$

hence (by (16))

$$\hat D_{\sigma(\xi)}u = D_{\sigma(\xi)}u$$ □

As an example, we look at a strictly pseudoconvex parallelilzable CR manifold $M$. Let $\{T_\alpha\}$ be a fixed global frame of $T_{1,0}(M)$. Let $F : T_{1,0}(M)\longrightarrow[0,\infty)$ be the complex Finsler structure given by

$$F(u) = |\zeta^1(u)|^2 + \cdots + |\zeta^n(u)|^2$$
$$u = \zeta^\alpha(u)T_\alpha(x),\ u\in T_{1,0}(M)_x,\ x\in M\,.$$

Then $H_{\alpha\bar\beta} = \frac{1}{2}\delta_{\alpha\beta}$, so that $F$ is convex. Let $D$ be the canonical connection determined by the data $(F,N)$. Let $\{\zeta_\alpha\}$ be the admissible (global) frame of $\mathcal{H}$ given by $(d_T\pi)\zeta_\alpha = T_\alpha\circ\pi$. Then

$$D_{\sigma\ \zeta_{\bar\alpha}}\tilde T_\beta = \sum_\rho\left(\Gamma^\rho_{\bar\alpha\beta}\circ\pi\right)\tilde T_\rho\ ,\quad D_{\sigma\ \zeta_\alpha}\tilde T_\beta = -\sum_\rho\left(\Gamma^{\bar\beta}_{\alpha\bar\rho}\circ\pi\right)\tilde T_\rho$$

$$D_{\partial/\partial\overline{\zeta}^\alpha}\tilde T_\beta = D_{\partial/\partial\zeta^\alpha}\tilde T_\beta = 0$$

$$2nD_{\sigma\ \xi}\tilde T_\beta = -\left(\Lambda_\theta B\right)\tilde T_\beta$$

where $\Gamma^\rho_{\bar\alpha\beta}$ are (among) the Christoffel symbols of the Tanaka-Webster connection and $\Lambda_\theta B$ may be computed from (9). If $M = \mathbf{H}_n$ (the Heisenberg group with the standard strictly pseudoconvex CR structure, cf. e.g. [8], p. 189) then $\Lambda_\theta B = 0$. Finally, note that on a parallelizable CR manifold there is a natural choice of complement of $T(\mathcal{F})$ in $T(T_{1,0}(M)_0)$, obtained by the injection $\alpha_\zeta : M\longrightarrow M\times\mathbf{C}^n_0$, $\alpha_\zeta(x) = (x,\zeta)$, $x\in M$, $\zeta\in\mathbf{C}^n_0 = \mathbf{C}^n\setminus\{0\}$, i.e.

$$N_u = d_{\pi(u)}(h\circ\alpha_{\zeta(u)})T_{\pi(u)}(M)$$

where $\zeta(u) = (\zeta^1(u), \cdots, \zeta^n(u))$ and $h : M \times \mathbf{C}_0^n \longrightarrow T_{1,0}(M)_0$ is the natural diffeomorphism $h(x, \zeta) = \zeta^\alpha T_\alpha(x)$.

Applications (of the canonical connection $D$) are delegated to a further paper. Let $\mathcal{L} \in \Gamma^\infty(\pi^* E)$ be the *Liouville vector* i.e. $\mathcal{L}(u) = (u, u)$, for any $u \in E_0$. Locally $\mathcal{L} = \zeta^j \tilde{s}_j$. Note that $\overline{\partial}_{\mathcal{F}} \mathcal{L} = 0$. We close by observing that $X \mapsto D_X \mathcal{L}$ is an isomorphism of $T_{1,0}(\mathcal{F})$ onto $\pi^* E$ (indeed, as a consequence of the complex homogeneity property of $F$, one has $D_{\gamma\, s} \mathcal{L} = s$ for any $s \in \Gamma^\infty(\pi^* E)$).

It is an open problem to build canonical connections for CR-holomorphic vector bundles $E$ over CR manifolds of CR codimension higher than 1, whether in the presence of a Hermitian structure on $E$, or a Hermitian structure on $\pi^* E$, associated with a convex complex Finsler structure. As remarked in the introduction, the construction of the Tanaka connection explicitely employs the Tanaka-Webster connection of the base pseudohermitian manifold; on the other hand, an analogue of the Tanaka-Webster connection, on a nondegenerate CR manifold of higher CR codimension, is already available, due to the work by R. Mizner, [14] (although limitted to the case where the conormal bundle $H(M)^\perp$ admits global frames).

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*Lavoro pervenuto alla redazione il 22 aprile 1999*
*ed accettato per la pubblicazione il 4 ottobre 1999.*
*Bozze licenziate il 12 gennaio 2000*

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