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Bounds for products of singular values of a matrix

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RIASSUNTO: *Assegnata una matrice complessa quadrata di ordine n con valori singolari $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A) > 0$, in questo articolo sono presentati limiti inferiori e superiori per $\prod_{i \in I} \sigma_i(A)$, $I \subset \{1, 2, \ldots, n\}$. In particolare sono migliorati alcuni noti limiti inferiori per il più piccolo valore singolare di A.*

ABSTRACT: *Given an order n complex matrix with singular values $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A) > 0$, in this paper some lower and upper bounds for $\prod_{i \in I} \sigma_i(A)$, $I \subset \{1, 2, \ldots, n\}$, are shown. In particular some note lower bounds for the smallest singular value of A are improved.*

1 – Introduction

Let A be a square complex matrix of order n, $n \geq 3$, and let $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A) > 0$ be the singular values of A. In this paper we show some lower and upper bounds for the product of k singular values of A, $k < n$, involving only the determinant of A and the 2$-$norms of the rows of A, that we denote by r_i, $i = 1, \ldots, n$. Since the singular values of A are the same of PA, where P is a permutation matrix in the following we suppose $r_1 \geq r_2 \geq \cdots \geq r_n > 0$, without loss of generality.

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The simplest inequalities are

$$\sigma_1(A)\sigma_2(A)\ldots\sigma_k(A) \le P_k \left(\frac{n}{k}\right)^{\frac{k}{2}} \prod_{i=1}^{k} r_i, \qquad P_k \le 1 \tag{1}$$

and

$$\sigma_{n-k+1}(A)\sigma_{n-k}(A)\ldots\sigma_n(A) \ge Q_k \left(\frac{n-k}{n}\right)^{\frac{n-k}{2}} \frac{|\det A|}{\prod_{i=1}^{n-k} r_i}, \qquad Q_k \ge 1. \tag{2}$$

Paying attention to the rows of A inequality (1) can be modified in the following way

$$\sigma_1(A)\sigma_2(A)\ldots\sigma_k(A) \le R_k \left(\frac{n}{k}\right)^{\frac{k}{2}} \prod_{i=1}^{k} r_i \left(\frac{k+\sum_{i=k+1}^{n} t_i}{n}\right)^{\frac{k}{2}}, \qquad R_k \le 1$$

where

$$t_i = \left(\frac{r_i}{r_k}\right)^2, \qquad i = k+1, \ldots, n.$$

While inequality (2) becomes

$$\begin{aligned} &\sigma_{n-k+1}(A)\sigma_{n-k}(A)\ldots\sigma_n(A) \ge \\ &\ge T_k \left(\frac{n-k}{n}\right)^{\frac{n-k}{2}} \frac{|\det A|}{\left(\frac{n-k+\sum_{i=n-k+1}^{n} t_i}{n}\right)^{\frac{n-k}{2}} \prod_{i=1}^{n-k} r_i}, \end{aligned} \tag{3}$$

with $T_k \ge 1$ and where

$$t_i = \left(\frac{r_i}{r_{n-k}}\right)^2, \qquad i = n-k+1, \ldots, n.$$

The constants P_k, Q_k, R_k and T_k depend on matrix A and an arbitrary parameter ψ, $0 \le \psi \le 1$. If $\psi = 0$ then $P_k = Q_k = R_k = T_k = 1$. For $k = 1$ and $\psi = 0$ (hence $Q_1 = 1$), inequality (2) becomes

$$\sigma_n(A) \ge \left(\frac{n-1}{n}\right)^{\frac{n-1}{2}} \frac{|\det A|}{\prod_{i=1}^{n-1} r_i},$$

that is the some lower bound for the smallest singular values given in [2] (considering only the rows of A). This bound is improved in [4]. We prefer to rewrite the bound given in [4], referring only to rows of A, as

$$\sigma_n(A) \geq \left(\frac{n-1}{n-1+t}\right)^{\frac{n-1}{2}} \left(1+\frac{1}{2}G\right) \frac{|\det A|}{\prod_{i=1}^{n-1} r_i}, \qquad t = t_n,$$

where

$$G = \frac{|\det A|^2}{r_{n-1}^2 \left(\prod_{i=1}^{n-1} r_i^2\right) \left(\frac{n-1+t}{n-1}\right)^n}.$$

The bound is better than bound (3) with $k = 1$ and $\psi = 0$. Moreover with an opportune choise for ψ, inequality (3) gives, for $k = 1$:

$$\sigma_n(A) \geq \left(\frac{n-1}{n-1+t}\right)^{\frac{n-1}{2}} \left(\frac{1-G}{1-\dfrac{n}{n-1}G}\right)^{\frac{n-1}{2}} \frac{|\det A|}{\prod_{i=1}^{n-1} r_i},$$

that improves the bound given in [4].

Moreover in this paper the following bound is proved

$$\sigma_n(A) \geq \min\left(\frac{r_{n-2}}{\gamma}, r_n\right) \sqrt{1-\sqrt{1-S^2}}$$

where

$$S = \frac{|\det A|}{\prod_{i=1}^{n} r_i} \leq 1.$$

and

$$\gamma = \sqrt{1+\sqrt{1-S^2}}.$$

Since $1 \leq \gamma \leq \sqrt{2}$, it is improved also another bound given in [4], which can be written

$$\sigma_n(A) \geq \min\left(\frac{r_{n-2}}{\sqrt{2}}, r_n\right) \sqrt{1-\sqrt{1-S^2}}.$$

The paper is organized as follows. In sections 2, 3, 4 and 5 starting from the arithmetic-geometric mean inequality bounds for the product of k among n positive numbers σ_i, $i = 1, \dots, n$, are derived. These bounds involve $\sum_{i=1}^n \sigma_i$ and $\prod_{i=1}^n \sigma_i$. In section 6 these results are applied to obtain bounnds for the product of k singular values of a matrix A involving the determinant and the Frobenius norm of A. Starting from these bounds and taking a nonsingular arbitrary matrix X new bounds are obtained rewriting the previous for matrix $X^{-1}A$ and using classical inequalities for the singular values of the product of two matrices. In section 7 for some particular choices for matrix X the results written previously are obtained. Finally in section 8 the bounds founds in this paper are compared with those given in [2] and [4].

Throughout the paper we use the following notations:

$$\Pi := \prod_{i=1}^n \sigma_i \qquad \sigma^2 := \sum_{i=1}^n \sigma_i^2$$

$$I \subset \{1, 2, \dots, n\}, I \neq \emptyset, \qquad I \neq \{1, 2, \dots, n\};$$

$\overline{I}$ the complementary set of I;

$$p(I) := \prod_{i \in I} \sigma_i \qquad s^2(I) := \sum_{i \in I} \sigma_i^2$$

$\|A\|_F$ the Frobenius norm of A.

When it is not essential to specify the set, we also write $p = p(I)$, $s = s(I)$ $\overline{p} = p(\overline{I})$ and $\overline{s} = s(\overline{I})$.

2 – The Main Lemma

LEMMA 2.1. *For $p := p(I)$, $s := s(I)$, $card(I) := k$, the following inequalities hold:*

$$p^2 \geq \frac{\Pi^2}{\left[\dfrac{\sigma^2 - k(p^2)^{1/k}}{n-k}\right]^{n-k}} \tag{4}$$

$$s^2 \geq \frac{k(\Pi^2)^{1/k}}{\left[\dfrac{\sigma^2 - s^2}{n-k}\right]^{\frac{n-k}{k}}}, \tag{5}$$

and the equality occurs if and only if $\sigma_1 = \sigma_2 = \cdots = \sigma_n$.

PROOF. From arithmetic-geometric mean inequality, we have

$$\Pi^2 \le \left(\frac{\sigma^2}{n}\right)^n; \qquad p^2 \le \left(\frac{s^2}{k}\right)^k; \qquad \overline{p}^2 \le \left(\frac{\overline{s}^2}{n-k}\right)^{n-k}; \tag{6}$$

with equality if and only if $\sigma_1 = \sigma_2 = \cdots = \sigma_n$. Furthermore, since

$$p\overline{p} = \Pi, \tag{7}$$

using the third inequality in (6), we have

$$p^2 \ge \frac{\Pi^2}{\left[\dfrac{\overline{s}^2}{n-k}\right]^{n-k}} = \frac{\Pi^2}{\left[\dfrac{\sigma^2 - s^2}{n-k}\right]^{n-k}}.$$

Now (4) follows from the second inequality in (6). Analogously, (5) is a consequence of $s^2 + \overline{s}^2 = \sigma^2$, and of the second inequality in (6). ∎

COROLLARY 2.1. *Put*

$$q_k = \frac{n-k}{k}\left(\Pi^2\right)^{\frac{1}{n-k}}$$

and

$$g_k(x) = x^{\frac{n}{k}} - \frac{\sigma^2}{k}x + q_k, \qquad k = 1, \ldots, n-1$$

we have

i) *the inequality in Lemma* 2.1 *is equivalent to*

$$g_k(x) \le 0, \qquad x > 0$$

with $x = (p^2)^{1/(n-k)}$;

ii) *the inequality in Lemma* 2.1 *is equivalent to*

$$g_k(x) \le 0, \qquad x > 0$$

with $x = (s^2/k)^{k/(n-k)}$.

∎

3 – On the function $g_k(x)$

Let $g_k(x)$ be the function defined in Corollary 2.1. Evidently $g_k''(x) > 0$ for $x > 0$, and

$$\widehat{x} = \left(\frac{\sigma^2}{n}\right)^{\frac{k}{n-k}} \tag{8}$$

is the unique point such that

$$g_k(\widehat{x}) = \min_{x \geq 0} g_k(x).$$

Also from the first inequality in (6):

$$g_k(\widehat{x}) = \frac{n-k}{k}\left[(\Pi^2)^{\frac{1}{n-k}} - \left(\frac{\sigma^2}{n}\right)^{\frac{n}{n-k}}\right] \leq 0. \tag{9}$$

Furthermore, $g_k(0) = q_k > 0$ and $\lim_{x\to+\infty} g_k(x) = +\infty$. Hence $g_k(x)$ admits only two zeros r_k', r_k'', with $r_k' \leq r_k''$. Consequently, from Corollary 2.1, the following bounds for p and s are established:

$$(r_k')^{n-k} \leq p^2 \leq (r_k'')^{n-k} \tag{10}$$

$$k(r_k')^{\frac{n-k}{k}} \leq s^2 \leq k(r_k'')^{\frac{n-k}{k}}. \tag{11}$$

LEMMA 3.1. *The following equalities hold*

$$(r_k')^{n-k}(r_{n-k}'')^k = \Pi^2 \tag{12}$$

$$k(r_k')^{\frac{n-k}{k}} + (n-k)(r_{n-k}'')^{\frac{k}{n-k}} = \sigma^2. \tag{13}$$

PROOF. A simple calculus reveals that

$$g_{n-k}(x) = \frac{x^{\frac{n}{n-k}}}{q_k} g_k\left(\frac{(\Pi^2)^{\frac{1}{n-k}}}{x^{\frac{k}{n-k}}}\right),$$

from which (12) follows evaluating $g_{n-k}(x)$ in r_{n-k}''. Now (13) is a consequence of (12) and $g_k(r_k') = 0$. ∎

REMARK. Exchanging the role of k and $n-k$ in (12) we have

$$\left(\frac{r''_k}{r'_k}\right)^{n-k} = \left(\frac{r''_{n-k}}{r'_{n-k}}\right)^{k}. \tag{14}$$

Now assuming

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$$

from (10) and (11) it follows

$$(r'_k)^{n-k} \leq \prod_{j=0}^{k-1} \sigma^2_{n-j} \leq \prod_{j=1}^{k} \sigma^2_j \leq (r''_k)^{n-k} \tag{15}$$

and

$$k(r'_k)^{\frac{n-k}{k}} \leq \sum_{j=0}^{k-1} \sigma^2_{n-j} \leq \sum_{j=1}^{k} \sigma^2_j \leq k(r''_k)^{\frac{n-k}{k}}. \tag{16}$$

REMARK. For $k=1$ and $k=n-1$ (15) furnish, respectively

$$\frac{\sigma_1^2}{\sigma_n^2} \leq \left(\frac{r''_1}{r'_1}\right)^{n-1}$$

and

$$\frac{\sigma_1^2}{\sigma_n^2} \leq \frac{r''_{n-1}}{r'_{n-1}}.$$

Equation (14) shows that these are the same bound.

4 – Lower and upper bounds for the zeros of $g_k(x)$

Let

$$x_0 = t\frac{(\Pi^2)^{\frac{1}{n-k}}}{\sigma^2}, \qquad t \geq 0.$$

Evaluating $g'_k(x)$ in $x = x_0$, we have

$$g'_k(x_0) = \frac{\sigma^2}{k}\left[\frac{t^{\frac{n-k}{k}}}{n^{\frac{n-k}{k}}}\left(\frac{\Pi^2}{\left(\frac{\sigma^2}{n}\right)^n}\right)^{\frac{1}{k}} - 1\right].$$

Therefore

$$g_k'(x_0) \le 0 \qquad \text{for } t \ge 0, \qquad t \le n,$$

and

$$g_k'(x_0) = 0 \qquad \text{if and only if } \sigma_1 = \cdots = \sigma_n, \text{ and } t = n.$$

Putting

$$(17) \qquad \rho = \frac{\Pi^2}{\left(\frac{\sigma^2}{n}\right)^n} \le 1, \qquad \text{and} \qquad \psi = \frac{t}{n}, \qquad 0 \le \psi \le 1.$$

it is

$$g'(x_0) < 0, \qquad \text{if } (\rho, \psi) \ne (1, 1).$$

Supposing $\sigma_i \ne \sigma_j$, for any $i \ne j$, it is $\rho < 1$. In the following we shall assume that $\rho < 1$. Then the following theorem holds.

THEOREM 4.1. *If $\rho < 1$, for every ψ, $0 \le \psi \le 1$, it is*

$$(18) \qquad (n-k)\frac{(\Pi^2)^{\frac{1}{n-k}}}{\sigma^2} C_k(\psi, \rho) \le r_k' \le r_k'' \le \left[\frac{\sigma^2}{kC_{n-k}(\psi, \rho)}\right]^{\frac{k}{n-k}}.$$

where

$$(19) \qquad C_k(\psi, \rho) = \frac{1 - \psi^{\frac{n}{k}} \rho^{\frac{1}{k}}}{1 - \psi^{\frac{n-k}{k}} \rho^{\frac{1}{k}}}, \qquad 0 \le \psi \le 1,\ 0 \le \rho < 1$$

PROOF. Since

$$0 = g_k(r_k') = g_k(x_0) + g_k'(x_0)(r_k' - x_0) + g_k''(\xi)\frac{(r_k' - x_0)^2}{2},$$
$$\xi \in (x_0, r_k') \ \text{ or } \ \xi \in (r_k', x_0).$$

and $g_k''(x) > 0$, for $x > 0$, it follows

$$x_0 - \frac{g_k(x_0)}{g_k'(x_0)} \le r_k'.$$

Thus, computing $g_k(x_0)$, it follows

$$(n-k)\frac{(\Pi^2)^{\frac{1}{n-k}}}{\sigma^2}C_k(\psi,\rho) \le r'_k$$

and, as a consequence of (12) we have

$$r''_k \le \left[\frac{\sigma^2}{kC_{n-k}(\psi,\rho)}\right]^{\frac{k}{n-k}}. \qquad \square$$

5 – On the function $C_k(\psi, \rho)$

For the function

$$C_k(\psi,\rho) = \frac{1-\psi^{\frac{n}{k}}\rho^{\frac{1}{k}}}{1-\psi^{\frac{n-k}{k}}\rho^{\frac{1}{k}}}, \qquad 0 \le \psi \le 1,\ 0 \le \rho < 1$$

we have:

a) $C_k(\psi,\rho) \ge 1$, and $C_k(\psi,\rho) = 1$ if and only if $\psi = 0$ or $\psi = 1$;

b) $C_k(\psi,\rho)$ assumes the maximum value in $\psi_k(\rho)$, unique solution in $[0,1]$ of the equation

$$C_k(\psi,\rho) = \frac{n}{n-k}\psi \tag{20}$$

or equivalently

$$\frac{k}{n-k}\psi^{\frac{n}{k}}\rho^{\frac{1}{k}} - \frac{n}{n-k}\psi + 1 = 0; \tag{21}$$

c) if $\psi_k = \psi_k(\rho)$ is the solution of (21), then

$$\lim_{\rho\to 1}\psi_k(\rho) = 1$$

and

$$\lim_{\rho\to 1}\psi'_k(\rho) = \infty;$$

d) from (20) and c):

$$\lim_{\rho\to 1} C_k(\psi_k,\rho) = \frac{n}{n-k}\lim_{\rho\to 1}\psi_k(\rho) = \frac{n}{n-k}.$$

Moreover we observe that, for n even and $k = n/2$, (21) reduces to

$$\rho^{\frac{2}{n}}\psi^2 - 2\psi + 1 = 0$$

from which

$$\psi_{n/2}(\rho) = \frac{1-\sqrt{1-\rho^{\frac{2}{n}}}}{\rho^{\frac{2}{n}}}. \tag{22}$$

Equation (22) can be considered as an approximation to $\psi_k(\rho)$, solution of (21) for every k.

Moreover it is easy to verify that for every function $\psi = \psi(\rho)$, such that

$$\lim_{\rho\to 1}\psi(\rho) = 1, \qquad \text{and} \qquad \lim_{\rho\to 1}\psi'(\rho) = \infty, \tag{23}$$

it is

$$\lim_{\rho\to 1} C_k(\psi(\rho),\rho) = \frac{n}{n-k}.$$

6 – Application to singular values

Let A be an order n nonsingular complex matrix with singular values

$$\sigma_{\max}(A) = \sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A) = \sigma_{\min}(A).$$

In this case

$$\Pi = |\det A| \qquad \sigma^2 = \|A\|_F^2$$

$$\rho = \frac{|\det A|^2}{\left(\frac{\|A\|_F^2}{n}\right)^n}.$$

Evidently $\rho = 1$ if and only if A is proportional to a unitary matrix. Now, from (15) and Theorem 4.1, we have, for an arbitrary function $\psi = \psi(\rho)$, $0 \leq \rho \leq 1$,

$$\begin{aligned}\left[\frac{n-k}{n}C_k(\psi,\rho)\right]^{\frac{n-k}{2}} \frac{|\det A|}{\left[\frac{\|A\|_F^2}{n}\right]^{\frac{n-k}{2}}} &\leq \sigma_n \ldots \sigma_{n-k+1} \leq \sigma_1\sigma_2\ldots\sigma_k \leq \\ &\leq \left[\frac{n}{kC_{n-k}(\psi,\rho)}\right]^{\frac{k}{2}} \left[\frac{\|A\|_F^2}{n}\right]^{\frac{k}{2}}.\end{aligned} \tag{24}$$

We note that if ψ is chosen to satisfy (23), then (24) is an equality when A is proportional to a unitary matrix.

In [1] it is shown a bound for the condition number of a matrix A involving the determinant and the Frobenius norm of A, as in inequality (24). Finally, to improve the bounds in (24) we remark that if A and B are two order n complex matrices (see [3], p. 193), then

$$\prod_{r=1}^{k} \sigma_{i_r+j_r-r}(AB) \le \prod_{r=1}^{k} \sigma_{i_r}(A) \prod_{r=1}^{k} \sigma_{j_r}(B), \qquad k = 1, \dots, n.$$

In particular, if $i_r = j_r = r$, $r = 1, \dots, k$, then

$$\prod_{r=1}^{k} \sigma_r(AB) \le \prod_{r=1}^{k} \sigma_r(A) \prod_{r=1}^{k} \sigma_r(B),$$

while, for $j_r = n - r + 1$, $i_r = r$, $r = 1, \dots, k$, we have

$$\prod_{r=1}^{k} \sigma_{n-r+1}(AB) \le \prod_{r=1}^{k} \sigma_r(A) \prod_{r=1}^{k} \sigma_{n-r+1}(B).$$

From the previous inequality it follows, for every nonsingular matrix X:

$$\prod_{r=1}^{k} \sigma_r(A) = \prod_{r=1}^{k} \sigma_r(XX^{-1}A) \le \prod_{r=1}^{k} \sigma_r(X) \prod_{r=1}^{k} \sigma_r(X^{-1}A),$$

and

$$\prod_{r=1}^{k} \sigma_{n-r+1}(X^{-1}A) \le \prod_{r=1}^{k} \sigma_r(X^{-1}) \prod_{r=1}^{k} \sigma_{n-r+1}(A).$$

Therefore

$$\sigma_1(A) \dots \sigma_k(A) \le \left[\frac{n}{kC_{n-k}(\psi, \rho(X^{-1}A))} \right]^{\frac{k}{2}} \prod_{r=1}^{k} \sigma_r(X) \left[\frac{\|X^{-1}A\|_F^2}{n} \right]^{\frac{k}{2}}, \tag{25}$$

and

$$\left[\frac{n-k}{n} C_k(\psi, \rho(X^{-1}A)) \right]^{\frac{n-k}{2}} \frac{|\det A|}{\frac{|\det X|}{\prod_{r=n-k+1}^{n} \sigma_r(X)} \left[\frac{\|X^{-1}A\|_F^2}{n} \right]^{\frac{n-k}{2}}} \le \tag{26}$$

$$\le \sigma_n(A) \dots \sigma_{n-k+1}(A)$$

with $0 \le \psi \le 1$.

7 – Some choices for the matrix X

For $i = 1, \dots, n$, let r_i be the euclidean norm of the i-th row of A and, with no loss of generality, we assume

$$r_{\max} = r_1 \geq r_2 \geq \cdots \geq r_n = r_{\min} > 0.$$

Put

$$X = \operatorname{diag}(r_1, \dots, \underbrace{r_{n-k}, \dots, r_{n-k}}_{k+1 \text{ times}})$$

and

$$t_i = \left(\frac{r_i}{r_{n-k}}\right)^2, \qquad i = n-k+1, \dots, n,$$

from (26) it follows

$$\text{(27)} \qquad \left[\frac{n-k}{n} C_k(\psi, \rho)\right]^{\frac{n-k}{2}} \frac{|\det A|}{\left[\frac{n-k+\sum_{i=n-k+1}^n t_i}{n}\right]^{\frac{n-k}{2}} \prod_{i=1}^{n-k} r_i} \leq \\ \leq \sigma_n(A) \dots \sigma_{n-k+1}(A)$$

where:

$$\rho = \rho(X^{-1}A) = \frac{|\det A|^2}{\left[\frac{n-k+\sum_{i=n-k+1}^n t_i}{n}\right]^n r_{n-k}^{2(k+1)} \prod_{i=1}^{n-k-1} r_i^2}.$$

Also, choosing

$$X = \operatorname{diag}(r_1, \dots, \underbrace{r_k, \dots, r_k}_{n-k+1 \text{ times}})$$

and putting

$$t_i = \left(\frac{r_i}{r_k}\right)^2, \qquad i = k+1, \dots, n,$$

from (25) it follows

$$\text{(28)} \quad \sigma_1(A) \dots \sigma_k(A) \leq \left[\frac{n}{k C_{n-k}(\psi, \rho(X^{-1}A))}\right]^{\frac{k}{2}} \prod_{i=1}^{k} r_i \left[\frac{k + \sum_{i=k+1}^n t_i}{n}\right]^{\frac{k}{2}}$$

where

$$\rho(X^{-1}A) = \frac{|\det A|^2}{\left[\dfrac{k+\sum_{i=k+1}^{n} t_i}{n}\right]^n r_k^{2(n-k+1)} \prod_{i=1}^{k-1} r_i^2}.$$

The bounds (27) and (28) can be vary sharp when $t_i << 1$, also choosing $C_k = C_{n-k} = 1$. Hence if

$$X = \mathrm{diag}(r_1, r_2, \ldots, r_n)$$

we obtain

$$\sigma_1(A)\ldots\sigma_k(A) \leq \left[\frac{n}{kC_{n-k}(\psi, \rho(X^{-1}A))}\right]^{\frac{k}{2}} \prod_{i=1}^{k} r_i \tag{29}$$

and

$$\left[\frac{n-k}{n} C_k(\psi, \rho(X^{-1}A))\right]^{\frac{n-k}{2}} \frac{|\det A|}{\prod_{i=1}^{n-k} r_i} \leq \sigma_n(A)\ldots\sigma_{n-k+1}(A) \tag{30}$$

where

$$\rho(X^{-1}A) = \frac{|\det A|^2}{\prod_{i=1}^{n} r_i^2}.$$

Remark. For $k = 1$ and $C_1(\psi, \rho) = 1$, equation (30) gives the same lower bound shown in [2] for $\sigma_{\min}$.

If $\psi = \psi(\rho)$ is chosen in order to satisfy (23), then relations (27), (28), (29) and (30) become equalities when A is unitary.

Equations (23) are satisfied if $\psi(\rho)$ is defined as in (22). After this choise of the function $\psi = \psi(\rho)$, relations (29) and (30) become equalities, when the rows of A are pairwise orthogonal.

In fact in this case $\sigma_i(A) = r_i$, for $i = 1, \ldots, n$, and $\rho(X^{-1}A) = 1$.

8 – An improvement for some known bounds

In [4] authors give the following lower bound for $\sigma_{\min}(A)$:

$$\left(\frac{n-1}{n-1+t}\right)^{\frac{n-1}{2}}\left(1+\frac{1}{2}G\right)\frac{|\det A|}{\prod_{i=1}^{n-1} r_i}\le \sigma_{\min}(A),$$

where

$$G=\frac{|\det A|^2}{r_{n-1}^2\left(\prod_{i=1}^{n-1} r_i^2\right)\left(\frac{n-1+t}{n-1}\right)^n}$$

and $t=t_n=r_n^2/r_{n-1}^2$. Instead, for $k=1$, (27) furnishes

$$\left[\frac{n-1}{n-1+t}\right]^{\frac{n-1}{2}}\left[C_1(\psi,\rho(X^{-1}A))\right]^{\frac{n-1}{2}}\frac{|\det A|}{\prod_{i=1}^{n-1} r_i}\le \sigma_{\min}(A)$$

where

$$\rho(X^{-1}A)=\left(\frac{n}{n-1}\right)^n G.$$

Putting $\psi=(n-1)/n$

$$\left[C_1\left(\frac{n-1}{n},\rho(X^{-1}A)\right)\right]^{\frac{n-1}{2}}=\left(\frac{1-G}{1-\dfrac{n}{n-1}G}\right)^{\frac{n-1}{2}}>$$

$$>\left(1+\frac{G}{n-1}\right)^{\frac{n-1}{2}}>1+\frac{1}{2}G,\quad \text{for } n\ge 3.$$

Again in [4] the following further bound for $\sigma_{\min}(A)$ is proved:

$$\min\left(\frac{r_{n-2}}{\sqrt{2}},r_n\right)\sqrt{1-\sqrt{1-S^2}}\le \sigma_{\min}(A) \tag{31}$$

where

$$S=\frac{|\det A|}{\prod_{i=1}^{n} r_i}\le 1.$$

Now we show that choosing

$$X = \mathrm{diag}(r_1, \dots, r_{n-2}, \gamma r_{n-1}, \gamma r_n), \qquad \gamma \ge 1$$

the bound (31) can be slightly improved.
In fact, since $\sigma_n(X) = \min\{r_{n-2}, \gamma r_n\}$, from (26):

$$\left[\frac{n-1}{n-1+b} C_1(\psi, \rho(X^{-1}A))\right]^{\frac{n-1}{2}} \min\left(\frac{r_{n-2}}{\gamma}, r_n\right) S \frac{1}{\gamma} \le \sigma_{\min}(A)$$

where:

$$\rho(X^{-1}A) = \frac{S^2}{\gamma^4 \left[\frac{n-1+b}{n}\right]^n}$$

and

$$b = \frac{2}{\gamma^2} - 1.$$

Imposing that

$$\frac{S^2}{\gamma^4} = b, \tag{32}$$

we have

$$C_1 = \frac{1 - \psi^n \frac{b}{\left(\frac{n-1+b}{n}\right)^n}}{1 - \psi^{n-1} \frac{b}{\left(\frac{n-1+b}{n}\right)^n}}.$$

From (21) it is easy to verify that C_1 assumes the maximum value in

$$\psi = \frac{n-1+b}{n}$$

for which (see (20)):

$$C_1 = \frac{n-1+b}{n-1}.$$

Now, from (32)

$$\gamma^2 = 1 + \sqrt{1 - S^2}$$

and therefore $1 \le \gamma \le \sqrt{2}$.

Finally

$$\sigma_{\min}(A) \geq \min\left(\frac{r_{n-2}}{\gamma}, r_n\right)\frac{S}{\gamma} = \min\left(\frac{r_{n-2}}{\gamma}, r_n\right)\sqrt{1-\sqrt{1-S^2}}.$$

REMARK. The bounds given in [2] and [4] involve also the column of A. But the bounds exhibited for $\prod_{i\in I}\sigma_i(A)$ and in particular for $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ involving the rows of A can be rewritten involving the columns of A. In fact it is enough to observe that $\sigma_i(A) = \sigma_1(A^*)$, for all i.

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