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Branches of solutions to a semilinear elliptic equation with singular coefficients on $\mathbb{R}^N$

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RIASSUNTO: *Diamo un risultato di biforcazione globale per un problema semilineare definito su* $\mathbb{R}^N$ *utilizzando risultati di Pejsachowicz e Rabier validi per applicazioni Fredholm d'indice zero di classe* $\mathcal{C}^1$.

ABSTRACT: *We give a global bifurcation result for a semilinear problem on* $\mathbb{R}^N$ *using a Theorem available for* $\mathcal{C}^1$*-Fredholm mappings of index zero stated by Pejsachowicz and Rabier.*

1 – Introduction

Given a mapping $f: \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfying
(H1) $f(\cdot, 0) \equiv 0$,
we are interested in the following nonlinear Schrödinger eigenvalue problem:

$$\begin{cases} -\triangle u + f(\cdot, u) - \lambda u = 0 & \text{on } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0, u \not\equiv 0. \end{cases} \tag{1.1}$$

This kind of problem has been investigated under various assumptions on f for example by AMBROSETTI and GAMEZ [1], EDELSON and

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STUART [5], MONTEFUSCO [15] ... Here we are interested in given some conditions on the function f which ensure the existence of global branches of solutions for Problem (1.1) bifurcating from a trivial solution $(\lambda_0, 0)$ in $\mathbb{R} \times W^{2,p}(\mathbb{R}^N)$ for $p > \max\{1, \frac{N}{2}\}$. Since the Laplacian operator does not have a compact inverse $\mathbb{R}^N$, we cannot reduce Problem (1.1) into a form which allows to apply the Global Bifurcation Theorem of Rabinowitz [19]. In [12] this difficulty was overcome by using a result on global bifurcation due to FITZPATRICK, PEJSACHOWICZ and RABIER [10]. The goal of this paper is to show how following the same approach it is possible to give a similar result under slightly more general hypotheses on the function f. The main improvements are:

1) As anticipated in [12], PEJSACHOWICZ and RABIER (see [17]) have extended to $\mathcal{C}^1$-Fredholm mappings a topological degree first defined in [10] for $\mathcal{C}^2$-Fredholm mappings. As a consequence, it is sufficient for our problem to assume that $s \mapsto f(x, s)$ is $\mathcal{C}^1$.
2) The hypotheses of equicontinuity in [12] are essentially replaced here by a growth condition.
3) In [12], we assumed $\partial_s f(\cdot, 0) \in \mathrm{L}^\infty(\mathbb{R}^N)$. Here, we require $\partial_s f(\cdot, 0) \in \mathrm{L}^p(\mathbb{R}^N) + \mathrm{L}^\infty(\mathbb{R}^N)$.

The results of this paper have been organized as follows.

In Section 2, we summarize the results of bifurcation for $\mathcal{C}^1$-Fredholm mappings we need.

In Section 3, we give a differentiability result for a class of Nemitsky operators defined between the space $\mathrm{L}_{p,\infty} := \{u \in \mathrm{L}^p(\mathbb{R}^N) + \mathrm{L}^\infty(\mathbb{R}^N) : \lim_{|x|\to\infty} u(x) = 0\}$ endowed with the norm $\|u\| := \|u\|_{L^p} + \|u\|_{L^\infty}$ and the space $\mathrm{L}^p(\mathbb{R}^N)$.

In Section4, we show with Proposition 4.1 that if in addition to (H1) we assume

(H2) 1) f is a Carathéodory function,

2) for a.e. $x \in \mathbb{R}^N$, $s \mapsto f(x, s)$ is of class $\mathcal{C}^1$,

3) $\partial_s f(\cdot, 0) \in \mathrm{L}^p(\mathbb{R}^N) + \mathrm{L}^\infty(\mathbb{R}^N)$ and there exists $R_0 > 0$ such that

$$\lim_{|s|\to 0} \|\partial_s f(x, s) - \partial_s f(x, 0)\|_{L^\infty(|x|>R_0)} = 0,$$

4) for every $R, \delta > 0$ there exists $\Psi := \Psi_{R,\delta} \in \mathrm{L}^p(|x| < R)$ such that
$$|\partial_s f(x,s)| \le \Psi(x) \qquad \forall |s| < \delta, \quad a.e\, |x| < R,$$

then the mapping $F : \mathbb{R} \times W^{2,p}(\mathbb{R}^N) \to \mathrm{L}^p(\mathbb{R}^N)$, $(\lambda, u) \mapsto -\triangle u + f(\cdot, u) - \lambda u$ is a $\mathcal{C}^1$ mapping whose zeros are clearly solutions of Problem (1.1). Setting $\alpha := \liminf_{|x|\to\infty} \partial_s f(x,0)$, we show that the operator $D_u F_{(\lambda,u)}$ is Fredholm of index zero on $(-\infty, \alpha) \times W^{2,p}(\mathbb{R}^N)$.

In Section 5, we show that $(\lambda_0, 0)$ is a bifurcation point for the equation $F(\lambda, u) = 0$ if moreover:

(H3) $\alpha > -\infty$, $\lambda_0 < \alpha$ and Ker $\left[D_u F_{(\lambda_0,0)}\right]$ is of odd dimension.

In Section 6, using a maximum principle, we give a sufficient condition **(H4)** on the mapping f which ensures that the restriction of the mapping F to $(-\infty, \beta) \times W^{2,p}(\mathbb{R}^N)$ is "boundedly proper" (where β will be made precise later on).

In Section 7, we show that the local result given in Section 5 is global if (H4) is assumed.

Notations

$B_R := \{x \in \mathbb{R}^N : |x| < R\}, \quad B_{R,\infty} := \{x \in \mathbb{R}^N : |x| > R\}$,

X, Y denote Banach spaces, I an open interval of $\mathbb{R}$, Ω an open subset of $\mathbb{R}^N$,

$\Phi_0(X,Y)$: Set of linear operator from the space X to Y which are Fredholm of index 0,

$\alpha := \liminf\limits_{|x|\to\infty} \partial_s f(x,0), \quad p_0 := \max\left\{1, \frac{N}{2}\right\}$.

2 – Bifurcation results for $\mathcal{C}^1$-Fredholm mapping

In this part, we summarize the notion of parity introduced in [7] and mention a local and a global bifurcation result valid for $\mathcal{C}^1$-Fredholm mappings published in [8] and [17].

A) Parity

If $T \in GL(X)$ is a compact perturbation of the identity, we let $deg_{LS}(T)$ be the Leray-Schauder degree of $T : U \to X$ with respect to 0, where U is any bounded neighborhood of the origin ([4]).

Definition 2.1. Let $A : [a, b] \to \Phi_0(X, Y)$ be a continuous path.

1) A continuous path $\eta : [a, b] \to GL(Y, X)$ will be called a parametrix for A if for each $\lambda \in [a, b]$, $\eta(\lambda) \circ A(\lambda)$ is a compact perturbation of the identity.
2) If $A(a)$ and $A(b)$ are invertible, then the parity of A in $[a, b]$, $\sigma(A,[a,b])$ is defined by :

$$\sigma(A, [a, b]) := deg_{LS}\,[\eta(a) \circ A(a)] \cdot deg_{LS}\,[\eta(b) \circ A(b)]\,.$$

3) We will denote by $\Sigma(A) := \{\lambda \in [a, b] : \operatorname{Ker} A(\lambda) \neq \{0\}\}$. When λ is an isolated point in $\Sigma(A)$, there exists a closed interval $J \subset [a, b]$ such that $J \cap \Sigma(A) = \{\lambda\}$. The parity of the restriction of A to J will be denoted by $\sigma(A, \lambda)$.

Remark 2.2.

1) The existence of a parametrix in Definition 2.1 is guaranteed by the compactness and contractibility of the interval $[a, b]$ (See Proposition 2.3, [6]).
2) The parity is independent of the choice of parametrix ([7]) and $\sigma(A, I) = \pm 1$ ([4], p. 64).

Proposition 2.3 (Theorem 6.18, [9]). *Let $A : [a, b] \to \Phi_0(X, Y)$ be continuous, be differentiable at λ_0 and satisfying*

$$A'(\lambda_0)[\operatorname{Ker} A(\lambda_0)] \oplus \operatorname{Ran} A(\lambda_0) = Y. \tag{2.2}$$

Then, λ_0 is isolated in $\Sigma(A)$ and $\sigma(A,\lambda_0) = (-1)^k$, where $k = \dim \operatorname{Ker} A(\lambda_0)$.

B) A Local and a global bifurcation result

Consider a mapping F such that:

$$F : I \times X \to Y, \quad (\lambda, x) \mapsto F(\lambda, x), \qquad \text{with } F(\cdot, 0) = 0. \tag{2.3}$$

Proposition 2.4 (Fitzpatrick and Pejsachowicz, [8]). *Let F be a mapping satisfying (2.3), be of class $\mathcal{C}^1$ and such that $D_xF_{(\lambda,0)} \in \Phi_0(X, Y)$ for all $\lambda \in I$. Consider the mapping $A : I \to \Phi_0(X, Y), \lambda \mapsto D_xF_{(\lambda,0)}$.*

Assume A continuous, $A(a), A(b) \in GL(X,Y)$ for some $a, b \in I$ $(a < b)$ and $\sigma(A,[a,b]) = -1$. Then, there exists $\lambda_0 \in (a,b)$ such that $(\lambda_0, 0)$ is a bifurcation point for the equation $F(\lambda, x) = 0$.

PROPOSITION 2.5. *Let F be a mapping satisfying the hypotheses of Proposition 2.4. Assume that the mapping $A : I \to \Phi_0(X,Y), \lambda \mapsto D_xF_{(\lambda,0)}$ is differentiable at λ_0 and satisfies (2.2). Then, $\sigma(A,\lambda_0) = -1$ iff $\dim \operatorname{Ker} A(\lambda_0)$ is of odd dimension and in such a case, $(\lambda_0, 0)$ is a bifurcation point for the equation $F(\lambda, x) = 0$.*

This result, a direct consequence of Proposition 2.4 and 2.3, is a local bifurcation result for $\mathcal{C}^1$-Fredholm maps, in the same spirit as the well-known Theorem of KRASNOSELSKY [14]. In order to give a global bifurcation theorem available for $\mathcal{C}^1$ Fredholm maps, we denote by

$$
\begin{aligned}
\mathcal{S} &= \{(\lambda, x) \in I \times X : F(\lambda, x) = 0 \text{ and } x \neq 0\},\\
\mathcal{Z}_{\lambda_0} &= \mathcal{S} \cup \{(\lambda_0, 0)\} \text{ (with the topology inherited from } \mathbb{R} \times X),\\
\mathcal{C}_{\lambda_0} &: \text{ connected component of } \mathcal{Z}_{\lambda_0} \text{ containing } (\lambda_0, 0)\\
p_1 &: \text{ the projection defined by } I \times X \to I, (\lambda, x) \mapsto \lambda.
\end{aligned}
$$

DEFINITION 2.6. A mapping $F : I \times X \to Y$ is said to be boundedly proper if the restriction of F to any subset of $I \times X$ which is bounded and closed in $\mathbb{R} \times X$ is proper.

PROPOSITION 2.7 (Pejsachowicz and Rabier, [17]). *Let F be a mapping satisfying (2.3). Assume that F is a $\mathcal{C}^1$ Fredholm mapping of index 0 which is boundedly proper. Moreover assume the existence of a point λ_0 isolated in $\Sigma\Big(D_xF_{(\lambda,0)}\Big)$ and such that $\sigma\Big(D_xF_{(\lambda,0)}, \lambda_0\Big) = -1$. Then $\mathcal{C}_{\lambda_0}$ has at least one of the following properties.*

1) *$\mathcal{C}_{\lambda_0}$ is unbounded,*
2) *the closure of $\mathcal{C}_{\lambda_0}$ contains a point of the form $(\lambda^*, 0)$ with $\lambda^* \in I$,*
3) *the closure of $p_1(\mathcal{C}_{\lambda_0})$ intersects the boundary of I.*

3 – Differentiability of Nemitsky operator $W^{2,p}(\mathbb{R}^N) \to \mathrm{L}^p(\mathbb{R}^N)$

PROPOSITION 3.1. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function.*

1) *If $u : \Omega \to \mathbb{R}$ is measurable, then the mapping $\Omega \to \mathbb{R}$, $x \mapsto g(x, u(x))$ is measurable.*
2) *If $g(x, \cdot) \in \mathcal{C}^1(\mathbb{R})$ a.e. $x \in \Omega$, then $\partial_s g$ is a Carathéodory function.*

PROOF. For part 1) we refer to Theorem 18.3, p. 152 in VAINBERG's book [21]. Part 2) is easy. ∎

For $p, q \in [1, \infty)$, it is shown in KRASNOSELSKY [14] that the Nemitsky operator $\tilde{g}$ is well defined from $\mathrm{L}^p(\Omega)$ to $\mathrm{L}^q(\Omega)$ iff g satisfies the following condition:

there exists a function $a \in \mathrm{L}^q(\Omega)$ and a constant $b > 0$ such that, for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$

$$|g(x, s)| \leq a(x) + b|s|^{p/q}.$$

Moreover, in such a case, the operator $\tilde{g}$ is automatically continuous.

A similar result exists for the case $p = \infty$, $q < \infty$.

DEFINITION 3.2. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory map. For $p \in [1, \infty)$, we say that g satisfies the property $(P_{\infty,p})$ if for all $M > 0$, there exists $\eta_M \in \mathrm{L}^p(\Omega)$ such that

$$|g(x, s)| \leq \eta_M(x) \qquad \forall |s| < M, \quad \text{a.e. } x \in \Omega.$$

PROPOSITION 3.3 (Theorem 3.1, [18]). *Let $q \geq 1$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory map. Then, the operator $\widetilde{g} : \mathrm{L}^\infty(\Omega) \to \mathrm{L}^q(\Omega)$, $u \mapsto g(\cdot, u)$ is well defined iff g satisfies the property $(P_{\infty,q})$. In such a case, the operator $\widetilde{g}$ is automatically continuous.*

Let $p \in [1, \infty)$ and consider

$$\mathrm{L}_{p,\infty} := \{u \in \mathrm{L}^p(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N) : \lim_{|x| \to \infty} |u(x)| = 0\}, \tag{3.4}$$

endowed with the norm $\|u\|_{L_{p,\infty}} := \|u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^\infty(\mathbb{R}^N)}$. The aim of what follows is to study the differentiability of Nemitsky operators defined from $\mathrm{L}_{p,\infty}(\Omega)$ to $\mathrm{L}^p(\Omega)$. With this aim, we introduce the space: $\mathrm{L}^p(\Omega) + \mathrm{L}^\infty(\Omega) := \{f_1 + f_2 : f_1 \in \mathrm{L}^p(\Omega) \text{ and } f_2 \in \mathrm{L}^\infty(\Omega)\}$.

Remark 3.4. Notice that if $f \in \mathrm{L}^p(\Omega) + \mathrm{L}^\infty(\Omega)$ and $g \in \mathrm{L}_{p,\infty}(\Omega)$, then $fg \in \mathrm{L}^p(\Omega)$.

Proposition 3.5. *Let $1 \le p < \infty$ and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a mapping such that*

a) *g is a Carathéodory map and $g(\cdot, 0) \in \mathrm{L}^p(\mathbb{R}^N)$,*
b) *for a.e. $x \in \mathbb{R}^N$, $s \mapsto g(x,s)$ is of class $\mathcal{C}^1$,*
c) *$\partial_s g(\cdot, 0) \in \mathrm{L}^p(\mathbb{R}^N) + \mathrm{L}^\infty(\mathbb{R}^N)$ and there exists $R_0 > 0$ such that*

$$\lim_{|s| \to 0} \|\partial_s g(x,s) - \partial_s g(x,0)\|_{L^\infty(B_{R_0,\infty})} = 0,$$

d) *for every $R > 0$, the restriction of $\partial_s g$ to $B_R \times \mathbb{R}$ satisfies the condition $(P_{\infty,p})$.*

Then, the Nemitsky operator $\widetilde{g} : \mathrm{L}_{p,\infty}(\mathbb{R}^N) \to \mathrm{L}^p(\mathbb{R}^N)$, $u \mapsto g(\cdot, u)$ is $\mathcal{C}^1$ Fréchet differentiable and for every $u \in \mathrm{L}_{p,\infty}(\mathbb{R}^N)$, the Fréchet derivative is given by:

$$D\widetilde{g}_{(u)} : \mathrm{L}_{p,\infty}(\mathbb{R}^N) \longrightarrow \mathrm{L}^p(\mathbb{R}^N) \qquad \xi \longmapsto \partial_s g(\cdot, u)\,\xi.$$

Proof Step 1. We show that $\widetilde{g}$ is well defined.

Using hypothesis b), we have for a.e. $x \in \mathbb{R}^N$: $g(x,s) = g(x,0) + \int_0^1 \frac{d}{dt}[g(x,ts)]\,dt$.

Thus, given $u \in \mathrm{L}_{p,\infty}(\mathbb{R}^N)$, for a.e. $x \in \mathbb{R}^N$, we have:

$$|g(x,u(x))| \le |g(x,0)| + |u(x)| \int_0^1 |\partial_s g(x, tu(x))|\,dt.$$

Denoting by $A(x) := \int_0^1 |\partial_s g(x, tu(x))| dt$, above inequality shows it is enough to prove that $A \in \mathrm{L}^p(\mathbb{R}^N) + \mathrm{L}^\infty(\mathbb{R}^N)$ (see hypothesis a) and also Remark 3.4).

By hypothesis c), and the fact that $\lim_{|x|\to\infty} |u(x)| = 0$, we deduce the existence of $R > 0$ (depending on g and u) such that: $|\partial_s g(x, tu(x)) - \partial_s g(x, 0)| \le 1$ (a.e. $x \in B_{R,\infty}$, for all $t \in [0, 1]$).

Hence, on $B_{R,\infty}$, we have

$$|A(x)| \le \int_0^1 |\partial_s g(x, tu(x)) - \partial_s g(x, 0)| dt + |\partial_s g(x, 0)| \le 1 + |\partial_s g(x, 0)|.$$

From hypothesis c), we have $\partial_s g(\cdot, 0) \in \mathrm{L}^p(\mathbb{R}^N) + \mathrm{L}^\infty(\mathbb{R}^N)$, which implies

$$A \in \mathrm{L}^p(B_{R,\infty}) + \mathrm{L}^\infty(B_{R,\infty}). \tag{3.5}$$

On B_R, hypothesis d) and the fact that $u \in \mathrm{L}^\infty(\mathbb{R}^N)$ imply the existence of $\eta \in \mathrm{L}^p(\mathbb{R}^N)$ such that $|\partial_s g(\cdot, tu)| \le \eta$ on B_R for all $t \in [0, 1]$. Hence,

$$A \in \mathrm{L}^p(B_R) + \mathrm{L}^\infty(B_R). \tag{3.6}$$

Relations (3.5) and (3.6) imply $A \in \mathrm{L}^p(\mathbb{R}^N) + \mathrm{L}^\infty(\mathbb{R}^N)$.

STEP 2. Given $u \in \mathrm{L}_{p,\infty}$, we verify that the linear mapping $D\tilde{g}_{(u)}$ given in the conclusion of this proposition is well defined and continuous.

By Proposition 3.1, we know that $\partial_s g$ is a Carathéodory map which implies the measurability of the mapping $\partial_s g(\cdot, u)\xi$.

From hypothesis c) and the fact that $\lim_{|x|\to\infty} |u(x)| = 0$, there exists $R > R_0$ such that

$$|\partial_s g(\cdot, u) - \partial_s g(\cdot, 0)| \le 1 \qquad \text{on } B_{R,\infty}.$$

Thus,

$$\begin{aligned}\|\partial_s g(\cdot,u)\xi\|_{L^p(B_{R,\infty})} &\le \|[\partial_s g(\cdot,u) - \partial_s g(\cdot,0)]\xi\|_{L^p(B_{R,\infty})} + \|\partial_s g(\cdot,0)\xi\|_{L^p(B_{R,\infty})} \le \\ &\le \|\xi\|_{L^p(B_{R,\infty})} + \|\partial_s g(\cdot, 0)\, \xi\|_{L^p(B_{R,\infty})} \le \\ &\le C\|\xi\|_{L_{p,\infty}(\mathbb{R}^N)}.\end{aligned}$$

On B_R, from hypothesis d) we have $\partial_s g(\cdot,u) \in \mathrm{L}^p(B_R)$ and thus,

$$\|\partial_s g(\cdot,u)\,\xi\|_{L^p(B_R)} \le C\|\xi\|_{L^\infty(B_R)} \le C\|\xi\|_{L_{p,\infty}(\mathbb{R}^N)}.$$

STEP 3. $\lim_{\|\xi\|_{L_{p,\infty}}\to 0} \frac{\|\tilde g(u+\xi)-\tilde g(u)-D\tilde g_{(u)}(\xi)\|_{L^p(\mathbb{R}^N)}}{\|\xi\|_{L_{p,\infty}}} = 0$ (i.e $\tilde g$ is Fréchet-differentiable).

We have,

$$\begin{aligned}\|\tilde g(u+\xi)-\tilde g(u)-D\tilde g_{(u)}(\xi)\|_{L^p(\mathbb{R}^N)} &= \\ &= \|g(\cdot,u+\xi)-g(\cdot,u)-\partial_s g(\cdot,u)\,\xi\|_{L^p(\mathbb{R}^N)} \le \\ &\le \left\|\int_0^1 \frac{d}{dt}[g(\cdot,u+t\xi)]\,dt - \partial_s g(\cdot,u)\,\xi\right\|_{L^p(\mathbb{R}^N)} = \\ &= \|\xi\Psi(u,\xi)\|_{L^p(\mathbb{R}^N)}\,,\end{aligned}$$

where we have set $\Psi(u,\xi) = \int_0^1 [\partial_s g(\cdot,u+t\xi) - \partial_s g(\cdot,u)]\,dt$. For every $R>0$, we have

$$\begin{aligned}\|\xi\Psi(u,\xi)\|_{L^p(\mathbb{R}^N)} \le \|\xi\|_{L^\infty(\mathbb{R}^N)}\,\|\Psi(u,\xi)\|_{L^p(B_R)} &+ \\ + \|\xi\|_{L^p}(\mathbb{R}^N)\,\|\Psi(u,\xi)\|_{L^\infty(B_{R,\infty})} &\le \\ \le \|\xi\|_{L_{p,\infty}}\Big(\|\Psi(u,\xi)\|_{L^p(B_R)} + \|\Psi(u,\xi)\|_{L^\infty(B_{R,\infty})}\Big)&.\end{aligned}$$

Thus for every $R>0$ and $\xi \ne 0$, we have

$$\begin{aligned}\frac{\|\tilde g(u+\xi)-\tilde g(u)-D\tilde g_{(u)}(\xi)\|_{L^p(\mathbb{R}^N)}}{\|\xi\|_{L_{p,\infty}}} &\le \\ \le \|\Psi(u,\xi)\|_{L^p(B_R)} + \|\Psi(u,\xi)\|_{L^\infty(B_{R,\infty})}&\end{aligned} \tag{3.7}$$

Let $\epsilon > 0$. We show that for R "large enough" and $\|\xi\|_{L_{p,\infty}}$ "small enough", the right hand-side of this last inequality becomes less than ϵ. From hypothesis c), and the fact that $\lim_{|x|\to\infty} |u(x)| = 0$, there exists $\delta, R > 0$ such that

$$|\partial_s g(x,s) - \partial_s g(x,0)| < \frac{\epsilon}{4} \qquad \text{a.e. } x \in B_{R,\infty},\ \forall |s| < \delta,$$

$$|u(x)| < \frac{\delta}{2} \qquad \text{a.e. } x \in B_{R,\infty}.$$

Let us consider $\xi \in L_{p,\infty}(\mathbb{R}^N)$ satisfying $\|\xi\|_{L_{p,\infty}} < \frac{\delta}{2}$. Then, for a.e. $x \in B_{R,\infty}$ and for all $t \in [0,1]$, we have $|u(x) + t\xi(x)| < \delta$ and by consequence,

$$\|\Psi(u,\xi)\|_{L^\infty(B_{R,\infty})} < \frac{\epsilon}{2} \qquad \forall \|\xi\|_{L_{p,\infty}} < \frac{\delta}{2}. \tag{3.8}$$

In B_R, we have

$$\begin{aligned}\|\Psi(u,\xi)\|^p_{L^p(B_R)} &= \int_{B_R} \Big| \int_0^1 [\partial_s g(\cdot, u+t\xi) - \partial_s g(\cdot,u)]dt \Big|^p dx \leq \\ &\leq \int_{B_R}\int_0^1 |\partial_s g(\cdot,u+t\xi) - \partial_s g(\cdot,u)|^p dt dx = \\ &= \int_0^1 \int_{B_R} |\partial_s g(\cdot,u+t\xi) - \partial_s g(\cdot,u)|^p dx dt = \\ &= \int_0^1 \|\partial_s g(\cdot,u+t\xi) - \partial_s g(\cdot,u)\|^p_{L^p(B_R)} dt.\end{aligned}$$

Thus,

$$\|\Psi(u,\xi)\|^p_{L^p(B_R)} \leq \int_0^1 \|\partial_s g(\cdot,u+t\xi) - \partial_s g(\cdot,u)\|^p_{L^p(B_R)} dt. \tag{3.9}$$

By hypothesis the restriction of the mapping $\partial_s g$ to $B_R \times \mathbb{R}$ satisfies the assumptions of Proposition 3.3. Thus, there exists $\widetilde{\delta} > 0$ such that: $\|\partial_s g(\cdot,u+t\xi) - \partial_s g(\cdot,u)\|_{L^p(B_R)} < \frac{\epsilon}{2}$, $\forall \|\xi\|_{L_{p,\infty}} < \widetilde{\delta}$.

From (3.9), we deduce

$$\|\Psi(u,\xi)\|_{L^p(B_R)} < \frac{\epsilon}{2} \qquad \forall \|\xi\|_{L_{p,\infty}} < \widetilde{\delta}, \quad \forall t \in [0,1]. \tag{3.10}$$

Thus, relations (3.7), (3.8) and (3.10) give for every $\|\xi\|_{L_{p,\infty}} < \min\{\delta, \widetilde{\delta}\}$

$$\frac{\|\tilde{g}(u+\xi) - \tilde{g}(u) - D\tilde{g}_{(u)}(\xi)\|_{L^p(\mathbb{R}^N)}}{\|\xi\|_{L_{p,\infty}}} < \epsilon.$$

STEP 4. Set $X := \mathrm{L}_{p,\infty}$ and $Y := \mathrm{L}^p(\mathbb{R}^N)$, we show that $D\tilde{g} : X \to \mathcal{L}(X;Y)$ is continuous.

Let $u, u_0 \in X$. For every $R > 0$, we have

$$\begin{aligned}
&\|D\tilde{g}_{(u)} - D\tilde{g}_{(u_0)}\|_{\mathcal{L}(X,Y)} = \\
&\qquad = \sup_{\|\xi\|_X \le 1} \|[\partial_s g(\cdot, u) - \partial_s g(\cdot, u_0)]\xi\|_{L^p(\mathbb{R}^N)} \le \\
&\qquad \le \|\partial_s g(\cdot, u) - \partial_s g(\cdot, u_0)\|_{L^p(B_R)} + \|\partial_s g(\cdot, u) - \partial_s g(\cdot, u_0)\|_{L^\infty(B_{R,\infty})}.
\end{aligned}$$

As in the second step we verify that this last expression tends to 0 when $\|u - u_0\|_X \to 0$. □

Since for $p > p_0$ the injection $i : W^{2,p}(\mathbb{R}^N) \hookrightarrow \mathrm{L}_{p,\infty}$ is of class $\mathcal{C}^\infty$, above proposition can be applied to prove the differentiability of Nemitsky operators defined from $W^{2,p}(\mathbb{R}^N)$ to $\mathrm{L}^p(\mathbb{R}^N)$.

4 – A functional framework

In this section we define a mapping $F : \mathbb{R} \times X \to Y$ of class $\mathcal{C}^1$, whose zeros are solutions of Problem (1.1). From Proposition 3.5, we have immediately :

PROPOSITION 4.1. *Let f be a mapping satisfying* (H1) *and* (H2). *Then, the mapping defined by*

$$(4.11)\quad F : \mathbb{R}\times W^{2,p}(\mathbb{R}^N) \longrightarrow \mathrm{L}^p(\mathbb{R}^N) \qquad (\lambda, u) \longmapsto -\triangle u + f(\cdot, u) - \lambda u$$

is well-defined, of class $\mathcal{C}^1$ and $D_u F_{(\lambda,u)}(\xi) = -\triangle\xi + (\partial_2 f(\cdot, u) - \lambda)\,\xi$.

EXAMPLE 4.2. A mapping of the kind $f(x, s) = \Big(p(x) + q(x)r(s)\Big)s$ satisfies all the hypotheses of Proposition 4.1 if:

$$(4.12)\qquad p, q \in \mathrm{L}^\infty(\mathbb{R}^N),$$

$$(4.13.)\qquad r(s)s \text{ is } \mathcal{C}^1(\mathbb{R})$$

REMARK 4.3.

1) Condition (4.13) is not satisfied if $r(s) = s^{\theta-1}$ with $0 < \theta < 1$. Thus, this kind of non linearity (studied for example in [15]) cannot be considered here.

2) It is easy to show that the hypotheses made on f in [12] to ensure the $\mathcal{C}^1$ differentiability of the mapping (4.11) are less general than the assumptions of Proposition 4.1.

To give results of bifurcation for the problem $F(\lambda, u) = 0$ based on the results presented in Section 2 we first discuss the Fredholm property of $D_u F_{(\lambda,u)}$ and then prove that this operator satisfy the condition (2.2).

PROPOSITION 4.4. *Let $p > p_0$, f be a mapping satisfying* (H1) *and* (H2)*. Then, the mapping* (4.11) *is Fredholm with index* 0 *on* $(-\infty, \alpha) \times W^{2,p}(\mathbb{R}^N)$ *(where* $\alpha := \liminf_{|x|\to\infty} \partial_s f(\cdot, 0)$*).*

PROOF. Let $u \in W^{2,p}(\mathbb{R}^N)$. We have by Proposition 4.1 that $D_u F_{(\lambda,u)} = -\triangle + \partial_s f(\cdot, u) - \lambda$. Setting,

$$\mu := \lambda - \alpha, \quad V := \partial_s f(\cdot, u) - \alpha, \quad V^+ = \max(V, 0), \quad V^- = -\min(V, 0)$$

we can write, $D_u F_{(\lambda,u)} = -\triangle + (V^+ - \mu) + V^-$. Hypothesis (H2) implies that $V \in \mathrm{L}^p(\mathbb{R}^N) + \mathrm{L}^\infty(\mathbb{R}^N)$ and $\liminf\limits_{|x|\to\infty} V(x) > -\infty$. We deduce from PAZY's book ([16]) that $-\triangle + (V^+ - \mu)$ is an isomorphism for every $\mu < 0$.

Moreover, from the fact that $V^- \in \mathrm{L}^p_{loc}(\mathbb{R}^N)$ and $\lim\limits_{|x|\to\infty} V^-(x) = 0$, we deduce that multiplication operator $W^{2,p}(\mathbb{R}^N) \longrightarrow \mathrm{L}^p(\mathbb{R}^N)$, $u \longmapsto V^- u$ is well-defined and compact.

Thus, $D_u F_{(\lambda,u)}$ is a compact perturbation of a Fredholm operator of index 0 for every $\mu < 0$. This implies that $D_u F_{(\lambda,u)}$ is again Fredholm of index 0 for every $\lambda < 0$ (see [13]). □

PROPOSITION 4.5. *Let $p > p_0$, f be a mapping satisfying* (H1)*,* (H2) *and consider the mapping* (4.11)*. Setting $A_{p,\lambda} := D_u F_{(\lambda,u)}$, we have*

$$\operatorname{Ker} A_{p,\lambda} \oplus \operatorname{Ran} A_{p,\lambda} = \mathrm{L}^p(\mathbb{R}^N) \qquad \forall \lambda < \alpha. \tag{4.14}$$

PROOF. Since the operator $A_{p,\lambda}$ is Fredholm with index 0 (Proposition 4.4), assertion (4.14) is equivalent to $\operatorname{Ker} A_{p,\lambda} \cap \operatorname{Ran} A_{p,\lambda} = \{0\}$. Consider then, $h \in \operatorname{Ker} A_{p,\lambda}$, and $g \in W^{2,p}(\mathbb{R}^N)$ with $A_{p,\lambda}(g) = h$. Roughly speaking we have: $\int_{\mathbb{R}^N} |h|^2 = \int_{\mathbb{R}^N} A_{p,\lambda}(g) h = \int_{\mathbb{R}^N} g A_{p,\lambda}(h) = 0$ (and each of these equalities is meaningful by classical regularity results). □

5 – A local bifurcation result

Under the assumptions (H1) and (H2), we know from Proposition 4.1 that the mapping F defined by (4.11) is of class $\mathcal{C}^1$ for $p > p_0$. We introduce now the following hypothesis

(H3) We assume $\alpha > -\infty$ and the existence of $\lambda_0 < \alpha$ such that Ker $[D_u F_{(\lambda_0,0)}]$ is of dimension odd (and thus in particular non-trivial).

PROPOSITION 5.1. *Let $p > p_0$, f be a mapping satisfying the hypotheses* (H1) *to* (H3)*, and let $\lambda_0 < \alpha$ be such that* Ker $[D_u F_{(\lambda_0,0)}]$ *is of odd dimension. Then,* $(\lambda_0, 0)$ *is a local bifurcation point for the problem* $F(\lambda, u) = 0$.

PROOF. Apply Proposition 4.1, 4.5, and Proposition 2.5. □

REMARK 5.2. If Ker $[D_u F_{(\lambda_0,0)}]$ is of dimension 1, then the conclusion of previous theorem follows from the well-known CRANDALL-RABINOWITZ Theorem [3].

6 – Properness

The aim of this section is to give an additional condition on the function f which ensures that the mapping (4.11) is boundedly proper in the sense of Definition 2.6. With this aim, for each $C \geq 0$, we introduce the following real number,

$$\beta(C) := \lim_{R\to\infty} \inf_{|x|\geq R, |s|\leq C} \left\{ \frac{f(x,s)}{s} \right\}.$$

We set $\beta = \inf_{C\geq 0} \beta(C)$ (note that $\beta \leq \alpha$) and we make the following hypothesis

(H4) $\beta > -\infty$.

Under the hypothesis, we are going to prove that the mapping (4.11) is boundedly proper on $(-\infty, \beta) \times W^{2,p}(\mathbb{R}^N)$. This will be derived mainly from the following result:

Proposition 6.1. *Let $p > p_0$, $E \subset W^{2,p}(\mathbb{R}^N)$ and $(V_u)_{u \in E}$ a family of function in* $\mathrm{L}^p(\mathbb{R}^N) + \mathrm{L}^\infty(\mathbb{R}^N)$ *such that:*

a) *there exists $a, R > 0$ such that $\inf_{|x| \geq R} V_u(x) \geq a > 0$, for all $u \in E$,*

b) *$M := \sup\{|u|_{L^\infty} : u \in E\}$ is finite,*

c) *there exists $\widetilde{w} \in \mathrm{L}^p(\mathbb{R}^N)$ such that $|-\triangle u + V_u u| \leq \widetilde{w}$ for all $u \in E$.*

Then, for all $u \in E$, we have $|u| \leq M e^{-\sqrt{a}(|x|-R)} + (-\triangle + a)^{-1}\widetilde{w}$.

Proof. The proof given in Theorem 3.2 of [12] for a family of function $V_u \in \mathrm{L}^\infty(\mathbb{R}^N)$ can be adapted immediately in our case by using the following maximum principle:

Maximum principle: Let $\Omega := \mathbb{R}^N$ or $\Omega := B_{R_0,\infty}$ for some $R_0 > 0$ and $p > p_0$. Let also $V \in \mathrm{L}^p(\Omega) + \mathrm{L}^\infty(\Omega)$ with $V \geq 0$. Then, the mapping $W^{2,p}(\Omega) \to \mathrm{L}^p(\Omega)$, $u \mapsto -\triangle u + Vu$ satisfies a maximum principle on Ω in the following sense:

$$u(\partial\Omega) \geq 0 \quad \text{and} \quad -\triangle u + Vu \geq 0 \quad \Longrightarrow \quad u \geq 0. \qquad \square$$

Corollary 6.2. *Let E and $(V_u)_{u \in E}$ be as in the above proposition. Then, there exists $w \in \mathrm{L}_{p,\infty}$ (in fact in $W^{2,p}(\mathbb{R}^N)$) such that on $\mathbb{R}^N$, we have $|u| \leq w$ for all $u \in E$.*

Lemma 6.3. *Let $p > p_0$, f be a mapping satisfying* (H1) *and* (H2), *F be defined by* 4.11 *and K be a compact subset of* $\mathrm{L}^p(\mathbb{R}^N)$. *Then, the following affirmations are equivalent:*

1) *$F^{-1}(K)$ is relatively compact in $\mathbb{R} \times W^{2,p}(\mathbb{R}^N)$,*

2) *$F^{-1}(K)$ is relatively compact in $\mathbb{R} \times \mathrm{L}_{p,\infty}$.*

Proof. The fact that 1) implies 2) derives from the continuity of $W^{2,p}(\mathbb{R}^N) \hookrightarrow \mathrm{L}_{p,\infty}$. Reciprocally, let $\{(\lambda_n, u_n)\}$ be a sequence in $F^{-1}(K)$. We have,

$$(6.15) \qquad (-\triangle + 1)u_n = F(\lambda_n, u_n) - f(\cdot, u_n) + (\lambda_n + 1)u_n.$$

Since K is compact we can assume that the sequence $F(\lambda_n, u_n)$ converges. By hypothesis, the sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty$ converges in $\mathrm{L}^p(\mathbb{R}^N)$

(up to a subsequence). Moreover Proposition 3.5 ensures the continuity of $\mathrm{L}_{p,\infty} \to \mathrm{L}^p(\mathbb{R}^N)$, $u \mapsto f(\cdot, u)$. Thus, the right hand side of (6.15) converges in $\mathrm{L}^p(\mathbb{R}^N)$. Since the operator $-\triangle + 1$ is an isomorphism from $W^{2,p}(\mathbb{R}^N)$ to $\mathrm{L}^p(\mathbb{R}^N)$ (see STEIN, [20]), we deduce that u_n converges in $W^{2,p}(\mathbb{R}^N)$. □

LEMMA 6.4. *Let $p > p_0$ and E be a bounded subset of $W^{2,p}(\mathbb{R}^N)$. Then, the two following assertions are equivalent,*

1) *E is relatively compact in* $\mathrm{L}_{p,\infty}$*,*
2) $\forall \epsilon > 0$*, there exists $R > 0$ such that* $\|u\|_{L^p(B_{R,\infty})} + \|u\|_{L^\infty(B_{R,\infty})} < \epsilon$ *for all $u \in E$.*

In particular, if there exists $\Psi \in \mathrm{L}_{p,\infty}$ such that $|u| \le \Psi$ for all $u \in E$, then E is relatively compact in $\mathrm{L}_{p,\infty}$.

PROOF. The fact that 1) implies 2) is clear. Reciprocally, let $q \in \{p, \infty\}$ and $(u_n)_{n=1}^{\infty}$ be a sequence of E. Since E is bounded, we can assume that this sequence converges weakly to $u \in W^{2,p}(\mathbb{R}^N)$. Let $\epsilon > 0$. There exists $R > 0$ such that $\|u\|_{L^q} + \|u_n\|_{L^q} < \frac{\epsilon}{2}$. Thus, on $B_{R,\infty}$ we have

$$\|u(x) - u_n(x)\|_{L^q(B_{R,\infty})} \le \|u(x)\|_{L^q(B_{R,\infty})} + \|u_n(x)\|_{L^q(B_{R,\infty})} < \epsilon.$$

Since $p > p_0$, $W^{2,p}(B_R)$ is compactly embedded in $\mathcal{C}(B_R)$. Thus, there exists $n_0 := n(\epsilon)$ such that

$$\|u - u_n\|_{L^q(B_R)} < \epsilon \qquad \forall n \ge n_0.$$

Thus, $\|u_n - u\|_{L^q(\mathbb{R}^N)} \le \|u_n - u\|_{L^q(B_R)} + \|u_n - u\|_{L^q(B_{R,\infty})} < \epsilon.$ □

PROPOSITION 6.5. *Let f be a mapping satisfying the hypotheses* (H1)*,* (H2) *and* (H4)*. Then, the mapping* (4.11) *is boundedly proper on* $(-\infty, \beta) \times W^{2,p}(\mathbb{R}^N)$.

PROOF. Let $[a,b] \subset (-\infty,\beta)$, B be a bounded closed subset of $[a,b] \times W^{2,p}(\mathbb{R}^N)$ and K be a compact subset of $\mathrm{L}^p(\mathbb{R}^N)$. We must prove that every sequence $\{(\lambda_n,u_n)\}$ of $F^{-1}(K) \cap B$ is relatively compact in $\mathbb{R} \times W^{2,p}(\mathbb{R}^N)$. Without loss of generality, we can suppose that

$$\lambda_n \to \lambda \leq b \quad \text{and} \quad F(\lambda_n,u_n) \to w \in K \quad (\text{strongly in } \mathrm{L}^p).$$

Theorem IV.9 of [2] ensures the existence of $\widetilde{w} \in \mathrm{L}^p(\mathbb{R}^N)$ such that (up to a subsequence)

$$|F(\lambda_n,u_n)| \leq \widetilde{w} \qquad \text{a.e on } \mathbb{R}^N, \forall n \in \mathbb{N}.$$

Setting $V_n := \frac{f(\cdot,u_n)}{u_n} - \lambda$, we can write

$$F(\lambda_n,u_n) = -\triangle u_n + \left(\frac{f(\cdot,u_n)}{u_n} - \lambda\right) u_n = -\triangle u_n + V_n u_n \leq \widetilde{w}.$$

From (H4), there exists $R > 0$ such that: $\inf_{|x|\geq R} V_n(x) > 0$, $\forall \lambda \leq b < \beta$, $\forall n \in \mathbb{N}$.
From Corollary 6.2 there exists $\Psi \in \mathrm{L}^p(\mathbb{R}^N)$ such that: $|u_n| \leq \Psi \; \forall n \in \mathbb{N}$.
Lemma 6.4 and 6.3 imply then that $\{(\lambda_n,u_n)\}$ is relatively compact in $\mathbb{R} \times W^{2,p}(\mathbb{R}^N)$. □

7 – A global bifurcation result

Applying Proposition 4.1, 4.4, 4.5, 6.5 and Theorem 2.7 we deduce immediately

PROPOSITION 7.1. *Let f satisfy hypotheses* (H1) *to* (H4), *and assume the existence of a value $\lambda_0 < \beta$ such that* $\dim \mathrm{Ker}(D_u F_{(\lambda_0,u)})$ *is odd. Then $\mathcal{C}_{\lambda_0}$ has at least one of the following properties:*

1) *$\mathcal{C}_{\lambda_0}$ is unbounded,*
2) *the closure of $\mathcal{C}_{\lambda_0}$ contains a point of the form $(\lambda^\star,0)$ with $\lambda^\star \neq \lambda_0$,*
3) $\sup_{(\lambda,u)\in\mathcal{C}_{\lambda_0}} \lambda = \beta$.

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