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Observability and controllability of Maxwell's equations

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RIASSUNTO: *Si considera un sistema di equazioni di Maxwell in un dominio limitato $\Omega \subset \mathbb{R}^3$ con frontiera regolare e si provano alcuni risultati di osservabilità e controllabilità interna e al bordo. Le diseguaglianze di osservabilità sono ottenute con la tecnica dei moltiplicatori e i risultati di controllabilità con il metodo HUM introdotto da J. L. Lions.*

ABSTRACT: *We consider a system of Maxwell's equations in a bounded domain $\Omega \subset \mathbb{R}^3$, with smooth boundary Γ. We prove some results of boundary and internal observability and controllability. The observability inequalities are obtained by using multiplier techniques and the controllability results by the Hilbert Uniqueness Method of J. L. Lions.*

1 – Introduction

Let Ω be an open bounded domain in $\mathbb{R}^3$ having boundary Γ of class C^1. In this paper we study, under suitable boundary conditions, the boundary and internal observability and controllability of Maxwell's equations:

$$D' - \operatorname{curl}(\mu B) = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \tag{1.1}$$

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$$(1.2) \qquad B' + \mathrm{curl}(\lambda D) = 0 \qquad \text{in} \quad \Omega \times (0, +\infty)$$

$$(1.3) \qquad \mathrm{div}\, D = \mathrm{div}\, B = 0 \qquad \text{in} \quad \Omega \times (0, +\infty)$$

$$(1.4) \qquad D(0) = D^0 \quad \text{and} \quad B(0) = B^0 \quad \text{in} \quad \Omega$$

where D, B are three-dimensional vector-valued functions of t, $x = (x_1, x_2, x_3)$; $\mu = \mu(x)$, $\lambda = \lambda(x)$ are scalar functions in $C^1(\overline{\Omega})$ bounded below by a positive constant and verifying suitable hypotheses; D^0, B^0 are the initial data in a suitable space.

If Γ is perfectly conducting, then the solution satisfies the boundary conditions

$$(1.5) \qquad \nu \times D = 0 \quad \text{and} \quad \nu \cdot B = 0 \quad \text{on} \quad \Gamma \times (0, +\infty)\ ,$$

where ν denotes the outward unit normal vector to Γ.

We define the energy of the solutions by the formula

$$(1.6) \qquad \mathcal{E}(t) := \frac{1}{2} \int_\Omega (\lambda |D(t)|^2 + \mu |B(t)|^2) dx\ .$$

Let us introduce the Hilbert space

$$(1.7) \qquad \mathcal{H}_0 = \{(\varphi, \psi) \in (L^2(\Omega))^6 : \mathrm{div}\, \varphi = \mathrm{div}\, \psi = 0,\ \nu \cdot \psi|_\Gamma = 0\}\ ,$$

equipped with the scalar product

$$\langle (\varphi_1, \psi_1), (\varphi_2, \psi_2) \rangle_0 = \int_\Omega (\lambda \varphi_1 \varphi_2 + \mu \psi_1 \psi_2) dx\ , \quad (\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \mathcal{H}_0\ .$$

Then, for every initial data $(D_0, B_0) \in \mathcal{H}_0$, the problem (1.1)-(1.4), (1.5) has a unique weak solution

$$(D, B) \in C(\mathbb{R}, \mathcal{H}_0)$$

whose energy is conserved. Hence we write simply $\mathcal{E}$ instead of $\mathcal{E}(t)$ in this case. Moreover, if (D_0, B_0) belongs to

$$(1.8) \qquad \mathcal{H}_1 = \{\ (\varphi, \psi) \in (H^1(\Omega))^6 \cap \mathcal{H}_0\ :\ \nu \times \varphi|_\Gamma = 0\ \}\ ,$$

then we have a strong solution

$$(D, B) \in C^1(\mathbb{R}, \mathcal{H}_0) \cap C(\mathbb{R}, \mathcal{H}_1)\ ,$$

satisfying the inequality

$$\|(D,B)\|_{L^\infty(\mathbb{R},\mathcal{H}_1)} \le c\|(D^0,B^0)\|_{\mathcal{H}_1} \tag{1.9}$$

for some constant $c > 0$. We refer to [2] or to [3] for a proof.

We recall that Ω is star-shaped with respect to the origin if

$$x \cdot \nu(x) \ge 0 \quad \text{for all} \quad x \in \Gamma\ . \tag{1.10}$$

First we study the boundary observability and controllability of (1.1)-(1.4) under the assumption that Ω is star-shaped with respect to the origin.

The exact boundary controllability of Maxwell's equations has been studied by Russell [17] for a circular cylindrical region, by Kime [5] for a spherical region, by Lagnese [11] and Kapitonov [4] for more general regions. Russell and Kime have solved the problem by the moment problem method. Lagnese [11] studied the exact boundary controllability by using the Hilbert Uniqueness Method introduced by J. L. Lions [12], while Kapitonov solved the problem using the energy decay of the solution of a system with Leontovich's boundary condition. Some other results of exact boundary controllability for Maxwell's equations have been proved by Komornik [7] (see also [6], [8] and [9]) and, using a different approach, by Nalin [15] and Phung [16]. In these papers the functions λ, μ are assumed to be constant, while here we consider λ, μ variable in the space, that is, we consider Maxwell's equations in a heterogeneus medium. We use the multiplier method to prove the observability inequality generalizing analogous inequality in Komornik [7]. We follow Lagnese [11] to prove the exact boundary controllability by using the Hilbert Uniqueness Method.

Next we study the internal observability and controllability of Maxwell's equations.

We define the set

$$\mathcal{N}_\varepsilon(V) = \bigcup_{x\in V}\{y \in \mathbb{R}^3 : |y-x| < \varepsilon\} \tag{1.11}$$

for any $V \subset \mathbb{R}^3$ and for any positive number $\varepsilon > 0$.

Following K. Liu [13] we assume that there exist domains $\Omega_j \subset \Omega$ with Lipschitz boundary $\Gamma_j, j = 1, \dots, N$ such that

$$\Omega_i \cap \Omega_j = \emptyset,\ 1 \le i < j \le N\ , \tag{1.12}$$

and we consider the set

$$G_\varepsilon = \Omega \cap \mathcal{N}_\varepsilon\Big(\Big(\Omega \setminus \bigcup_{j=1}^{N} \Omega_j\Big) \cup \Gamma\Big)\ . \tag{1.13}$$

In particular, if $N = 1$ and $\Omega_1 = \Omega$, then G_ε is the neighborhood $\mathcal{N}_\varepsilon(\Gamma)$ of the boundary Γ of Ω.

We prove an observability inequality by using the piecewise multiplier method introduced by Liu and Yamamoto [14] to study the exact internal controllability of wave equations and, always following Lagnese [11], we prove the exact controllability when the control acts on G_ε. Our theorems generalize earlier results of Qi Zhou [19] and Xu Zhang [18], by considering λ, μ variable in Ω while they have assumed $\lambda \equiv \mu \equiv 1$. The approach of Zhou is based on Liu's "frequency domain method" [13], and does not give an explicit estimate of the time T necessary for the controllability.

Finally, we give another result of observability and controllability in the spaces introduced by Ladyzhenskaya and Solonnikov [10] and used in [11] by Lagnese. In the last section we consider Maxwell's equations in a homogeneous medium, that is, we assume that λ, μ are constant. Further, we assume that the domain Ω is star-shaped.

Due to the finite speed of propagation, the boundary and internal controllability for the system of Maxwell's equations is possible only if the time T is large enough. In this paper we give explicit estimates of the time T required to have osservability and controllability. These estimates are optimal when the functions λ and μ are constant.

2 – Boundary observability and controllability

In this section we obtain a result of boundary observability and controllability under suitable hypotheses on λ, μ. Let us denote

$$m(x) \equiv x, \quad x \in \Omega, \tag{2.1}$$

and set

$$R = \sup_{x \in \Omega} |x| \ . \tag{2.2}$$

We prove the following theorem.

THEOREM 2.1. *Assume* (1.10) *and consider* $\lambda, \mu \in C^1(\overline{\Omega})$ *satisfying the following hypotheses:*

$$\lambda(x), \mu(x) > 0, \qquad \forall\, x \in \overline{\Omega} \tag{2.3}$$

$$L = \inf_{\Omega} \sqrt{\lambda(x)\mu(x)} > 0 \ , \tag{2.4}$$

$$\mu - m \cdot \nabla\mu \geq c_0\mu, \qquad \forall\, x \in \Omega \tag{2.5}$$

$$\lambda - m \cdot \nabla\lambda \geq c_0\lambda, \qquad \forall\, x \in \Omega \tag{2.6}$$

where L *and* c_0 *are strictly positive constants. Set*

$$T_0 = \frac{2R}{Lc_0}, \tag{2.7}$$

then for every time $T > T_0$ *and for every pair* $(D^0, B^0) \in \mathcal{H}_1$ *the solution of* (1.1)-(1.4), (1.5) *satisfies the inequality*

$$\int_0^T \int_\Gamma |B|^2 d\Gamma dt \geq c\mathcal{E} \quad , \tag{2.8}$$

where c *is a positive constant independent on the initial data.*

REMARK 2.1. If λ and μ are constant then, choosing $c_0 = 1$, the time T_0 is optimal, that is we cannot have observability inequality if $T < T_0$ according to the finite speed propagation. In the case of $\lambda \equiv \mu \equiv 1$ the observability inequality holds for every $T > 2R$.

REMARK 2.2. The hypotheses (2.5) and (2.6) are satisfied for example if the functions λ, μ, verify the conditions $x \cdot \nabla\mu(x) \leq 0$ and $x \cdot \nabla\lambda(x) \leq 0$ in Ω. In particular this occurs if $\lambda = \lambda(|x|)$ and $\mu = \mu(|x|)$ are radially decreasing.

To prove Theorem 2.1 we need a preliminary lemma.

LEMMA 2.2. *Assume that* $(D^0, B^0) \in \mathcal{H}_1, q \in (C^1(\overline{\Omega}))^3$ *and let* $T > 0$. *Then the solution of* (1.1)-(1.4), (1.5) *satisfies the identity*

$$\begin{aligned}\left[2\int_\Omega (D\times B)\cdot q dx\right]_0^T = & \int_0^T\int_\Gamma (\lambda|D|^2 - \mu|B|^2)(q\cdot\nu)d\Gamma\, dt + \\ & + \int_0^T\int_\Omega \{(\operatorname{div} q)(\lambda|D|^2 + \mu|B|^2) + \\ & - 2\sum_{i,k=1}^3 q_{i,k}(\lambda D_i D_k + \mu B_i B_k)\}dx\, dt + \\ & + \int_0^T\int_\Omega \{-(q\cdot\nabla\lambda)|D|^2 - (q\cdot\nabla\mu)|B|^2\}dx\, dt\,,\end{aligned} \tag{2.9}$$

where D_i, B_i, q_i, *for* $i = 1, 2, 3$ *are the scalar components of* D, B, q, *and* $q_{i,k} = \frac{\partial q_i}{\partial x_k}$ *for* $i, k = 1, 2, 3$.

PROOF OF LEMMA 2.2. The identity (2.9) is obtained by the multiplier method (see [8], [12]) and generalizes the analogous formula in [6], [7]. Writing for brevity $D_{i,k} = \frac{\partial D_i}{\partial x_k}$, $B_{i,k} = \frac{\partial B_i}{\partial x_k}$ for $i, k = 1, 2, 3$ $\lambda_{,k} = \frac{\partial \lambda}{\partial x_k}$ and $\mu_{,k} = \frac{\partial \mu}{\partial x_k}$ for $k = 1, 2, 3$, we have

$$D_1' = \mu_{,2}B_3 - \mu_{,3}B_2 + \mu B_{3,2} - \mu B_{2,3} \quad \text{in} \quad \Omega\times(0,+\infty)\,, \tag{2.10}$$
$$D_2' = \mu_{,3}B_1 - \mu_{,1}B_3 + \mu B_{1,3} - \mu B_{3,1} \quad \text{in} \quad \Omega\times(0,+\infty)\,, \tag{2.11}$$
$$D_3' = \mu_{,1}B_2 - \mu_{,2}B_1 + \mu B_{2,1} - \mu B_{1,2} \quad \text{in} \quad \Omega\times(0,+\infty)\,, \tag{2.12}$$
$$B_1' = \lambda_{,3}D_2 - \lambda_{,2}D_3 + \lambda D_{2,3} - \lambda D_{3,2} \quad \text{in} \quad \Omega\times(0,+\infty)\,, \tag{2.13}$$
$$B_2' = \lambda_{,1}D_3 - \lambda_{,3}D_1 + \lambda D_{3,1} - \lambda D_{1,3} \quad \text{in} \quad \Omega\times(0,+\infty)\,, \tag{2.14}$$
$$B_3' = \lambda_{,2}D_1 - \lambda_{,1}D_2 + \lambda D_{1,2} - \lambda D_{2,1} \quad \text{in} \quad \Omega\times(0,+\infty)\,. \tag{2.15}$$

Therefore

$$\begin{aligned}2(D_1B_2q_3)' = & \,2(\lambda_{,1}D_3 - \lambda_{,3}D_1)D_1q_3 + 2(\mu_{,2}B_3 - \mu_{,3}B_2)B_2q_3 + \\ & + 2(\lambda D_{3,1} - \lambda D_{1,3})D_1q_3 + 2(\mu B_{3,2} - \mu B_{2,3})B_2q_3 = \\ = & \,2q_3(\lambda_{,1}D_3D_1 - \lambda_{,3}D_1^2) + 2q_3(\mu_{,2}B_3B_2 - \mu_{,3}B_2^2) + \\ & + 2q_3\lambda D_{3,1}D_1 - q_3\lambda(D_1^2)_3 + 2q_3\mu B_{3,2}B_2 - q_3\mu(B_2^2)_3\ ,\end{aligned}$$

and analogously

$$\begin{aligned}2(D_2B_1q_3)' &= 2(\lambda_{,3}D_2-\lambda_{,2}D_3)D_2q_3+2(\mu_{,3}B_1-\mu_{,1}B_3)B_1q_3+\\&\quad+2(\lambda D_{2,3}-\lambda D_{3,2})D_2q_3+2(\mu B_{1,3}-\mu B_{3,1})B_1q_3=\\&=-2q_3(\lambda_{,2}D_3D_2-\lambda_{,3}D_2^2)-2q_3(\mu_{,1}B_3B_1-\mu_{,3}B_1^2)+\\&\quad-2q_3\lambda D_{3,2}D_2+q_3\lambda(D_2^2)_3-2q_3\mu B_{3,1}B_1+q_3\mu(B_1^2)_3\ .\end{aligned}$$

If we integrate by parts their difference in $\Omega\times(0,T)$, denoting by ν_1,ν_2,ν_3 the scalar components of ν, we obtain

$$\begin{aligned}&2\int_0^T\int_\Omega((D_1B_2q_3)'-(D_2B_1q_3)')dx\,dt=\\&=\int_0^T\int_\Gamma(2q_3\mu B_3B_2\nu_2-q_3\mu B_2^2\nu_3+2q_3\lambda D_1D_3\nu_1-q_3\lambda D_1^2\nu_3+\\&\quad-q_3\mu B_1^2\nu_3+2q_3\mu B_1B_3\nu_1-q_3\lambda D_2^2\nu_3+2q_3\lambda D_2D_3q_3\nu_2)d\Gamma\,dt+\\&\quad+\int_0^T\int_\Omega(-2q_3\mu B_3B_{2,2}-2q_{3,2}\mu B_2B_3+q_{3,3}\mu B_2^2+\\&\quad-2q_3\lambda D_{1,1}D_3-2q_{3,1}\lambda D_1D_3+q_{3,3}\lambda D_1^2+\\&\quad+q_{3,3}\mu B_1^2-2q_3\mu B_3B_{1,1}-2q_{3,1}\mu B_1B_3+\\&\quad+q_{3,3}\lambda D_2^2-2q_3\lambda D_{2,2}D_3-2q_{3,2}\lambda D_2D_3)dx\,dt+\\&\quad+\int_0^T\int_\Omega(-q_3\mu_{,3}B_2^2-q_3\lambda_{,3}D_1^2-q_3\mu_{,3}B_1^2-q_3\lambda_{,3}D_2^2)dx\,dt\ ,\end{aligned}$$

which may be rewritten as follows:

$$\begin{aligned}2\Big[\int_\Omega(D_1B_2q_3-D_2B_1q_3)dx\Big]_0^T&=\int_0^T\int_\Gamma\{-q_3\nu_3(\mu B_1^2+\mu B_2^2+\lambda D_1^2+\lambda D_2^2)+\\&\quad+2q_3\nu_2(\mu B_2B_3+\lambda D_2D_3)+2q_3\nu_1(\mu B_1B_3+\lambda D_1D_3)\}d\Gamma\,dt+\\&\quad+\int_0^T\int_\Omega\{q_{3,3}(\mu B_1^2+\mu B_2^2+\lambda D_1^2+\lambda D_2^2)+\\&\quad-2q_3B_3(\mu B_{1,1}+\mu B_{2,2})-2q_3D_3(\lambda D_{1,1}+\lambda D_{2,2})+\\&\quad-2q_{3,2}(\lambda D_2D_3+\mu B_2B_3)-2q_{3,1}(\lambda D_1D_3+\mu B_1B_3)\}dx\,dt+\\&\quad+\int_0^T\int_\Omega\{-q_3\mu_{,3}(B_1^2+B_2^2)-q_3\lambda_{,3}(D_1^2+D_2^2)\}dx\,dt\ .\end{aligned}$$

We observe that

$$\int_\Omega -2q_3\mu B_3(B_{1,1}+B_{2,2})dx = \int_\Omega 2q_3\mu B_3 B_{3,3}dx = \int_\Omega q_3\mu(B_3^2)_3 dx = \\ = \int_\Gamma q_3\mu B_3^2\nu_3 d\Gamma - \int_\Omega q_{3,3}\mu B_3^2 dx - \int_\Omega q_3\mu_{,3}B_3^2 dx$$

and analogously

$$\int_\Omega -2q_3\lambda D_3(D_{1,1}+D_{2,2})dx = \int_\Omega 2q_3\lambda D_3 D_{3,3}dx = \int_\Omega q_3\lambda(D_3^2)_3 dx = \\ = \int_\Gamma q_3\lambda D_3^2\nu_3 d\Gamma - \int_\Omega q_{3,3}\lambda D_3^2 dx - \int_\Omega q_3\lambda_{,3}D_3^2 dx\,.$$

Therefore

$$\begin{aligned}\Big[2\int_\Omega (D_1B_2q_3 - D_2B_1q_3)dx\Big]_0^T = \\ &= \int_0^T \int_\Gamma \{-q_3\nu_3(\mu B_1^2+\mu B_2^2-\mu B_3^2+\lambda D_1^2+\lambda D_2^2-\lambda D_3^2)+ \\ &\quad + 2q_3\nu_2(\mu B_2B_3+\lambda D_2D_3)+2q_3\nu_1(\mu B_1B_3+\lambda D_1D_3)\}d\Gamma\, dt+ \\ &\quad + \int_0^T \int_\Omega \{q_{3,3}(\mu B_1^2+\mu B_2^2-\mu B_3^2+\lambda D_1^2+\lambda D_2^2-\lambda D_3^2)+ \\ &\quad - 2q_{3,2}(\lambda D_2D_3+\mu B_2B_3)-2q_{3,1}(\lambda D_1D_3+\mu B_1B_3)\}dx\, dt+ \\ &\quad + \int_0^T \int_\Omega (-q_3\mu_{,3}|B|^2 - q_3\lambda_{,3}|D|^2)dx\, dt\ .\end{aligned}$$

By permutation of the indices $1, 2, 3$ we obtain two analogous identities, and summing the three identities we obtain

$$\begin{aligned}\Big[2\int_\Omega (D\times B)\cdot q\Big]_0^T = &\int_0^T \int_\Gamma \{-(\lambda|D|^2+\mu|B|^2)(q\cdot\nu)+ \\ &+ 2\mu(q\cdot B)(\nu\cdot B)+2\lambda(q\cdot D)(\nu\cdot D)\}d\Gamma\, dt+ \\ &+ \int_0^T \int_\Omega \{(\operatorname{div} q)(\lambda|D|^2+\mu|B|^2)+ \\ &- 2\sum_{i,k=1}^3 (\lambda D_iD_k+\mu B_iB_k)\}dx\, dt+ \\ &+ \int_0^T \int_\Omega \{-(q\cdot\nabla\lambda)|D|^2-(q\cdot\nabla\mu)|B|^2\}dx\, dt\,.\end{aligned} \tag{2.16}$$

Now we can use the boundary condition (1.5) to obtain the formula (2.9). ∎

Remark 2.3. We observe that the formula (2.16) is independent on the boundary condition (1.5) and on the initial condition (1.4).

Proof of Theorem 2.1. If in (2.9) we assume $q(x) \equiv m(x)$, then we obtain

$$
\begin{aligned}
&c_0 \int_0^T \int_\Omega (\mu|B|^2 + \lambda|D|^2) dx\, dt \leq \\
&\leq \int_0^T \int_\Gamma (m\cdot\nu)(\mu|B|^2 - \lambda|D|^2) d\Gamma\, dt - \left[2\int_\Omega (D\times B)\cdot m dx \right]_0^T .
\end{aligned} \tag{2.17}
$$

We observe that

$$
\left[2\int_\Omega (D\times B)\cdot m dx \right]_0^T \leq \frac{2R}{L} \int_\Omega (\lambda|D|^2 + \mu|B|^2) dx = \frac{4R}{L}\mathcal{E} , \tag{2.18}
$$

hence

$$
2c_0 T\mathcal{E} - \frac{4R}{L}\mathcal{E} \leq \int_0^T \int_\Gamma (m\cdot\nu)(\mu|B|^2 - \lambda|D|^2) d\Gamma\, dt .
$$

Using the condition (1.10) and recalling definition (2.7) we have

$$
\int_0^T \int_\Gamma \mu|B|^2 d\Gamma dt \geq c\mathcal{E} \tag{2.19}
$$

for every $T > T_0$ and therefore (2.8) follows because μ has a positive infimum. ∎

Now, applying the HUM of J.L. Lions (see [12]) we deduce from Theorem 2.1 an exact controllability result concerning the system

$$
D' - \operatorname{curl}(\mu B) = B' + \operatorname{curl}(\lambda D) = 0 \quad \text{in} \quad \Omega\times(0,+\infty) \tag{2.20}
$$

$$
\operatorname{div} D = \operatorname{div} B = 0 \quad \text{in} \quad \Omega\times(0,+\infty) \tag{2.21}
$$

$$
\nu\times B = J \quad \text{on} \quad \Gamma\times(0,+\infty) \tag{2.22}
$$

$$
D(0) = D^0 \quad \text{and} \quad B(0) = B^0 \quad \text{in} \quad \Omega . \tag{2.23}
$$

Let $(\varphi^0, \psi^0) \in \mathcal{H}_1$ and let (φ, ψ) be the solution of (1.1)-(1.4), (1.5). From (1.9) and the standard trace theorem

$$
H^1(\Omega) \hookrightarrow L^2(\Gamma)
$$

it follows that

$$\Big(\int_0^T \int_\Gamma \mu\lambda|\psi|^2 d\Gamma\, dt\Big)^{\frac{1}{2}} \leq c\|(\varphi^0,\psi^0)\|_{\mathcal{H}_1}\ . \tag{2.24}$$

From the previous theorem we have that

$$\|(\varphi^0,\psi^0)\|_{\mathcal{F}_1} = \Big(\int_0^T \int_\Gamma \mu\lambda|\psi|^2 d\Gamma\, dt\Big)^{\frac{1}{2}} \tag{2.25}$$

is a norm on $\mathcal{H}_1$. Let $\mathcal{F}_1$ be the completion of $\mathcal{H}_1$ with respect to this norm and let $\mathcal{F}_1'$ be its dual space with respect to the scalar product $\langle\cdot,\cdot\rangle_0$. We prove the following result.

THEOREM 2.3. *Assume* (1.10) *and let* T_0, λ, μ *be as in Theorem* 2.1. *Then for every time* $T > T_0$ *and for every pair* $(B^0, -D^0) \in \mathcal{F}_1'$ *there exists a function* $J \in L^2(0,T;L^2(\Gamma)^3)$, *such that*

$$J\cdot\nu = 0 \quad \text{a. e. on} \quad \Gamma\times(0,T)$$

and the solution of (2.20)-(2.23) *with initial data* (D^0,B^0) *satisfies*

$$D(T) = 0 \quad \textit{and} \quad B(T) = 0 \quad \textit{in} \quad \Omega\ . \tag{2.26}$$

PROOF OF THEOREM 2.3. Let $T > T_0$ be a fixed time. Let $(\varphi^0,\psi^0) \in \mathcal{F}_1$ and let (φ,ψ) be the solution of the system

$$\varphi' - \operatorname{curl}(\lambda\psi) = \psi' + \operatorname{curl}(\mu\varphi) = 0 \quad \text{in} \quad \Omega\times(0,T)\ , \tag{2.27}$$

$$\operatorname{div}\varphi = \operatorname{div}\psi = 0 \quad \text{in} \quad \Omega\times(0,T)\ , \tag{2.28}$$

$$\nu\times\varphi = 0 \quad \text{and} \quad \nu\cdot\psi = 0 \quad \text{on} \quad \Gamma\times(0,T)\ , \tag{2.29}$$

$$\varphi(0) = \varphi^0 \quad \text{and} \quad \psi(0) = \psi^0 \quad \text{in} \quad \Omega\ . \tag{2.30}$$

Since λ and μ verify the same properties, it is clear that the result stated for system (1.1)-(1.4), (1.5) also applies to problem (2.27)-(2.30).

Now we proceed formally. Everything will be justified in Remark 2.4 below. Consider the problem

$$
\begin{aligned}
&(2.31) \qquad D' - \mathrm{curl}(\mu B) = B' + \mathrm{curl}(\lambda D) = 0 && \text{in} \quad \Omega \times (0,T)\ ,\\
&(2.32) \qquad \mathrm{div}\, D = \mathrm{div}\, B = 0 && \text{in} \quad \Omega \times (0,T)\ ,\\
&(2.33) \qquad \nu \times B = \psi|_{\Gamma\times(0,T)} && \text{on} \quad \Gamma \times (0,T)\ ,\\
&(2.34) \qquad D(T) = B(T) = 0 && \text{in} \quad \Omega\ .
\end{aligned}
$$

We will see below that system (2.31)-(2.34) has a unique solution in a weak sense. Let us consider the expression

$$
(2.35) \qquad \int_0^T \int_\Omega \{\lambda(D' - \mathrm{curl}(\mu B))\psi - \mu(B' + \mathrm{curl}(\lambda D))\varphi\} dx\, dt = 0\ .
$$

After integrations by parts from (2.35) we obtain

$$
(2.36) \qquad
\begin{aligned}
&-\int_\Omega \lambda D(0)\psi(0)dx + \int_\Omega \mu B(0)\varphi(0)dx - \int_0^T \times\\
&\qquad\times \int_\Gamma \{(\mu B \times \lambda\psi)\cdot\nu + (\lambda D \times \mu\varphi)\cdot\nu\} d\Gamma dt = 0
\end{aligned}
$$

where we used (2.31). From (2.36) and boundary conditions (2.29), (2.33) we obtain

$$
-\int_\Omega \lambda D(0)\psi(0)dx + \int_\Omega \mu B(0)\varphi(0)dx = \int_0^T \int_\Gamma \mu\lambda|\psi|^2 d\Gamma dt\ ,
$$

that we rewrite as

$$
(2.37) \qquad \langle (B(0), -D(0)), (\varphi^0, \psi^0)\rangle_{\mathcal{F}_1', \mathcal{F}_1} = \int_0^T \int_\Gamma \mu\lambda|\psi|^2 d\Gamma\, dt\ .
$$

Introducing the linear operator $\Lambda : (\varphi^0, \psi^0) \to (B(0), -D(0))$, (2.37) becomes

$$
(2.38) \qquad \langle \Lambda(\varphi^0, \psi^0), (\varphi^0, \psi^0)\rangle_{\mathcal{F}_1', \mathcal{F}_1} = \int_0^T \int_\Gamma \mu\lambda|\psi|^2 d\Gamma\, dt\ .
$$

Then from (2.38), using (2.8), it follows that $\Lambda : \mathcal{F}_1 \to \mathcal{F}_1'$ is an isomorphism. Consequently for every pair $(B^0, -D^0)$ in $\mathcal{F}_1'$ if we choose

$$(\varphi^0, \psi^0) = \Lambda^{-1}\left(B^0, -D^0\right),$$

then the unique solution of (2.20)-(2.23) with initial data (D^0, B^0) satisfies (2.26). This completes the Proof of Theorem 2.3. □

REMARK 2.4. We need to explain the sense in which (2.31)-(2.34) and (2.20)-(2.23) are to be understood using the transposition method. We proceed as LAGNESE in [11]. Let us consider the problem (2.20)-(2.23) with $J \in L^2((0,T); L^2(\Gamma)^3)$ and initial data such that $(B^0, -D^0) \in \mathcal{H}_1'$. Let $(\varphi^0, \psi^0) \in \mathcal{H}_1$ and let (φ, ψ) be the solution of (2.27)-(2.30). Let $t > 0$ be fixed. Consider the expression

$$\int_0^t \int_\Omega \{\lambda(D' - \operatorname{curl}(\mu B))\psi - \mu(B' + \operatorname{curl}(\lambda D))\varphi\} dx\, dt = 0 \ .$$

Integrating by parts we obtain

$$\begin{aligned} \int_\Omega \lambda D(t)\psi(t) dx - \int_\Omega \mu B(t)\varphi(t) dx = \int_\Omega \lambda D^0 \psi^0 dx + \\ - \int_\Omega \mu B^0 \varphi^0 dx + \int_0^t \int_\Gamma \mu\lambda J \psi d\Gamma ds \ . \end{aligned} \tag{2.39}$$

Formula (2.39) is the definition of the solution of the system (2.20)-(2.23). In fact,

$$\begin{aligned} &\left| \langle (B^0, -D^0), (\varphi^0, \psi^0) \rangle_{\mathcal{H}_1', \mathcal{H}_1} - \int_0^t \mu\lambda J \psi \, d\Gamma ds \right| \le \\ &\le \|(B^0, -D^0)\|_{\mathcal{H}_1'} \|(\varphi^0, \psi^0)\|_{\mathcal{H}_1} + \\ &\qquad + c \|J\|_{L^2((0,T);L^2(\Gamma)^3)} \|\psi\|_{L^2((0,T);L^2(\Gamma)^3)} \le \\ &\le \|(B^0, -D^0)\|_{\mathcal{H}_1'} \|(\varphi^0, \psi^0)\|_{\mathcal{H}_1} + \\ &\qquad + c' \|J\|_{L^2((0,T);L^2(\Gamma)^3)} \|(\varphi^0, \psi^0)\|_{\mathcal{H}_1} \ , \end{aligned} \tag{2.40}$$

for some constants $c\ , c' > 0$. Since the map $(\varphi^0, \psi^0) \to (\varphi(t), \psi(t))$ is an isomorphism from $\mathcal{H}_1$ to $\mathcal{H}_1$ it follows from (2.40) that there exists a unique pair $(B, -D) \in C([0,T]; \mathcal{H}_1')$ that satisfies (2.39). This pair is, by definition, the solution of (2.20)-(2.23). Now let $T > T_0$ and let $(B^0, -D^0) \in \mathcal{F}_1'$. From (2.39) with $t = T$ we obtain

$$|\langle (B(T), -D(T)), (\varphi(T), \psi(T)) \rangle_{\mathcal{F}_1', \mathcal{F}_1}| \leq$$
$$\leq \left[\|(B^0, -D^0)\|_{\mathcal{F}_1'} + \|J\|_{L^2((0,T);L^2(\Gamma)^3)} \right] \|(\varphi^0, \psi^0)\|_{\mathcal{F}_1} .$$

From the definition of $\mathcal{F}_1$ we have that $(\varphi^0, \psi^0) \in \mathcal{F}_1$ if and only if $(\varphi(T), \psi(T)) \in \mathcal{F}_1$, further the map $(\varphi^0, \psi^0) \to (\varphi(T), \psi(T))$ is an isometry. For the problem (2.31)-(2.34) it follows that there exists a unique solution $(B, -D) \in C([0, T]; \mathcal{F}_1')$.

3 – Internal observability and controllability (I)

In this section we prove a result of internal observability and controllability. Let $\varepsilon > 0$ and let G_ε be as in (1.13). We denote by R_j the radius of the smallest ball that contains Ω_j and by x_0^j its center, that is,

$$R_j = \sup_{x \in \Omega_j} |x - x_0^j| \ , \tag{3.1}$$

for $j = 1, \dots, N$, and we set

$$m^j(x) = x - x_0^j \ , \tag{3.2}$$

$$\overline{R} = \sup_{1 \leq j \leq N} R_j \ . \tag{3.3}$$

We have the following result.

Theorem 3.1. *Let G_ε be as above. Consider $\lambda, \mu \in C^1(\overline{\Omega})$ satisfying hypotheses* (2.3), (2.4) *and*

$$\mu - m^j \cdot \nabla \mu \geq c_0 \mu, \quad \forall\, x \in \Omega_j,\ 1 \leq j \leq N, \tag{3.4}$$

$$\lambda - m^j \cdot \nabla \lambda \geq c_0 \lambda, \quad \forall\, x \in \Omega_j,\ 1 \leq j \leq N, \tag{3.5}$$

for some positive constant c_0. Set

$$T_\varepsilon = \frac{2\overline{R} - 2\varepsilon}{c_0 L}, \tag{3.6}$$

then for every $T > T_\varepsilon$ and for every pair $(D^0, B^0) \in \mathcal{H}_1$ the solution of (1.1)-(1.4), (1.5) *satisfies the inequalities*

$$c_1\mathcal{E} \le \int_0^T \int_{G_\varepsilon} (|D(t)|^2 + |B(t)|^2) dx dt \le c_2\mathcal{E} \tag{3.7}$$

where c_1, c_2, are suitable positive constants independent on the initial data.

REMARK 3.1. If we consider $N = 1$ and $\Omega_1 = \Omega$ and the functions λ, μ verifying hypotheses $(2.3) - (2.6)$ then G_ε is an ε-neighborhood of the boundary Γ of Ω.

REMARK 3.2. In the case of previous Remark if λ and μ are constant then, choosing $c_0 = 1$, the time T_ε is optimal, that is we cannot have observability inequality if $T < T_\varepsilon$ due to the finite speed propagation. In particular if $\lambda \equiv \mu \equiv 1$ the observability inequality holds for every $T > (2R - 2\varepsilon)$.

PROOF OF THEOREM 3.1. As in [14], let $\varepsilon_0, \varepsilon_1$ be positive constants such that $\varepsilon_1 < \varepsilon_0 < \varepsilon$, and define the sets

$$G_0 = \Omega \cap \mathcal{N}_{\varepsilon_0}\Big(\Big(\Omega \setminus \bigcup_{j=1}^N \Omega_j\Big) \cup \Gamma\Big) \tag{3.8}$$

and

$$G_1 = \Omega \cap \mathcal{N}_{\varepsilon_1}\Big(\Big(\Omega \setminus \bigcup_{j=1}^N \Omega_j\Big) \cup \Gamma\Big) \ . \tag{3.9}$$

For every $1 \le j \le N$ we can choose a function ϕ^j such that

$$\phi^j \in C_0^\infty(\mathrm{I\!R}^3), 0 \le \phi^j \le 1 \ , \tag{3.10}$$

$$\phi^j \equiv 1 \quad \text{in} \quad \overline{\Omega}_j \setminus G_0 \ , \tag{3.11}$$

$$\phi^j \equiv 0 \quad \text{in} \quad \Omega_j \cap G_1 \ . \tag{3.12}$$

For every $1 \le j \le N$ we can apply the formula (2.9) in Ω_j with $q(x) :=$

$\phi^j(x)m^j(x)$, where m^j is defined by (3.2), and we obtain

$$\begin{aligned}
&\Big[2\int_{\Omega_j}(D\times B)(\phi^j m^j)dx\Big]_0^T=\\
&=\int_0^T\int_{\Omega_j}\{(\lambda|D|^2+\mu|B|^2)\operatorname{div}(\phi^j m^j)+\\
&\quad -2\sum_{i,k=1}^{3}\frac{\partial(\phi^j m_i^j)}{\partial x_k}(\lambda D_iD_k+\mu B_iB_k)\}dx\,dt+\\
&\quad +\int_0^T\int_{\Omega_j}\{-(\phi^j m^j\cdot\nabla\lambda)|D|^2-(\phi^j m^j\cdot\nabla\mu)|B|^2\}dx\,dt\\
&\qquad\qquad\qquad\qquad\qquad\qquad\forall\, j=1,\dots,N\ ,
\end{aligned}\tag{3.13}$$

that is

$$\begin{aligned}
&\Big[2\int_{\Omega_j}(D\times B)(\phi^j m^j)dx\Big]_0^T=\\
&=\int_0^T\int_{\Omega_j\setminus G_0}(\lambda|D|^2+\mu|B|^2)dx\,dt+\\
&\quad+\int_0^T\int_{\Omega_j\setminus G_0}\{-(m^j\cdot\nabla\lambda)|D|^2-(m^j\cdot\nabla\mu)|B|^2\}dx\,dt+\\
&\quad+\int_0^T\int_{\Omega_j\cap(G_0\setminus G_1)}\{(\lambda|D|^2+\mu|B|^2)\operatorname{div}(\phi^j m^j)+\\
&\quad-2\sum_{i,k=1}^{3}\frac{\partial(\phi^j m_i^j)}{\partial x_k}(\lambda D_iD_k+\mu B_iB_k)\}dx\,dt+\\
&\quad+\int_0^T\int_{\Omega_j\cap(G_0\setminus G_1)}\{-(\phi^j m^j\cdot\nabla\lambda)|D|^2-(\phi^j m^j\cdot\nabla\mu)|B|^2\}dx\,dt\,.
\end{aligned}\tag{3.14}$$

Therefore using the conditions (3.4), (3.5) from (3.14) we have

$$\begin{aligned}
&c_0\int_0^T\int_{\Omega_j\setminus G_0}(\lambda|D|^2+\mu|B|^2)dx\,dt-\Big[2\int_{\Omega_j}(D\times B)(\phi^j m^j)dx\Big]_0^T\le\\
&\le\tilde{c}_j\int_0^T\int_{\Omega_j\cap(G_0\setminus G_1)}(|D|^2+|B|^2)dx\,dt
\end{aligned}\tag{3.15}$$

where $\tilde{c}_j$ is a suitable constant.

Setting

$$\tilde{c} = \max_{1\leq j\leq N} \tilde{c}_j \tag{3.16}$$

we obtain

$$\begin{aligned}&\sum_{j=1}^{N} c_0 \int_0^T \int_{\Omega_j\setminus G_0} (\lambda|D|^2 + \mu|B|^2)dx\,dt+\\ &\quad -\sum_{j=1}^{N}\left[2\int_{\Omega_j}(D\times B)(\phi^j m^j)dx\right]_0^T \leq\\ &\leq \tilde{c}\sum_{j=1}^{N}\int_0^T\int_{\Omega_j\cap(G_0\setminus G_1)}(|D|^2+|B|^2)dx\,dt\ .\end{aligned} \tag{3.17}$$

Next we observe that

$$\begin{aligned}&\Big|\sum_{j=1}^{N}\Big[2\int_{\Omega_j}(D\times B)(\phi^j m^j)dx\Big]_0^T\Big| \leq\\ &\quad\leq \sum_{t=0,T}\sum_{j=1}^{N}\frac{2}{L}\int_{\Omega_j}\sqrt{\lambda}|D|\sqrt{\mu}|B|(R_j-\varepsilon_1)dx \leq\\ &\quad\leq \sum_{t=0,T}\frac{\overline{R}-\varepsilon_1}{L}\int_{\Omega_j}(\lambda|D|^2+\mu|B|^2)dx \leq 2\mathcal{E}\frac{2\overline{R}-2\varepsilon_1}{L}.\end{aligned}$$

Therefore we deduce from (3.17) the inequality

$$c_0\int_0^T\int_{\Omega\setminus G_0}(\lambda|D|^2+\mu|B|^2)dx\,dt - 2\mathcal{E}\frac{2\overline{R}-2\varepsilon_1}{L} \leq \tilde{c}\int_0^T\int_{\Omega\cap G_0}(|D|^2+|B|^2)dx\,dt\,.$$

Adding

$$c_0\int_0^T\int_{\Omega\cap G_0}(\lambda|D|^2+\mu|B|^2)dx\,dt$$

and using the condition (3.15) we obtain

$$2\mathcal{E}\Big(c_0T-\frac{2\overline{R}-2\varepsilon_1}{L}\Big) \leq c^*\int_0^T\int_{G_\varepsilon}(|D|^2+|B|^2)dx\,dt\ , \tag{3.18}$$

for some suitable positive constant c^*.

Since the inequality (3.18) holds for every $\varepsilon_1 < \varepsilon$, the first inequality in (3.7) follows for every $T > (2\overline{R} - 2\varepsilon)/c_0 L$. The second inequality in (3.7) obviously holds and the proof of Theorem 3.1 is completed. ∎

Applying HUM again, Theorem 3.1 leads to an exact controllability result concerning

$$D' - \operatorname{curl}(\mu B) = \chi_{Q_\varepsilon} h \quad \text{in} \quad \Omega \times (0, +\infty) \tag{3.19}$$

$$B' + \operatorname{curl}(\lambda D) = \chi_{Q_\varepsilon} k \quad \text{in} \quad \Omega \times (0, +\infty) \tag{3.20}$$

$$\operatorname{div} D = \operatorname{div} B = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \tag{3.21}$$

$$\nu \times D = 0 \quad \text{on} \quad \Gamma \times (0, +\infty) \tag{3.22}$$

$$D(0) = D^0 \quad \text{and} \quad B(0) = B^0 \quad \text{in} \quad \Omega \tag{3.23}$$

where, for fixed T,

$$Q_\varepsilon = G_\varepsilon \times (0, T) \tag{3.24}$$

and χ_{Q_ε} is the characteristic function of the set Q_ε.

Let $(\varphi^0, \psi^0) \in \mathcal{H}_1$ and let (φ, ψ) be the solution of (1.1)-(1.4)-(1.5). From the previous theorem it follows that

$$\|(\varphi^0, \psi^0)\|_{\mathcal{G}_1} = \Big(\int_0^T \int_{G_\varepsilon} (\mu|\varphi(t)|^2 + \lambda|\psi(t)|^2) dx\, dt \Big)^{\frac{1}{2}} \tag{3.25}$$

is a norm on $\mathcal{H}_1$. Let $\mathcal{G}_1$ be the completion of $\mathcal{H}_1$ with respect to this norm and let $\mathcal{G}_1'$ be its dual space with respect to the scalar product $\langle \cdot, \cdot \rangle_0$. We have

Theorem 3.2. *Let G_ε be as above and let $T_\varepsilon, \lambda, \mu$ be as in Theorem* 3.1. *Then for every time $T > T_\varepsilon$ and for every pair $(B^0, -D^0) \in \mathcal{G}_1'$ there exist two functions $h, k \in L^2(0, T; L^2(G_\varepsilon)^3)$ such that the solution of* (3.19)-(3.23) *with initial data (D^0, B^0) satisfies the final condition*

$$D(T) = 0, \ B(T) = 0 \quad in \quad \Omega\,. \tag{3.26}$$

PROOF OF THEOREM 3.2. Let $T > T_\varepsilon$ be a fixed time. As in § 2, we apply the Hilbert Uniqueness Method of Lions. Let $(\varphi^0, \psi^0) \in \mathcal{G}_1$ and let (φ, ψ) the solution of the system

$$\varphi' - \operatorname{curl}(\lambda\psi) = \psi' + \operatorname{curl}(\mu\varphi) = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{3.27}$$

$$\operatorname{div}\varphi = \operatorname{div}\psi = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{3.28}$$

$$\nu \times \psi = 0 \quad \text{and} \quad \nu \cdot \varphi = 0 \quad \text{on} \quad \Gamma \times (0,T)\ , \tag{3.29}$$

$$\varphi(0) = \varphi^0 \quad \text{and} \quad \psi(0) = \psi^0 \quad \text{in} \quad \Omega\ . \tag{3.30}$$

We consider the following problem:

$$D' - \operatorname{curl}(\mu B) = \chi_{Q_\varepsilon}\psi \quad \text{in} \quad \Omega \times (0,T)\ , \tag{3.31}$$

$$B' + \operatorname{curl}(\lambda D) = -\chi_{Q_\varepsilon}\varphi \quad \text{in} \quad \Omega \times (0,T)\ , \tag{3.32}$$

$$\operatorname{div} D = \operatorname{div} B = 0 \quad \text{in} \quad \Omega \times (0,T)\ , \tag{3.33}$$

$$\nu \times D = 0 \quad \text{on} \quad \Gamma \times (0,T)\ , \tag{3.34}$$

$$D(T) = B(T) = 0 \quad \text{in} \quad \Omega\ . \tag{3.35}$$

Proceeding formally (we will precise everything in Remark 3.3), we have

$$\begin{aligned}&\int_0^T \int_\Omega \{\lambda(D' - \operatorname{curl}(\mu B))\psi - \mu(B' + \operatorname{curl}(\lambda D))\varphi\} dx\, dt = \\ &\quad = \int_0^T \int_{G_\varepsilon} (\mu|\varphi|^2 + \lambda|\psi|^2) dx\, dt\ .\end{aligned} \tag{3.36}$$

Hence, after integrations by parts

$$-\int_\Omega \lambda D(0)\psi(0) dx + \int_\Omega \mu B(0)\varphi(0) dx = \int_0^T \int_{G_\varepsilon} (\mu|\varphi|^2 + \lambda|\psi|^2) dx\, dt\,.$$

Therefore, as in the proof of Theorem 2.3 from hypotheses on λ, μ and from the observability inequality in (3.7) we have that for every $(B^0, -D^0) \in \mathcal{G}_1'$ we can choose $(\varphi^0, \psi^0) \in \mathcal{G}_1$ such that the solution of (3.19)-(3.23) with initial data (D^0, B^0) verifies (3.26). $\square$

REMARK 3.3. Analogously to the previous section, the solutions of (3.31)-(3.35) and (3.19)-(3.23) are understood by using the transposition method. Consider the problem (3.19)-(3.23) with $h,\ k \in L^2((0,T);$

$L^2(G_\varepsilon)^3$) and initial data $(B^0, -D^0) \in \mathcal{H}_1'$. Let $(\varphi^0, \psi^0) \in \mathcal{H}_1$ and let (φ, ψ) the solution of (3.27)-(3.30). Let $t > 0$ be fixed. We can define the solution of (3.19)-(3.23) by the formula

$$
\begin{aligned}
(3.37)\qquad & \int_\Omega \lambda D(t)\psi(t)dx - \int_\Omega \mu B(t)\varphi(t)dx = \\
& = \int_\Omega \lambda D(0)\psi(0)dx - \int_\Omega \mu B(0)\varphi(0)dx + \int_0^t \int_{G_\varepsilon} (\mu|\varphi|^2 + \lambda|\psi|^2)dx\,dt\,.
\end{aligned}
$$

As in Remark 2.4 we can see that there exists a unique pair $(B, -D) \in C([0,T]; \mathcal{H}_1')$ that satisfies (3.37) and this pair is, by definition, the solution of the problem (3.19)-(3.23). Moreover if $T > T_\varepsilon$ the system (3.31)-(3.35) has a unique solution $(B, -D) \in C([0,T]; \mathcal{G}_1')$.

4 – Internal observability and controllability (II)

Let us now assume that λ, μ are positive constants and that Ω is star-shaped with respect to the origin, that is, (1.10) holds. Let us introduce the function spaces

$$J_\tau^*(\Omega), J_\nu^*(\Omega), J(\Omega), \hat{J}(\Omega),$$

as in Ladyzhenskaya and Solonnikov [10] (see also Lagnese [11]). We define

$$(4.1)\qquad J(\Omega) = \text{ closure of } \{\xi/\xi \in (C^\infty(\overline{\Omega}))^3, \operatorname{div}\xi = 0\} \text{ in } (L^2(\Omega))^3\ ,$$

$$(4.2)\qquad \hat{J}(\Omega) = \text{ closure of } \{\xi/\xi \in (C_0^\infty(\Omega))^3, \operatorname{div}\xi = 0\} \text{ in } (L^2(\Omega))^3\ ,$$

and

$$
\begin{aligned}
(4.3)\qquad & J^k(\Omega) = J(\Omega) \cap (H^k(\Omega))^3, \\
(4.4)\qquad & J_\tau^k(\Omega) = \{\xi/\xi \in J^k(\Omega), \nu \times \xi = 0 \text{ on } \Gamma\}\ , \\
(4.5)\qquad & J_\nu^k(\Omega) = \{\xi/\xi \in J^k(\Omega), \nu \cdot \xi = 0 \text{ on } \Gamma\}\ ,
\end{aligned}
$$

with the topology inherited from $(H^k(\Omega))^3$. Further, we introduce

$$J_\tau^*(\Omega) = \{\xi/\xi \in J_\tau^2(\Omega),\ \nu \cdot \operatorname{curl} \xi = 0 \text{ on } \Gamma\}\ , \tag{4.6}$$

$$J_\nu^*(\Omega) = \{\xi/\xi \in J_\nu^2(\Omega), \nu \times \operatorname{curl} \xi = 0 \text{ on } \Gamma\}\ , \tag{4.7}$$

with the topology in each space inherited from $(H^2(\Omega))^3$. The spaces above are known to have the following properties (see [10], [11]):

$$J_\tau^*(\Omega) \subset J_\tau^1(\Omega) \subset J(\Omega), \quad J_\nu^*(\Omega) \subset J_\nu^1(\Omega) \subset \hat{J}(\Omega)\ , \tag{4.8}$$

with each space dense and continuously imbedded in the one that follows it. Further, we can renorm $J_\tau^1(\Omega)$ and $J_\nu^1(\Omega)$ by setting

$$\|\xi\|_{J_\tau^1(\Omega)} = \|\operatorname{curl} \xi\|, \qquad \|\xi\|_{J_\nu^1(\Omega)} = \|\operatorname{curl} \xi\|\ , \tag{4.9}$$

which is equivalent to the $(H^1(\Omega))^3$ norm on these spaces.

THEOREM 4.1. *Assume* (1.10), *let* λ, μ *be constant and let* $G_\varepsilon = \mathcal{N}_\varepsilon(\Gamma)$. *Set*

$$T_\varepsilon = \frac{2R - 2\varepsilon}{L}\ , \tag{4.10}$$

with R *defined as in* (2.2). *Then for every* $T > T_\varepsilon$ *and for every pair* $(D^0, B^0) \in J_\tau^*(\Omega) \times J_\nu^*(\Omega)$ *the solution of* (1.1)-(1.4), (1.5) *satisfies the inequality*

$$\int_0^T \int_{G_\varepsilon} (|\operatorname{curl} D(t)|^2 + |\operatorname{curl} B(t)|^2) dx dt \geq c\mathcal{E}\ , \tag{4.11}$$

for some positive constant c, independent on the initial data.

PROOF OF THEOREM 4.1. We consider the system (1.1)-(1.4), (1.5) with $(D, B) \in J_\tau^*(\Omega) \times J_\nu^*(\Omega)$.

Set

$$\tilde{D} = \operatorname{curl} D, \qquad \tilde{B} = \operatorname{curl} B\ , \tag{4.12}$$

then we have

$$\tilde{D}' - \mu \ \operatorname{curl} \tilde{B} = 0 \quad \text{in} \quad \Omega \times (0,T) \ , \tag{4.13}$$

$$\tilde{B}' + \lambda \ \operatorname{curl} \tilde{D} = 0 \quad \text{in} \quad \Omega \times (0,T) \ , \tag{4.14}$$

and obviously

$$\operatorname{div} \tilde{D} = 0, \ \operatorname{div} \tilde{B} = 0 \quad \text{in} \quad \Omega \times (0,T) \ . \tag{4.15}$$

Further, if $(D,B) \in J_\tau^*(\Omega) \times J_\nu^*(\Omega)$, then for $(\tilde{D},\tilde{B})$ defined as in (4.12) we have

$$\nu \times \tilde{B} = 0 \quad \text{and} \quad \nu \cdot \tilde{D} = 0 \quad \text{on} \quad \Gamma \ . \tag{4.16}$$

Since $(\tilde{D},\tilde{B})$ verifies a conservative system, we can apply the results obtained by the multiplier method in § 3 and we have for $T > (2R-2\varepsilon)/L$

$$c_1 \tilde{\mathcal{E}} \le \int_0^T \int_{G_\varepsilon} (|\tilde{D}|^2 + |\tilde{B}|^2) dx dt \le c_2 \tilde{\mathcal{E}} \tag{4.17}$$

where

$$\tilde{\mathcal{E}} \equiv \tilde{\mathcal{E}}(t) = \int_\Omega (\lambda |\tilde{D}|^2 + \mu |\tilde{B}|^2) dx = \int_\Omega (\lambda |\operatorname{curl} D|^2 + \mu |\operatorname{curl} B|^2) dx \ . \tag{4.18}$$

Then from (4.9) and (4.18) it follows that

$$\int_0^T \int_{G_\varepsilon} (|\operatorname{curl} D|^2 + |\operatorname{curl} B|^2) dx \, dt \ge c\mathcal{E} \tag{4.19}$$

for some constant c, that is the observability inequality. □

Theorem 4.1 implies the following controllability theorem concerning the system

$$D' - \operatorname{curl}(\mu B) = \chi_{Q_\varepsilon} w \quad \text{in} \quad \Omega \times (0,+\infty) \tag{4.20}$$

$$B' + \operatorname{curl}(\lambda D) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{4.21}$$

$$\operatorname{div} D = \operatorname{div} B = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{4.22}$$

$$\nu \times D = 0 \quad \text{on} \quad \Gamma \times (0,+\infty) \tag{4.23}$$

$$D(0) = D^0 \quad \text{and} \quad B(0) = B^0 \quad \text{in} \quad \Omega \tag{4.24}$$

where Q_ε is given by (3.24).

THEOREM 4.2. *Under the assumptions of Theorem* 4.1, *for every time* $T > T_\varepsilon$ *and for every pair* $(D^0, B^0) \in J(\Omega) \times \hat{J}(\Omega)$ *there exists a function* $w \in L^2((0,T); J(\Omega))$ *such that the solution of* (4.20)-(4.24) *satisfies*

$$D(T) = B(T) = 0 \; . \tag{4.25}$$

PROOF. Let us consider the system

$$\begin{aligned}
&(4.26) \qquad \varphi' - \lambda \operatorname{curl} \psi = \psi' + \mu \operatorname{curl} \varphi = 0 && \text{in} \quad \Omega \times (0,T) \; , \\
&(4.27) \qquad \operatorname{div} \varphi = \operatorname{div} \psi = 0 && \text{in} \quad \Omega \times (0,T) \; , \\
&(4.28) \qquad \nu \times \psi = 0, \nu \cdot \varphi && \text{on} \quad \Gamma \times (0,T) \; , \\
&(4.29) \qquad \varphi(0) = \varphi^0, \psi(0) = \psi^0 && \text{in} \quad \Omega \; ,
\end{aligned}$$

where $(\varphi^0, \psi^0) \in J(\Omega) \times \hat{J}(\Omega)$, and the problem

$$\begin{aligned}
&(4.30) \quad D' - \mu \operatorname{curl} B = \chi_Q \Big(\operatorname{curl} \operatorname{curl} \psi - \frac{d}{dt} \psi' \Big) && \text{in} \quad \Omega \times (0,T) \; , \\
&(4.31) \quad B' + \lambda \operatorname{curl} D = 0 && \text{in} \quad \Omega \times (0,T) \; , \\
&(4.32) \quad \operatorname{div} D = \operatorname{div} B = 0 && \text{in} \quad \Omega \times (0,T) \; , \\
&(4.33) \quad \nu \times D = 0 && \text{on} \quad \Gamma \times (0,T) \; , \\
&(4.34) \quad D(T) = B(T) = 0 && \text{in} \quad \Omega \; ,
\end{aligned}$$

where $\frac{d}{dt}$ denotes the derivative in the duality. As in the proof of Theorem 2.3 and Theorem 3.2, we obtain that we can choose suitable (φ^0, ψ^0) such that the solution of (4.30)-(4.33) wich initial data $(D^0, B^0) \in J^*_\tau(\Omega) \times J^*_\nu(\Omega)$ satisfies the final condition (4.34). □

REMARK 4.1. As in the previous sections the solutions of (4.20)-(4.24) and (4.30)-(4.34) are understood by using the transposition method.

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