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From critical exponents to blow-up rates for parabolic problems

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Riassunto: *Si espongono alcuni risultati sulla velocità di blow-up per tre sistemi parabolici e per l'equazione di Chipot-Weissler. Si ottengono questi risultati da noti teoremi del tipo di Fujita.*

Abstract: *In this paper, we derive new results on blow-up rates for three parabolic systems and for the Chipot-Weissler equation from known Fujita-type theorems.*

1 – Introduction

The aim of the paper is to establish new results on blow-up rates for parabolic systems using known Fujita-type theorems. Since the classical paper [15] appeared, critical exponents have attracted considerable attention. For surveys of a large number of Fujita-type theorems we refer to [25] and its sequel [9].

As far as we know, the first blow-up rate result derived from a Fujita-type theorem appeared in [19]. There, positive solutions of the problem

$$
(1.1)\qquad \begin{aligned} u_t &= \Delta u, && x \in \Omega, && t > 0,\\ \frac{\partial u}{\partial \nu} &= u^p, && x \in \partial\Omega, && t > 0, \end{aligned}
$$

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are studied, here $\Omega \subset \mathbb{R}^N$ is bounded and $p > 1$. It is known that every positive solution u of (1.1) blows up in a finite time $T > 0$ (see [12]) and it is shown in [19] that for every positive solution u of (1.1) there is a constant $C > 0$ such that

$$u(x,t) \le C(T-t)^{-\frac{1}{2(p-1)}} \tag{1.2}$$

provided $p \le 1 + 1/N$. The number $p = 1 + 1/N$ is the critical exponent for the problem

$$\begin{aligned} u_t &= \Delta u, && x \in \mathbb{R}^N_+ = \{x_1 > 0\}, && t > 0, \\ -\frac{\partial u}{\partial x_1} &= u^p, && x_1 = 0, && t > 0, \\ u(\cdot,0) &= u_0 \ge 0, && u_0 \in L^\infty(\mathbb{R}^N_+), && \end{aligned} \tag{1.3}$$

in the sense that all nontrivial solutions of (1.3) blow up in finite time for $1 < p \le 1 + 1/N$ while nontrivial global solutions exist for $p > 1 + 1/N$ (see [8]).

Here, and in what follows, blow-up is meant in the following sense: There exists $0 < T < \infty$ such that $u(\cdot,t)$ is bounded for $0 < t < T$ and

$$\limsup_{t \to T} \|u(\cdot,t)\|_{L^\infty} = \infty.$$

To prove (1.2) for $p \le 1 + 1/N$ in [19], a scaling argument is used to obtain a contradiction with finite time blow-up of all positive solutions of (1.3).

We illustrate the wide applicability of this method by establishing blow-up rates for three parabolic systems and for the Chipot-Weissler equation.

One of the systems considered here is of the form

$$\begin{aligned} u_t &= \Delta u + v^p, & v_t &= \Delta v + u^q, & & x \in \mathbb{R}^N, \quad t > 0, \\ u(\cdot,0) &= u_0 \ge 0, & v(\cdot,0) &= v_0 \ge 0, & & u_0, v_0 \in L^\infty(\mathbb{R}^N), \end{aligned} \tag{1.4}$$

$p, q > 0$. This system was studied before i [2], [10], and [11], for example. It was shown in [10] that all positive solutions of (1.4) blow up in finite time for

$$(1.5) \qquad pq > 1, \qquad \frac{\max(p,q)+1}{pq-1} \geq \frac{N}{2},$$

while positive global solutions exist for $(\max(p,q)+1)/(pq-1) < N/2$. We shall use this result to prove that if (1.5) holds then for every positive solution (u,v) of (1.4) there is a constant $C > 0$ such that

$$(1.6) \qquad u(x,t) \leq C(T-t)^{-\frac{p+1}{pq-1}}, \qquad v(x,t) \leq C(T-t)^{-\frac{q+1}{pq-1}},$$

here T is the blow-up time. This improves the corresponding result (Theorem 1) in [2] where the strict inequality

$$\frac{\max(p,q)+1}{pq-1} > \frac{N}{2}$$

is required. Also, our proof is much shorter.

For certain radial monotone solutions of the system from (1.4) on a ball, the rate (1.6) was proved in [3] for some $p, q \geq 1$, $pq > 1$. In [7] it was shown that (1.6) holds for time-increasing solutions of the system from (1.4) on a bounded domain (with the Dirichlet boundary condition) for all $p, q \geq 1$, $pq > 1$.

The next system we study is

$$(1.7) \qquad \begin{array}{lll} u_t = \Delta u, & v_t = \Delta v, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = v^p, & \dfrac{\partial v}{\partial \nu} = u^q, & x \in \partial\Omega, \quad t > 0, \\ u(\cdot,0) = u_0 \geq 0, & v(\cdot,0) = v_0 \geq 0, & u_0, v_0 \in L^\infty(\Omega), \end{array}$$

$p, q > 0$, and $\Omega \subset \mathbb{R}^N$ is bounded or $\Omega = \mathbb{R}^N_+$. This system was considered before in [5], [6], [7], [8], [21], [26], [30], and [37], for instance. The following Fujita-type result was established in [8] for $\Omega = \mathbb{R}^N_+$: All nontrivial solutions of (1.7) blow up in finite time if

$$(1.8) \qquad pq > 1, \qquad \frac{\max(p,q)+1}{pq-1} \geq N,$$

while nontrivial global solutions exist if $(\max(p,q)+1)/(pq-1) < N$. We employ the global nonexistence result to show that if (1.8) holds then

$$u(x,t) \le C(T-t)^{-\frac{p+1}{2(pq-1)}}, \qquad v(x,t) \le C(T-t)^{-\frac{q+1}{2(pq-1)}}, \tag{1.9}$$

for some $C > 0$. The blow-up rate (1.9) of solutions of (1.7) was established before for $p, q \ge 1$, $pq > 1$, in the case when Ω is a ball, u, v are radially symmetric, and $u_r, u_t \ge 0$ (see [7] and also [31] for a similar result for a more general system). More recently, the restrictions $p, q \ge 1$ and $u_r \ge 0$ have been removed in [26]. In the case $\Omega = \mathbb{R}_+$ it was shown in [8] that (1.9) holds for some suitable solutions of (1.7) if $pq > 1$, $p, q \ge 1$. This result was improved in [37] by removing the restriction $p, q \ge 1$ and allowing a larger class of solutions.

The third system we are interested in is

$$\begin{aligned} &u_t = \Delta u + v^p, && v_t = \Delta v, && x \in \mathbb{R}^N_+, \quad t > 0,\\ &-\frac{\partial u}{\partial x_1} = 0, && -\frac{\partial v}{\partial x_1} = u^q, && x_1 = 0, \quad t > 0,\\ &u(\cdot,0) = u_0 \ge 0, && v(\cdot,0) = v_0 \ge 0, && u_0, v_0 \in L^\infty(\mathbb{R}^N_+), \end{aligned} \tag{1.10}$$

where $p, q > 0$. It was studied before in [14] where it was shown that all nontrivial solutions of (1.10) blow up in finite time if

$$pq > 1, \qquad \frac{\max(p+2, 2q+1)}{pq-1} > N, \tag{1.11}$$

or

$$pq > 1, \qquad \frac{\max(p+2, 2q+1)}{pq-1} = N, \qquad p, q \ge 1; \tag{1.12}$$

whereas nontrivial global solutions exist if $\max(p+2, 2q+1)/(pq-1) < N$. We use this result to prove that if (1.11) or (1.12) holds then

$$u(x,t) \le C(T-t)^{-\frac{p+2}{2(pq-1)}}, \qquad v(x,t) \le C(T-t)^{-\frac{2q+1}{2(pq-1)}}, \tag{1.13}$$

provided $u_{x_1}, v_{x_1} \le 0$.

The Chipot-Weissler equation

$$u_t = \Delta u - |\nabla u|^q + u^p, \qquad p > 1, \quad q \geq 1, \tag{1.14}$$

was introduced in [4]. There, and in [22], [13], [28], [29], [32], [33], and [35] it was studied on a bounded domain $\Omega \subset \mathbb{R}^N$ with the Dirichlet boundary condition. The main issue in those works was to determine for which p and q blow-up in finite time (in the L^∞-norm) may occur. It turns out (see [35]) that it occurs if and only if $p > q$. Equation (1.14) in $\mathbb{R}^N$ was considered in [1], [35], and [36] from a similar point of view. In this case, blow-up in finite time is also known to occur when $p > q$ (see [35]) but unbounded solutions always exist (see [36]). However, as far as we know, the only previous study of the blow-up profile for (1.14) was performed in [34] for $q = 2p/(p+1)$.

In this paper we establish the blow-up rate of blowing-up solutions of (1.14) with the initial condition

$$u(\cdot, 0) = u_0 \geq 0, \qquad u_0, |\nabla u_0| \in L^\infty(\mathbb{R}^N), \tag{1.15}$$

when

$$q < \frac{2p}{p+1}, \qquad p \leq 1 + \frac{2}{N}.$$

It turns out that in this case the rate is the same as for

$$u_t = \Delta u + u^p, \qquad x \in \mathbb{R}^N, \quad p > 1, \tag{1.16}$$

this means

$$u(x,t) \leq C(T-t)^{-\frac{1}{p-1}}, \tag{1.17}$$

here $T < \infty$ is the blow-up time.

We also prove that

$$|\nabla u(x,t)| \leq C(T-t)^{-\frac{p+1}{2(p-1)}}. \tag{1.18}$$

To prove (1.17) and (1.18) we use a scaling argument and the fact that all positive solutions of (1.16) blow up in finite time if $p \leq 1 + \frac{2}{N}$.

This was shown in [15] for $p < 1 + \frac{2}{N}$, and in [18], [23] for $p = 1 + \frac{2}{N}$. It was also shown in [15 that global positive solutions exist if $p > 1 + \frac{2}{N}$.

We remark that the exponents in the estimates (1.6), (1.9), (1.13) and (1.17) are optimal because there exist solutions that blow up precisely with the rate from these inequalities. This is obvious for system (1.4) and equation (1.14) since in those cases one can take spatially homogeneous solutions. For system (1.7), selfsimilar solutions which blow up with the rate from (1.9) can be found in [8]. For system (1.10), selfsimilar solutions which decrease in x_1 and blow up with the rate from (1.13) are given in this paper.

Finally, let us remark that we do not assume any symmetry or monotonicity of solutions that is often required if one uses other methods (cf. [3], [7], [8], [26], [31], and [37], for example). On the other hand, we impose restrictions on the exponents p and q.

2 – System (1.4)

In this section we prove the following:

THEOREM 2.1. *If $p, q > 0$, $pq > 1$ and $\max(\alpha, \beta) \geq \frac{N}{2}$, where $\alpha = \frac{p+1}{pq-1}$ and $\beta = \frac{q+1}{pq-1}$, then for every positive solution (u, v) of* (1.4) *there is a constant C such that*

$$u(x,t) \leq C(T-t)^{-\alpha}, \quad v(x,t) \leq C(T-t)^{-\beta} \tag{2.1}$$

in $\mathbb{R}^N \times (0,T)$, where $T < \infty$ is the blow-up time.

PROOF. Let

$$M_u(t) := \sup_{\mathbb{R}^N \times (0,t]} u \qquad \text{and} \qquad M_v(t) := \sup_{\mathbb{R}^N \times (0,t]} v, \qquad t \in (0,T).$$

Clearly, M_u and M_v are positive, continuous and nondecreasing functions on $(0,T)$. At least one of them diverges as $t \nearrow T$. We show later that there is $\delta \in (0,1)$ such that

$$\delta \leq M_u^{-\frac{1}{2\alpha}}(t) M_v^{\frac{1}{2\beta}}(t) \leq \delta^{-1}, \qquad t \in \Big(\frac{T}{2}, T\Big), \tag{2.2}$$

and, consequently, both M_u and M_v have to diverge as $t \nearrow T$. To establish the blow-up rates (2.1) we use a scaling argument similar as in [16] and [17]. The proof is divided into several steps.

STEP 1: *Scaling.* a) If M_u diverges as $t \nearrow T$ the following procedure can be applied. Given $t_0 \in (0, T)$ choose $(\hat{x}, \hat{t}) \in \mathbb{R}^N \times (0, t_0]$ such that

$$u(\hat{x}, \hat{t}) \geq \frac{1}{2} M_u(t_0). \tag{2.3}$$

Let

$$\lambda = \lambda(t_0) := M_u^{-\frac{1}{2\alpha}}(t_0) \tag{2.4}$$

be a scaling factor and define the rescaled functions φ^λ and ψ^λ by

$$\varphi^\lambda(y, s) := \lambda^{2\alpha} u(\lambda y + \hat{x}, \lambda^2 s + \hat{t}), \tag{2.5}$$

$$\psi^\lambda(y, s) := \lambda^{2\beta} v(\lambda y + \hat{x}, \lambda^2 s + \hat{t}), \tag{2.6}$$

where $(y, s) \in \mathbb{R}^N \times (-\lambda^{-2}\hat{t}, \lambda^{-2}(T - \hat{t}))$.

Then $(\varphi^\lambda, \psi^\lambda)$ is a solution of the system

$$\varphi_s = \Delta\varphi + \psi^p, \quad \psi_s = \Delta\psi + \varphi^q \tag{2.7}$$

in $\mathbb{R}^N \times (-\lambda^{-2}\hat{t}, \lambda^{-2}(T - \hat{t}))$.

If we restrict s to $s \in (-\lambda^{-2}\hat{t}, 0]$, then clearly

$$0 \leq \varphi^\lambda \leq 1 \quad \text{and} \quad 0 \leq \psi^\lambda \leq M_u^{-\frac{\beta}{\alpha}}(t_0) M_v(t_0) \tag{2.8}$$

in $\mathbb{R}^N \times (-\lambda^{-2}\hat{t}, 0]$, and $\varphi^\lambda(0, 0) \geq \frac{1}{2}$.

b) If M_v diverges as $t \nearrow T$ we can proceed in the same way changing the role of u and v.

STEP 2: *Schauder's estimates.* We need interior Schauder's estimates of the functions φ and ψ on the sets

$$S_K := \{y \in \mathbb{R}^N : |y| \leq K\} \times [-K, 0],$$

where (φ, ψ) is a solution of the system (2.7) in S_{2K} satisfying there

$$0 \le \varphi \le B \quad \text{and} \quad 0 \le \psi \le B. \tag{2.9}$$

We claim that for any $K \in [1, \infty)$, $B \in (0, \infty)$ and $\sigma > 0$ small enough there is a constant $C = C(K, B, \sigma)$ such that

$$\|\varphi\|_{C^{2+\sigma, 1+\frac{\sigma}{2}}(S_K)} \le C \tag{2.10}$$

$$\|\psi\|_{C^{2+\sigma, 1+\frac{\sigma}{2}}(S_K)} \le C. \tag{2.11}$$

Most of the argument is standard. Applying the parabolic interior regularity theory (cf. [24]), we obtain uniform estimates of the $C^{\sigma, \frac{\sigma}{2}}$-norms. If, in addition, $\sigma < p$, ($\sigma < q$, respectively), the uniform estimates of ψ^p (φ^q) in the $C^{\sigma, \frac{\sigma}{2}}$-norms follow. Now the parabolic interior Schauder's estimates imply (2.10) and (2.11).

STEP 3: *The proof of* (2.2). a) Let us prove the lower bound of (2.2) first.
Suppose that this lower bound were false. Then there would exist a sequence $t_j \nearrow T$ such that $M_u^{-\frac{1}{2\alpha}}(t_j) M_v^{\frac{1}{2\beta}}(t_j) \to 0$. Then clearly M_u diverges as $t \nearrow T$. For each t_j in the role of t_0 from part a) of Step 1 we scale about the corresponding point $(\hat{x}_j, \hat{t}_j)$ with the scaling factor

$$\lambda_j := M_u^{-\frac{1}{2\alpha}}(t_j)$$

to obtain the corresponding rescaled solution $(\varphi^{\lambda_j}, \psi^{\lambda_j})$ of (2.7). It holds that

$$0 \le \varphi^{\lambda_j} \le 1 \quad \text{and} \quad 0 \le \psi^{\lambda_j} \le M_u^{-\frac{\beta}{\alpha}}(t_j) M_v(t_j)$$

in $\mathbb{R}^N \times (-\lambda_j^{-2} \hat{t}_j, 0]$, and $\varphi^{\lambda_j}(0,0) \ge \frac{1}{2}$.

Clearly $\lambda_j \to 0$ and $\hat{t}_j \to T$.

Using our uniform Schauder's estimates derived in Step 2 to $(\varphi^{\lambda_j}, \psi^{\lambda_j})$, compactness and diagonal arguments yield a subsequence converging to a solution (φ, ψ) of (2.7) in $\mathbb{R}^N \times (-\infty, 0]$ such that $\psi \equiv 0$ (as $0 \le \psi \le \lim_{j \to \infty} M_u^{-\frac{\beta}{\alpha}}(t_j) M_v(t_j)$) and $\varphi(0,0) \ge \frac{1}{2}$. But that is impossible. This contradiction completes the proof of the lower bound of (2.2).

b) To prove the upper bound of (2.2) we can proceed similarly as in the part a) changing the role of u and v.

STEP 4: *Estimate on doubling of M_u.* As M_u is continuous and diverges as $t \nearrow T$, for any given $t_0 \in (0,T)$ we can define t_0^+ by

$$t_0^+ := \max\{t \in (t_0,T) : M_u(t) = 2M_u(t_0)\}. \tag{2.12}$$

Take $\lambda = \lambda(t_0)$ as in (2.4).We claim that

$$\lambda^{-2}(t_0)(t_0^+ - t_0) \le D, \qquad t_0 \in \Big(\frac{T}{2}, T\Big), \tag{2.13}$$

where the constant $D \in (0,\infty)$ is independent of t_0.

Suppose that this estimate were false. Then there would exist a sequence $t_j \nearrow T$ such that $\lambda^{-2}(t_j)(t_j^+ - t_j) \to \infty$.

For each t_j we scale about the corresponding point $(\hat{x}_j, \hat{t}_j)$ (with $\hat{t}_j \le t_j$) with the scaling factor $\lambda_j = \lambda(t_j) = M_u^{-\frac{1}{2\alpha}}(t_j)$ to obtain the corresponding rescaled solution $(\varphi^{\lambda_j}, \psi^{\lambda_j})$ of (2.7), as in the part a) of Step 3.

Clearly $\varphi^{\lambda_j}(0,0) \ge \frac{1}{2}$, and in $\mathbb{R}^N \times (-\lambda_j^{-2}\hat{t}_j, \lambda_j^{-2}(t_j^+ - \hat{t}_j)]$ it holds that

$$0 \le \varphi^{\lambda_j} \le 2 \quad \text{and} \quad 0 \le \psi^{\lambda_j} \le 2^{\frac{\beta}{\alpha}} \delta^{-2\beta},$$

where $\delta \in (0,1)$ is from (2.2).

(To verify the second inequality, notice that for

$$(y,s) \in \mathbb{R}^N \times (-\lambda_j^{-2}\hat{t}_j, \lambda_j^{-2}(t_j^+ - \hat{t}_j))$$

we have

$$\psi^{\lambda_j}(y,s) \le \lambda_j^{2\beta} M_v(t_j^+) \le \lambda_j^{2\beta} \delta^{-2\beta} M_u^{\frac{\beta}{\alpha}}(t_j^+)$$

by (2.2), and $\lambda_j^{2\beta} M_u^{\frac{\beta}{\alpha}}(t_j^+) = 2^{\frac{\beta}{\alpha}}$.)

The uniform Schauder's estimates derived in Step 2 for $(\varphi^{\lambda_j}, \psi^{\lambda_j})$ yield a subsequence converging in $C^{2+\sigma, 1+\frac{\sigma}{2}}_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R})$ to a solution (φ, ψ) of (2.7) in $\mathbb{R}^N \times \mathbb{R}$ such that

$$0 \le \varphi \le 2 \quad \text{and} \quad 0 \le \psi \le 2^{\frac{\beta}{\alpha}} \delta^{-2\beta}$$

in $\mathbb{R}^N \times \mathbb{R}$, and $\varphi(0,0) \geq \frac{1}{2}$.

This is a contradiction, as it was shown in [10] that all positive solutions of (2.7) under our assumptions blow up in finite time. This completes the proof of (2.13).

STEP 5: *Rate estimates.* This step is analogous to Step 3 of the proof of Theorem 2.1 in [19].

As in Step 4, for any $t_0 \in (\frac{T}{2}, T)$ we define $t_1 := t_0^+ \in (t_0, T)$ such that

$$M_u(t_1) = 2M_u(t_0)$$

and, due to (2.13),

$$t_1 - t_0 \leq D M_u^{-\frac{1}{\alpha}}(t_0).$$

Now we can use t_1 as the new t_0 and obtain $t_2 \in (t_1, T)$ such that

$$M_u(t_2) = 2M_u(t_1) = 2^2 M_u(t_0),$$

$$t_2 - t_1 \leq D M_u^{-\frac{1}{\alpha}}(t_1) = 2^{-\frac{1}{\alpha}} D M_u^{-\frac{1}{\alpha}}(t_0).$$

Continuing this process we obtain a sequence $t_j \nearrow T$ such that

$$t_{j+1} - t_j \leq 2^{-\frac{j}{\alpha}} D M_u^{-\frac{1}{\alpha}}(t_0), \quad j = 0, 1, 2, \ldots$$

If we add these inequalities we get

$$T - t_0 \leq (1 - 2^{-\frac{1}{\alpha}})^{-1} D M_u^{-\frac{1}{\alpha}}(t_0)$$

and

$$M_u(t_0) \leq (1 - 2^{-\frac{1}{\alpha}})^{-\alpha} D^{\alpha} (T - t_0)^{-\alpha} \quad \text{for } t_0 \in \Big(\frac{T}{2}, T\Big) \tag{2.14}$$

follows. Using (2.2) we obtain

$$M_v(t_0) \leq \delta^{-2\beta} M_u(t_0)^{\frac{\beta}{\alpha}}, \quad t_0 \in \Big(\frac{T}{2}, T\Big),$$

$$M_v(t_0) \leq \mathrm{const}\, (T - t_0)^{-\beta} \quad \text{for } t_0 \in \Big(\frac{T}{2}, T\Big)$$

follows as well from the above estimate of M_u.

3 – System (1.7)

In this section we consider the system (1.7) where p, $q > 0$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain in $\mathbb{R}^N$ with $\partial\Omega \in C^2$ or $\Omega = \mathbb{R}^N_+$.

We prove the following:

THEOREM 3.1. *If $pq > 1$ and $\max(\alpha,\beta) \geq \frac{N}{2}$, where $\alpha = \frac{p+1}{2(pq-1)}$ and $\beta = \frac{q+1}{2(pq-1)}$, then for every positive solution (u,v) of (1.7) there is a constant $C > 0$ such that the estimates (2.1) hold true in $\Omega \times (0,T)$, where $T < \infty$ is the blow-up time.*

PROOF. We will closely follow the scheme of the proof of Theorem 2.1. Let

$$M_u(t) := \sup_{\overline{\Omega}\times(0,t]} u \qquad \text{and} \qquad M_v(t) := \sup_{\overline{\Omega}\times(0,t]} v, \qquad t \in (0,T).$$

STEP 1: *Scaling.* a) Given $t_0 \in (0,T)$ with $M_u(t_0) > M_u(0+)$, there exists $(\hat{x},\hat{t}) \in \partial\Omega \times (0,t_0]$ such that (2.3) holds. This can be easily seen in case of bounded $\Omega \subset \mathbb{R}^N$, by the maximum principle. We can even require $u(\hat{x},\hat{t}) = M_u(t_0)$ in this case. If $\Omega = \mathbb{R}^N_+$, we employ the Phragmén-Lindelöf principle (cf. [27]) instead. Let $\lambda = \lambda(t_0)$ be defined by (2.4). Now define

$$\varphi^\lambda(y,s) := \lambda^{2\alpha} u(\lambda R y + \hat{x}, \lambda^2 s + \hat{t}),$$

$$\psi^\lambda(y,s) := \lambda^{2\beta} v(\lambda R y + \hat{x}, \lambda^2 s + \hat{t})$$

for $(y,s) \in \overline{\Omega}_\lambda \times (-\lambda^{-2}\hat{t}, \lambda^{-2}(T-\hat{t}))$, where

$$\Omega_\lambda := \{y \in \mathbb{R}^N : \lambda R y + \hat{x} \in \Omega\}$$

and R is an orthonormal transformation in $\mathbb{R}^N$ that maps $(-1,0,0,\dots,0)$ into the outer normal vector to Ω at $\hat{x}$. Clearly, $(-1,0,0,\dots,0)$ is the outer normal vector to Ω_λ at 0, and Ω_λ approaches (locally) the halfspace $\mathbb{R}^N_+$ as $\lambda \to 0+$.

Now $(\varphi^\lambda, \psi^\lambda)$ is a solution of the system

$$\begin{aligned} &\varphi_s = \Delta\varphi, && \psi_s = \Delta\psi, && y \in \Omega_\lambda, \quad s \in (-\lambda^{-2}\hat{t}, \lambda^{-2}(T-\hat{t})), \\ &\frac{\partial\varphi}{\partial\nu} = \psi^p, && \frac{\partial\psi}{\partial\nu} = \varphi^q, && y \in \partial\Omega_\lambda,\ s \in (-\lambda^{-2}\hat{t}, \lambda^{-2}(T-\hat{t})). \end{aligned}$$

If we restrict s to $s \in (-\lambda^{-2}\hat{t}, 0]$, then again (2.8) holds true in $\Omega_\lambda \times (-\lambda^{-2}\hat{t}, 0]$, and $\varphi^\lambda(0,0) \geq \frac{1}{2}$.

b) Changing the role of u and v the same scaling procedure can be applied.

STEP 2: *Schauder's estimates.* We need parabolic interior - boundary Schauder's estimates of the functions φ and ψ on the sets $(\overline{\Omega}_\lambda \times \mathbb{R}) \cap S_K$, where (φ, ψ) is a solution of

$$\begin{aligned} &\varphi_s = \Delta\varphi, && \psi_s = \Delta\psi, && (y,s) \in (\Omega_\lambda \times \mathbb{R}) \cap S_{2K}, \\ &\frac{\partial\varphi}{\partial\nu} = \psi^p, && \frac{\partial\psi}{\partial\nu} = \varphi^q, && (y,s) \in (\partial\Omega_\lambda \times \mathbb{R}) \cap S_{2K}, \end{aligned}$$

satisfying there (2.12).

We claim that for any $\sigma > 0$ small enough, $B \in (0, \infty)$ and $K \in [1, \infty)$, there are $\lambda_0 = \lambda(\Omega, K) > 0$ and $C = C(\Omega, \sigma, K, B) > 0$ such that if $\lambda \in (0, \lambda_0)$ and (φ, ψ) are as above, then

$$\|\varphi\|_{C^{1+\sigma, \frac{1+\sigma}{2}}((\overline{\Omega}_\lambda \times \mathbb{R}) \cap S_K)} \leq C, \qquad \text{and} \qquad \|\psi\|_{C^{1+\sigma, \frac{1+\sigma}{2}}((\overline{\Omega}_\lambda \times \mathbb{R}) \cap S_K)} \leq C.$$

The argument is again standard. Applying the regularity theory up to the boundary for the heat equation with a Neumann boundary condition, we obtain uniform estimates in the Hölder $C^{\sigma, \frac{\sigma}{2}}$-norms (or one can simply represent the solutions in terms of Green's function for this boundary value problem and obtain the Hölder estimates immediately). Similar estimates of ψ^p (φ^q) follow, if, additionally, $\sigma < p$ ($\sigma < q$, respectively). Now the standard parabolic estimates in $C^{1+\sigma, \frac{1+\sigma}{2}}$-norms follow (cf. [24], Chapter 4, formulae (2.3), (2.16-2.20) for more details).

STEP 3: *The proof of* (2.2). This is almost the same as the corresponding part in the proof of Theorem 2.1. For example, to prove the lower bound of (2.2), we proceed by contradiction. Suppose that this lower bound is false. Then there exists a sequence of corresponding rescaled solutions converging locally in $C^{1+\sigma,\frac{1+\sigma}{2}}$-norms to a solution (φ,ψ) of system (1.7) in $\mathbb{R}^N_+ \times (-\infty,0]$ such that $\psi \equiv 0$ and $\varphi(0,0) \geq \frac{1}{2}$, which is impossible. This contradiction proves the lower bound of (2.2). The upper bound can be proved similarly, changing the role of u and v.

STEP 4: *Estimate on doubling of* M_u. Given $t_0 \in (0,T)$, define t_0^+ by (2.12). To prove the bound (2.13) we proceed by contradiction in the same way as in the proof of Theorem 2.1. If this bound were false then there would exist a sequence of rescaled solutions converging, due to our uniform Schauder's estimates and compactness, to a positive global bounded solution (φ,ψ) of system (1.7) in $\mathbb{R}^N_+$.

This is a contradiction, as it was shown in [8] that all positive solutions of the system in $\mathbb{R}^N_+$ are nonglobal.

STEP 5: *Rate estimates.* Now the rate estimates (2.1) can be derived as in the proof of Theorem 2.1. They immediately follow from (2.2) and (2.13) which have been derived in Step 3 and Step 4, respectively.

COROLLARY 3.2. *Under the assumptions of Theorem 3.1 blow-up may occur only on* $\partial\Omega$. *More precisely, if* $\Omega' \subset \Omega$ *is such that* $\overline{\Omega'} \subset \Omega$, *then*

$$\sup_{0<t<T} \Big(\|u(\cdot,t)\|_{C(\overline{\Omega'})} + \|v(\cdot,t)\|_{C(\overline{\Omega'})} \Big) < \infty. \tag{3.1}$$

PROOF. This follows immediately from Theorem 3.1 and [20, Theorem 4.1].

REMARK. It was shown in [6] that (3.1) holds if Ω is a ball, $p,q > 1$ and u,v are radially symmetric. This result was improved in [30] where it was established that (3.1) holds if Ω is a ball and $p,q \geq 1$, $pq > 1$ or if Ω is bounded $p,q \geq 1$, $pq > 1$ and $u_t, v_t \geq 0$.

4 – System (1.10)

In this section we prove the following:

THEOREM 4.1. *If* $p, q > 0$, $pq > 1$ *and either* $\max(\alpha,\beta) > \frac{N}{2}$ *or* $\max(\alpha,\beta) = \frac{N}{2}$, p, $q \geq 1$, *where*

$$\alpha = \frac{p+2}{2(pq-1)} \quad \text{and} \quad \beta = \frac{2q+1}{2(pq-1)},$$

then for every positive solution (u,v) *of* (1.10) *satisfying* u_{x_1}, $v_{x_1} \leq 0$ *there is a constant* $C > 0$ *such that the estimates* (2.1) *hold true in* $\mathbb{R}^N_+ \times (0,T)$, *where* $T < \infty$ *is the blow-up time.*

PROOF. We again proceed as in the proof of Theorem 2.1. Let

$$M_u(t) := \sup_{\overline{\mathbb{R}^N_+}\times(0,t]} u \qquad \text{and} \qquad M_v(t) := \sup_{\overline{\mathbb{R}^N_+}\times(0,t]} v, \quad t \in (0,T).$$

STEP 1: *Scaling.* a) Given $t_0 \in (0,T)$ choose $(\hat{x},\hat{t}) \in \partial\mathbb{R}^N_+ \times (0,t_0]$ such that (2.3) holds. (The choice of $\hat{x} \in \partial\mathbb{R}^N_+$ is possible due to $u_{x_1} \leq 0$.) Let $\lambda = \lambda(t_0)$ by defined by (2.4) and φ^λ, ψ^λ be defined by (2.5) and (2.6). Then $(\varphi^\lambda,\psi^\lambda)$ is a solution of system (1.10) in $\mathbb{R}^N_+ \times (-\lambda^{-2}\hat{t}, \lambda^{-2}(T-\hat{t}))$.

If we restrict s to $s \in (-\lambda^{-2}\hat{t}, 0]$ then (2.8) holds true. Clearly $\varphi^\lambda(0,0) \geq \frac{1}{2}$.

b) Changing the role of u and v we can proceed similarly.

STEP 2: *Schauder's estimates.* We need interior - boundary Schauder's estimates of the functions φ and ψ on the sets

$$S^+_K := \{y \in \overline{\mathbb{R}^N_+} : |y| \leq K\} \times [-K,0],$$

where (φ,ψ) is a solution of the system

$$\begin{aligned} &\varphi_s = \Delta\varphi + \psi^p, && \psi_s = \Delta\psi, && \text{in } \{y \in \mathbb{R}^N_+ : |y| \leq 2K\}\times[-2K,0],\\ &-\frac{\partial\varphi}{\partial y_1} = 0, && -\frac{\partial\psi}{\partial y_1} = \varphi^q, && \text{on } \{y \in \partial\overline{\mathbb{R}^N_+} : |y| \leq 2K\}\times[-2K,0], \end{aligned}$$

satisfying (2.9) there.

We claim that for any $\sigma > 0$ small enough, $B \in (0,\infty)$ and $K \in [1,\infty)$, there is a constant $C = C(K,B,\sigma)$ such that the functions φ and ψ as above satisfy

$$\|\varphi\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(S_K^+)} \le C, \qquad \text{and} \qquad \|\psi\|_{C^{1+\sigma,\frac{1+\sigma}{2}}(S_K^+)} \le C.$$

To prove the above estimates one can use the same arguments as in the proofs of Theorem 2.1 and Thoerem 3.1.

Steps 3-5. We proceed as in the proof of Theorem 2.1. In order to prove (2.13) by contradiction we now use the global nonexistence result from [14].

Our next aim is to show that the estimate (1.13) is optimal. To do this we look for selfsimilar solutions of (1.10) with $N = 1$. They are of the form

$$u(x,t)=(T-t)^{-\alpha}U(y), \quad v(x,t) = (T-t)^{-\beta}V(y), \quad y=\frac{x}{\sqrt{T-t}}, \tag{4.1}$$

where (U,V) satisfies

$$\begin{aligned} U'' - \frac{y}{2}U' - \alpha U = V^p, \qquad V'' - \frac{y}{2}V' - \beta V = 0, \qquad y > 0,\\ U'(0) = 0, \qquad -V'(0) = U^q(0). \end{aligned} \tag{4.2}$$

Proposition 4.2. *There exist a selfsimilar solution (u,v) of (1.10) which is of the form (4.1) where (U,V) is a bounded positive solution of (4.2). Moreover, it holds that $u_x, v_x < 0$ for $x > 0$.*

Proof. We shall find a positive bounded solution of

$$\begin{aligned} W'' - \frac{y}{2}W' - \Big(\alpha + \frac{1}{2}\Big)W - (V^p)' = 0, \qquad y > 0,\\ W(0) = 0, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} V'' - \frac{y}{2}V' - \beta V = 0, \qquad y > 0,\\ -V'(0) = \Big(\int_0^\infty W(y)\,dy\Big)^q < \infty. \end{aligned}$$

Then (U, V) will be a solution of (4.2) if

$$U(y) = \int_y^\infty W(z)\,dz. \tag{4.4}$$

Let G_γ denote the unique positive bounded solution of

$$G'' - \frac{y}{2}G' - \gamma G = 0, \qquad y > 0,$$
$$G'(0) = -1.$$

The function G_γ can be expressed by an explicit formula and

$$G_\gamma(y) = Ky^{-2\gamma}(1 + O(y^{-2})) \qquad \text{as} \quad y \to \infty$$

for some $K > 0$. Then

$$V(y) = cG_\beta(y), \qquad c = \left(\int_0^\infty W(y)\,dy\right)^q.$$

We are now looking for a solution of (4.3) of the form $W(y) = F(y)G_{\alpha+\frac{1}{2}}(y)$ where $F(0) = 0$. The equation for $f(y) = F'(y)$ is:

$$G_{\alpha+\frac{1}{2}}f' + \left(2G'_{\alpha+\frac{1}{2}} - \frac{y}{2}G_{\alpha+\frac{1}{2}}\right)f - c^p\left(G_\beta^p\right)' = 0.$$

It has a bounded solution of the form

$$f(y) = c^p e^{\frac{y^2}{4}} G^{-2}_{\alpha+\frac{1}{2}}(y)\int_y^\infty e^{-\frac{s^2}{4}}G_{\alpha+\frac{1}{2}}(s)\left(-G_\beta^p\right)'(s)\,ds,$$

hence,

$$W(y) = c^p G_{\alpha+\frac{1}{2}}(y)\int_0^y e^{\frac{\sigma^2}{4}} G^{-2}_{\alpha+\frac{1}{2}}(\sigma)\left(\int_\sigma^\infty e^{-\frac{s^2}{4}}G_{\alpha+\frac{1}{2}}(s)\left(-G_\beta^p\right)'(s)\,ds\right)d\sigma.$$

The function

$$h(\sigma) = e^{\frac{\sigma^2}{4}} G^{-2}_{\alpha+\frac{1}{2}}(\sigma)\int_\sigma^\infty e^{-\frac{s^2}{4}}G_{\alpha+\frac{1}{2}}(s)\left(-G_\beta^p\right)'ds$$

is bounded on $[0,1]$ and on $[1,\infty)$ we have

$$h(\sigma) \le C_1 e^{\frac{\sigma^2}{4}} \sigma^{4\alpha+2} \int_\sigma^\infty e^{-\frac{s^2}{4}} s^{-2\alpha-2p\beta-2}\, ds \le \\ \le C_2 e^{\frac{\sigma^2}{4}} \sigma^{4\alpha+2} \int_\sigma^\infty \Big(- e^{-\frac{s^2}{4}} s^{-2\alpha-2p\beta-3}\Big)'\, ds \le C_3 \sigma^{-2\alpha-2p\beta-1}$$

for some positive constants C_i. Therefore $W(0+)=0$ and for $y \in [1,\infty)$ we obtain

$$W(y) \le C_4 G_{\alpha+\frac12}(y) \Big(1 + \int_1^y \sigma^{2\alpha-2p\beta-1}\, d\sigma \Big) \le \\ \le C_5 y^{-2\alpha-1} \Big(1 + \int_1^y \sigma^{-3}\, d\sigma \Big) \le C_6 y^{-2\alpha-1}.$$

Similarly we obtain that W'' and yW' are integrable at infinity. Hence, we can integrate the equation

$$W'' - \frac{y}{2} W' - \Big(\alpha + \frac12\Big) W - \Big(V^p\Big)' = 0$$

to see that if U is given by (4.4) then (U,V) is a solution of (4.2). Since

$$A = \int_0^\infty G_{\alpha+\frac12}(y) \Big[\int_0^y e^{\frac{\sigma^2}{4}} G^{-2}_{\alpha+\frac12}(\sigma) \Big(\int_\sigma^\infty e^{-\frac{s^2}{4}} G_{\alpha+\frac12}(s) \Big(-G_\beta^p\Big)' ds \Big) d\sigma \Big] dy < \infty,$$

it follows that

$$c = \Big(\int_0^\infty W(y)\, dy \Big)^q = \Big(c^p A \Big)^q, \qquad \text{hence} \qquad c = A^{-\frac{q}{pq-1}}.$$

5 – The Chipot-Weissler equation

In this section we consider equation (1.14) in $\mathbb{R}^N$ with an initial condition as in (1.15). We consider $1 \le q < 2p/(p+1) (< 2)$ therefore solutions may cease to exist in finite time only if they blow up in the sup-norm (cf. [24]).

We prove the following:

Theorem 5.1. *If* $1 < p \le 1 + \frac{2}{N}$ *and* $1 \le q < \frac{2p}{p+1}$ *then for every solution* u *of* (1.14), (1.15) *which blows up at a finite time* T *there is a constant* C *such that the estimates*

$$u(x,t) \le C(T-t)^{-\frac{1}{p-1}}, \qquad |\nabla u(x,t)| \le C(T-t)^{-\frac{p+1}{2(p-1)}} \tag{5.1}$$

hold in $\mathbb{R}^N \times (0,T)$.

Proof. Let

$$M_u(t) := \sup_{\mathbb{R}^N \times (0,t]} \left(u + |\nabla u|^{\frac{2}{p+1}} \right), \qquad t \in (0,T).$$

Clearly M_u is a positive, continuous and nondecreasing function on $(0,T)$ which diverges as $t \nearrow T$. Put $\alpha = \frac{1}{p-1}$.

Step 1: *Scaling.* Given $t_0 \in (0,T)$ choose $(\hat{x},\hat{t}) \in \mathbb{R}^N \times (0,t_0]$ such that

$$u(\hat{x},\hat{t}) + |\nabla(\hat{x},\hat{t})|^{\frac{2}{p+1}} \ge \frac{1}{2} M_u(t_0). \tag{5.2}$$

Let $\lambda = \lambda(t_0)$ be as in (2.4) and define the rescaled function φ^λ in $\mathbb{R}^N \times (-\lambda^{-2}\hat{t}, \lambda^{-2}(T-\hat{t}))$ by (2.5).

Clearly, $\varphi = \varphi^\lambda$ is a solution of

$$\varphi_s = \Delta\varphi - \lambda^\delta |\nabla\varphi|^q + \varphi^p,$$

where $\delta = 2\alpha + 2 - (2\alpha+1)q = [\frac{2p}{p+1} - q]\frac{p+3}{p+1} > 0$.

Define further t_0^+ by (2.12).

Clearly,

$$u(x,t) + |\nabla u(x,t)|^{\frac{2}{p+1}} \le 2M_u(t_0)$$

for $(x,t) \in \mathbb{R}^N \times (0,t_0^+]$ and, consequently,

$$\varphi^\lambda + |\nabla\varphi^\lambda|^{\frac{2}{p+1}} \le 2$$

in $\mathbb{R}^N \times (-\lambda^{-2}\hat{t}, \lambda^{-2}(t_0^+ - \hat{t}))$.

By (5.2) we have

$$\varphi^{\lambda}(0,0) + |\nabla\varphi^{\lambda}(0,0)|^{\frac{2}{p+1}} \geq \frac{1}{2}\,.$$

The crucial point in the proof of the blow-up rate estimates (5.1) will again be to prove that there is a constant $D \in (0,\infty)$ such that (2.13) holds.

From (2.13) we obtain, as in Step 5 of the proof of Theorem 2.1, that (2.14) holds. But this is equivalent with (5.1).

To prove (2.13) we will proceed by contradiction. Suppose that this estimate were false. Then there would exist a sequence $t_j \nearrow T$, $t_j > \frac{T}{2}$, such that

$$\lambda^{-2}(t_j)(t_j^+ - t_j) \to \infty.$$

For each t_j we scale the function u about the corresponding point $(\hat{x}_j, \hat{t}_j)$ (with $\hat{t}_j \leq t_j$ and $\hat{t}_j \to T$) with the scaling factor $\lambda_j = \lambda(t_j) := M_u^{-\frac{1}{2\alpha}}(t_j)$ to obtain a function φ^{λ_j} satisfying

$$\varphi_s^{\lambda_j} - \Delta\varphi^{\lambda_j} = -\lambda_j^{\delta}|\nabla\varphi^{\lambda_j}|^q + (\varphi^{\lambda_j})^p, \tag{5.3}$$

and

$$\varphi^{\lambda_j} + |\nabla\varphi^{\lambda_j}|^{\frac{2}{p+1}} \leq 2,$$

in $\mathbb{R}^N \times (-\lambda_j^{-2}\hat{t}_j, \lambda_j^{-2}(t_j^+ - \hat{t}_j))$ and

$$\varphi^{\lambda_j}(0,0) + |\nabla\varphi^{\lambda_j}(0,0)|^{\frac{2}{p+1}} \geq \frac{1}{2}\,.$$

As for any $K > 0$ the functions φ^{λ_j}, $\nabla\varphi^{\lambda_j}$ and the right hand side of (5.3) are uniformly bounded in $\mathbb{R}^N \times (-K, K)$ independently of $j \geq j(K)$, we obtain (locally) uniform estimates in $C^{1+\sigma,\frac{1+\sigma}{2}}$-norms (cf. [24]).

Consequently, we have (locally) uniform estimates in Hölder norms on the right hand side of (5.3), and the Schauder's estimates

$$\|\varphi^{\lambda_j}\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\{y\in\mathbb{R}^N:|y|\leq K\}\times[-K,K])} \leq C_K, \quad j \geq j(K)$$

with C_K independent on λ_j, $j \geq j(K)$, follow as in the proof of Theorem 2.1, Step 2.

Using these estimates and compactness we can find a subsequence converging to a positive solution φ of the equation

$$\varphi_s - \Delta\varphi = \varphi^p \tag{5.4}$$

in $\mathbb{R}^N \times \mathbb{R}$ such that

$$\varphi + |\nabla\varphi|^{\frac{2}{p+1}} \leq 2, \quad \text{and} \quad \varphi(0,0) + |\nabla\varphi(0,0)|^{\frac{2}{p+1}} \geq \frac{1}{2}\,.$$

This is a contradiction, as it was shown in [15], [18] and [23] that all positive solutions of (5.4) in $\mathbb{R}^N$ are nonglobal, if $p \leq 1 + \frac{2}{N}$. That completes the proof.

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