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Interface dynamics and Stefan problem from a microscopic conservative model

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RIASSUNTO: *Si considera un sistema stocastico di spins accoppiato a un processo di diffusione lineare. L'accoppiamento è tale che si ha campo localmente conservato. Gli stati di equilibrio sono le corrispondenti misure di Gibbs canoniche. Viene dimostrato che, in un limite di scala diffusivo, il campo conservato converge alla soluzione di un'equazione di diffusione non lineare. Per alcuni valori dei parametri il sistema esibisce una transizione di fase: in tale caso l'equazione macroscopica degenera e diviene la formulazione debole del problema di Stefan a due fasi.*

ABSTRACT: *We consider a stochastic spin system coupled to a linear diffusion process. The coupling is such that there is a locally conserved quantity. The equilibrium states are the corresponding canonical Gibbs measures. We prove that, under a diffusive scaling limit, the macroscopic density of the conserved quantity solves a non–linear diffusion equation. For certain values of the parameters a phase transition occurs; in this case the macroscopic equation degenerates and is the weak formulation of the two phases Stefan problem.*

0 – Introduction

Let us consider a pure material, say water, which can be in either of two phases, say liquid and solid, and occupies a region $\Omega \subset \mathbb{R}^d$. We prepare an initial condition in which there is a sharp interface $\Gamma_0 \subset \Omega$

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between the two phases and ask for its evolution. The interface dynamics is rather complicated and depends on the physical conditions of the experiment. A simple mathematical model is given by the Stefan problem which is, with the appropriate initial and boundary condition, the following free boundary problem

$$\begin{cases} \partial_t u(t,r) = \nabla \cdot \Big(\kappa(u)\nabla u(t,r)\Big) & \text{if } r \in \Omega \setminus \Gamma(t) \\ \ell\, v(t) = \Big[\kappa(u)\dfrac{\partial u}{\partial n}\Big]_{\Gamma(t)} & \\ u(t,r) = 0 & \text{if } r \in \Gamma(t) \end{cases} \tag{0.1}$$

in which $\kappa(u)$ is the diffusion coefficient, $\ell > 0$ the latent heat, $v(t)$ the normal velocity of $\Gamma(t)$ and $[\frac{\partial u}{\partial n}]_\Gamma$ is the jump of the normal derivative of u across Γ. The scalar field $u(t,r)$ is interpreted as the temperature of the material and we set the melting temperature equal to 0; therefore $u(t,r) < 0$ in the solid phase and $u(t,r) > 0$ in the liquid phase. Accordingly $\Gamma(t)$ is the phase boundary at time t.

The aim of this paper is the derivation of the Stefan problem from a microscopic model of phase field type. We shall deal with the *weak* formulation of (0.1) for which there is an existence and uniqueness result, see e.g. [7]. As a matter fact the existence result will also follow from the derivation. Any classical solution of (0.1) is also a weak solution, but the reverse is, in general, false: the interface may develop singularities after a finite time and/or the set $\{r : u(t,r) = 0\}$ may be of positive Lebesgue measure (mushy region), see [7] and references therein. We stress that we prove convergence to the weak formulation of the Stefan problem for all times. On the other hand, for simplicity, we restrict ourselves to periodic boundary condition, i.e. Ω is the d–dimensional torus S^d.

Generally speaking, phase field models describe the kinetics of phase transitions when the presence of latent heat is taken into account. This effect is usually negligible for metallic alloys which conduct heat well, but may be relevant for other types of materials. These models are usually formulated in term of a system of (non-linear) P.D.E. which, in the simplest case, takes the following form

$$\partial_t m(t,r) = \Delta m(t,r) - V'(m(t,r)) + \phi(t,r) \tag{0.2}$$

$$\partial_t (m(t,r) + a\phi(t,r)) = c\Delta\phi(t,r) \tag{0.3}$$

where $t \geq 0$, $r \in \Omega \subset \mathbb{R}^d$, Δ is the Laplacian, $a, c > 0$ and $V(m)$ a symmetric double well potential; a typical example is the function $V(m) = (m^2 - 1)^2$. Eq. (0.2) is a Ginzburg-Landau equation for the order parameter m (i.e. $m \sim -1$ (resp. $m \sim 1$) in the solid (resp. liquid) phase) with an external field $\phi(t, r)$ which is interpreted as the temperature. Eq. (0.3) describes the relaxation of the locally conserved quantity $m(t, r) + a\phi(t, r)$. We refer to [8] for a general overview on phase field equations and their derivation from thermodynamic considerations.

A most interesting question is the analysis of the singular limits of the phase field equations. According to the limiting procedure different interface motions can be obtained; among them there are the Hele-Shaw model and the Stefan problem. As the Allen-Cahn equation can be viewed as a limit of the phase field equation also the motion by curvature can be included in the possible singular limits. This class of problems has been recently widely studied in the literature, see [2] and references therein.

The phase field equations are to be considered as a phenomenological (mesoscopic) description of the physical system in which a *coarse graining* procedure has been taken in order to replace the microscopic variables with the order parameter m. It seems therefore quite natural to look for a derivation of the interface dynamics directly from a microscopic model without passing through the mesoscopic level of the phase field equations. In the case of the motion by curvature this program has been successfully completed starting from a stochastic Ising model with Kac potential [3].

With the above motivation, in this paper we introduce a simple *microscopic* and *stochastic* model of phase field type. The space structure is discretized and instead of the continuous order parameter $m(t, r)$ we have a spin variable $\sigma = \{\sigma(x), x \in \Lambda \subset \mathbb{Z}^d\}$ where $\sigma(x) = \pm 1$. The external field is described by a continuous charge $\phi(x)$ attached to each $x \in \Lambda$. The microscopic evolution is defined by a Glauber dynamics (stochastic Ising model) for the spin variable σ coupled to a systems of linear diffusions for the charge ϕ. The dynamics is constructed in such a way there is a locally conserved quantity. This will be realized by compensating the spin flip $\sigma(x) \mapsto -\sigma(x)$ (representing a change of phase at the site x) by adding to the field $\phi(x)$ the fraction $\lambda\sigma(x)$. Of course this is a rather crude model for the actual transition in real materials, however it could catch some of its essential features.

Since there is a conserved quantity, the relaxation to equilibrium is not exponentially fast, as it happens in the usual Glauber dynamics (away from the phase transition region). Our main result is the hydrodynamic limit (reduced description) for the model above described. We scale space and time diffusively and look for a limiting equation describing the evolution of the conserved quantity. We find a non-linear diffusion equation; its diffusion coefficient is obtained from the thermodynamic of the model. This result is proven in the whole range of the parameters, including the phase transition region. When phase transition occurs the limiting equation is however degenerate, i.e. the diffusion coefficient vanishes. In this case the equation is to be interpreted in the weak sense and it is precisely the weak formulation of the Stefan problem (0.1).

This result covers the case when a macroscopic interface is associated (see eq. (1.15) below for a formal definition) to the initial distribution of the microscopic process. According to the Lifshitz theory, the phase segregation phenomena in which a macroscopic droplet is nucleated from the microscopic dynamics take place instead on a longer time scale and therefore are not observed in the hydrodynamic limit, see [11]. We finally remark that, by introducing a Kac (long range) potential and considering the Lebowitz-Penrose limit, a non-local version of the mesoscopic phase field eqs. (0.2)-(0.3) can be derived from this microscopic model [9].

From a technical point of view, the hydrodynamic limit is proven by applying the entropy techniques introduced in [6] and developed in [10] to cover the case when the invariant measure is not a product measure. The structure of the paper is as follows. In the next Section we define precisely the model and state the hydrodynamic limit. In Section 2 we prove an ergodic theorem for the canonical Gibbs state; it is then used, in Section 3, to prove the "one and two blocks estimates". Finally, in Section 4, we obtain a large deviation result in equilibrium and conclude the proof of the hydrodynamic limit.

1 – Notation and results

For any positive integer N, let S_N be the periodic lattice $\{j : j = 0, 1, ..., N\}$ with 0 and N identified. We denote by S_N^d the product of d copies of S_N. The microscopic state space is

$$X_N \doteq \{-1, 1\}^{S_N^d} \times \mathbb{R}^{S_N^d} \tag{1.1}$$

The elements of X_N are denoted by $\xi = (\sigma, \phi)$, where for any $x \in S_N^d$, $\sigma(x) \in \{-1, 1\}$ is a spin variable and $\phi(x) \in \mathbb{R}$ a continuous charge. The energy of the configuration $\xi \in X_N$ is given by

$$H_N(\xi) = H_N^{\mathrm{s}}(\sigma) + H_N^{\mathrm{c}}(\phi) \tag{1.2}$$

where $\xi = (\sigma, \phi)$ and

$$H_N^{\mathrm{s}}(\sigma) \doteq -\frac{1}{2} \sum_{\substack{x,y \in S_N^d \\ |x-y|=1}} \sigma(x)\sigma(y), \qquad H_N^{\mathrm{c}}(\phi) \doteq \sum_{x \in S_N^d} \phi(x)^2 \tag{1.3}$$

For any $x \in S_N^d$ we define a local operator, the "compensated spin flip", $\delta_x : X_N \leftarrow X_N$ by

$$\delta_x(\sigma, \phi)(y) \doteq \begin{cases} (-\sigma(x), \phi(x) + \lambda\sigma(x)) & \text{if } y = x \\ (\sigma(y), \phi(y)) & \text{otherwise} \end{cases} \tag{1.4}$$

where $\lambda > 0$ is a given parameter. The corresponding variation of the energy is

$$\delta_x H_N(\xi) \doteq H_N(\delta_x\xi) - H_N(\xi) = \sum_{y:|y-x|=1} \sigma(x)\sigma(y) + 2\lambda\sigma(x)\phi(x) + \lambda^2 \tag{1.5}$$

The microscopic dynamics is constructed by means of two elementary processes. The former involves only the ϕ component of ξ and it is given by a linear Ginzburg-Landau process. More precisely, given $\beta > 0$, we consider the diffusion whose generator is

$$\begin{aligned} L^{(0)} = \frac{1}{2} &\sum_{\substack{x,y \in S_N^d \\ |x-y|=1}} \Big(\frac{\partial}{\partial\phi(x)} - \frac{\partial}{\partial\phi(y)}\Big)^2 + \\ &- \sum_{\substack{x,y \in S_N^d \\ |x-y|=1}} \beta(\phi(x) - \phi(y))\Big(\frac{\partial}{\partial\phi(x)} - \frac{\partial}{\partial\phi(y)}\Big) \end{aligned} \tag{1.6}$$

The latter is a jump Markov process which involves both the spins σ and the charges ϕ. Let

$$c_\beta(x, \xi) \doteq \frac{1}{2} \exp\Big[-\frac{\beta}{2}\delta_x H(\xi)\Big] \tag{1.7}$$

Then let $L^{(1)}$ be the Markov generator which acts on continuous and compactly supported functions on X_N as

$$L^{(1)} f(\xi) \doteq \sum_{x \in S_N^d} c_\beta(x,\xi)[f(\delta_x \xi) - f(\xi)] \tag{1.8}$$

Note that, on the space of functions which depend only on the spin variables, $L^{(1)}$ reduces to the generator of a Glauber dynamics with an external field $\lambda\phi(x)$. But in our case the external field is itself a dynamical variable.

The dynamics is then defined by the Markov process on X_N with generator

$$L_N \doteq N^2 (L^{(0)} + L^{(1)}) \tag{1.9}$$

in which we speed up the dynamics by a factor N^2; this realizes the above mentioned diffusive scaling. We remark that for N fixed we are in a bounded volume and therefore the existence of the Markov process generated by (1.9) is trivial.

The generator L_N is reversible with respect to the Gibbs measure ν_N defined by

$$\nu_N(d\xi) \doteq \frac{1}{Z_N} \exp[-\beta H_N^{\mathrm{s}}(\sigma)] \rho_N(d\phi) \tag{1.10}$$

where Z_N is the normalizing constant and $\rho_N(d\phi)$ is the Gaussian measure given by

$$\rho_N(d\phi) \doteq \Big(\frac{\beta}{\pi}\Big)^{N^d/2} \exp[-\beta H_N^{\mathrm{c}}(\phi)] d\phi \tag{1.11}$$

in which $d\phi$ is the Lebesgue measure on $\mathbb{R}^{S_N^d}$.

For any $\Lambda \subseteq S_N^d$, let

$$\omega(\Lambda) \doteq \sum_{x\in\Lambda} \omega(x), \qquad \omega(x) \doteq \sigma(x) + 2\lambda^{-1}\phi(x) \tag{1.12}$$

It is easy to check that ω is a locally conserved quantity. In particular the dynamics preserves the total charge $\omega(S_N^d)$. Our main result is the derivation of the hydrodynamic equation for the macroscopic charge density.

We consider the Markov process ξ_t with an initial distribution given by a probability density f_N^0 with respect to the reference measure ν_N. The marginal of the process at time t has then a density f_N^t with respect to ν_N which is obtained by solving the forward equation

$$\frac{\partial f_N^t}{\partial t} = L_N f_N^t, \qquad f_N^t\Big|_{t=0} = f_N^0 \tag{1.13}$$

To each $\omega \doteq \{\omega(x), x \in S_N^d\}$, $\omega(x)$ as in (1.12), we associate the empirical measure

$$\mu_N = \frac{1}{N^d} \sum_{x \in S_N^d} \omega(x) \delta_{x/N} \tag{1.14}$$

which is a random signed measure on the d-dimensional torus S^d (i.e. the product of d copies of the interval $[0,1]$ with 0 and 1 identified). We say that a function $q_0(\theta)$ is the asymptotic macroscopic charge density associated to μ_N if, for every smooth function J on S^d,

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_{x \in S_N^d} J(x/N)\omega(x) = \int d\theta\, J(\theta) q_0(\theta) \tag{1.15}$$

where the limit is in probability and $d\theta$ is the Lebesgue measure on S^d.

We shall assume that the initial density f_N^0 satisfies the entropy bound

$$\frac{1}{N^d} \int \nu_N(d\xi)\, f_N^0(\xi) \log f_N^0(\xi) \le C \tag{1.16}$$

for some $C > 0$ uniformly in N. Furthermore, we require convergence at time 0 by assuming that for some continuous function $q_0(\theta)$ and every $\delta > 0$,

$$\lim_{N\to\infty} \int_{E_{N,\delta}^0} \nu_N(d\xi)\, f_N^0(\xi) = 0 \tag{1.17}$$

where

$$E^0_{N,\delta} \doteq \Big\{\xi\,:\,\Big|\frac{1}{N^d}\sum_{x\in S^d_N} J(x/N)\omega(x) - \int d\theta\, J(\theta)q_0(\theta)\Big| \ge \delta\Big\} \tag{1.18}$$

Under the conditions (1.16) and (1.17) we shall prove that in the limit $N\to\infty$ the locally conserved quantity has a deterministic behaviour which is ruled by the solution of a nonlinear diffusion equation with initial condition q_0. In order to write the macroscopic equation we need to introduce the thermodynamic functions. The pressure is given by

$$p(\beta,\alpha) \doteq \lim_{N\to\infty}\frac{1}{\beta N^d}\log\int \nu_N(d\xi)\,\exp\Big[\beta\alpha\sum_{x\in S^d_N}\omega(x)\Big] \tag{1.19}$$

and we define h as the convex conjugate of p, that is

$$h(\beta,q) \doteq \sup_\alpha\{\alpha q - p(\beta,\alpha)\} \tag{1.20}$$

We can now state precisely our main result.

THEOREM 1.1. *Let the initial density f^0_N satisfies* (1.16) *and* (1.17). *Then for any $t\ge 0$, every smooth function J and each $\delta>0$,*

$$\lim_{N\to\infty}\int_{E^t_{N,\delta}}\nu_N(d\xi)\, f^t_N(\xi) = 0 \tag{1.21}$$

where

$$E^t_{N,\delta} \doteq \Big\{\xi\,:\,\Big|\frac{1}{N^d}\sum_{x\in S^d_N} J(x/N)\omega(x) - \int d\theta\, J(\theta)q(t,\theta)\Big| \ge \delta\Big\} \tag{1.22}$$

and $q(t,\theta)$ is the (*unique*) *weak solution of the nonlinear diffusion equation*

$$\begin{cases} \dfrac{\partial q}{\partial t}(t,\theta) &= 2\beta\Delta_\theta F\big(q(t,\theta)\big)\\ q(0,\theta) &= q_0(\theta)\end{cases} \tag{1.23}$$

where

$$F(q) \doteq \frac{2}{\lambda^2}\frac{\partial h}{\partial q}(\beta,q) \tag{1.24}$$

By standard results in Statistical Mechanics (see the next Section), $F(q)$ is a continuous and non decreasing function of q. Furthermore, if $d \geq 2$, there exists a $\beta_c \in (0, \infty)$, the inverse critical temperature and, for $\beta > \beta_c$, $m_\beta > 0$, the spontaneous magnetization, such that $F(q)$ is constant for $q \in [-m_\beta, m_\beta]$, strictly increasing and smooth for $|q| > m_\beta$. In this case the eq. (1.23) degenerates and is the appropriate weak formulation of the Stefan problem (0.1), which is most naturally formulated for the internal energy q and not for the temperature $u = 2\beta F(q)$, see [7, I.3]. The coefficients ℓ, κ in (0.1) are thus obtained from the microscopic interaction.

2 – Canonical gibbs measures and ergodic theorem

In this section we prove an ergodic theorem for the canonical Gibbs measures. Let us first recall some standard facts about Gibbs measures in the context of our model.

The (infinite volume) configuration space is $X \doteq \{-1, 1\}^{\mathbb{Z}^d} \times \mathbb{R}^{\mathbb{Z}^d}$; its elements are denoted by $\xi = (\sigma, \phi)$, where $\sigma \in \{-1, 1\}^{\mathbb{Z}^d}$ is a spin configuration and $\phi \in \mathbb{R}^{\mathbb{Z}^d}$ a continuous charge. X is naturally endowed with the product topology which makes it a Polish space. We denote by τ_x, $x \in \mathbb{Z}^d$, the shift operator on X, that is $\tau_x \xi(y) \doteq \xi(y + x)$ for any $\xi \in X$ and $y \in \mathbb{Z}^d$. We also denote by $\mathcal{M}(X)$ the space of probability measures on X and by $\mathcal{M}_\tau(X)$ the space of shift invariant probability measures on X. Both $\mathcal{M}(X)$ and $\mathcal{M}_\tau(X)$ are endowed with the topology of weak convergence.

For any finite subset $\Lambda \subset \mathbb{Z}^d$ (we shall write $\Lambda \subset\subset \mathbb{Z}^d$ to indicate Λ is finite) we define $X_\Lambda \doteq \{-1, 1\}^\Lambda \times \mathbb{R}^\Lambda$ and we denote by $\xi_\Lambda = (\sigma_\Lambda, \phi_\Lambda)$ the restriction of $\xi = (\sigma, \phi)$ to Λ, that is $\xi_\Lambda \in X_\Lambda$ and $\xi_\Lambda(x) = \xi(x)$ for any $x \in \Lambda$. The energy of the configuration ξ_Λ in the presence of a constant external field α is given by

$$H_{\Lambda,\alpha}(\xi_\Lambda) = H^{\mathrm{s}}_\Lambda(\sigma_\Lambda) + H^{\mathrm{c}}_\Lambda(\phi_\Lambda) - \alpha \sum_{x \in \Lambda} \omega(x) \tag{2.1}$$

where $\omega(x) = \sigma(x) + 2\lambda^{-1}\phi(x)$ (see (1.12)) and

$$H^{\mathrm{s}}_\Lambda(\sigma_\Lambda) \doteq -\frac{1}{2} \sum_{\substack{x,y \in \Lambda \\ |x-y|=1}} \sigma(x)\sigma(y), \qquad H^{\mathrm{c}}_\Lambda(\phi_\Lambda) \doteq \sum_{x \in \Lambda} \phi(x)^2 \tag{2.2}$$

while its energy inclusive of the interaction with a given boundary condition $\bar{\xi} = (\bar{\sigma}, \bar{\phi}) \in X$ is

$$H_{\Lambda,\alpha}(\xi_\Lambda|\bar{\xi}) \doteq H_{\Lambda,\alpha}(\xi_\Lambda) - \sum_{\substack{x\in\Lambda, y\in\Lambda^c \\ |x-y|=1}} \sigma(x)\bar{\sigma}(y) \tag{2.3}$$

We remark that the energy $H_{\Lambda,\alpha}(\xi_\Lambda|\bar{\xi})$ depends only on the spin component $\bar{\sigma}$ of the boundary condition $\bar{\xi}$. Note moreover that

$$H_{\Lambda,\alpha}(\xi_\Lambda|\bar{\xi}) = H^{\mathrm{s}}_{\Lambda,\alpha}(\sigma_\Lambda|\bar{\sigma}) + H^{\mathrm{c}}_{\Lambda,\alpha}(\phi_\Lambda) \tag{2.4}$$

where

$$H^{\mathrm{c}}_{\Lambda,\alpha}(\phi_\Lambda) \doteq H^{\mathrm{c}}_{\Lambda}(\phi_\Lambda) - \alpha \sum_{x\in\Lambda} 2\lambda^{-1}\phi(x) \tag{2.5}$$

in which $H^{\mathrm{s}}_{\Lambda,\alpha}(\sigma_\Lambda|\bar{\sigma})$ is the Hamiltonian of the Ising Model in the presence of an external magnetic field α and with boundary condition $\bar{\sigma}$ (an expression analogous to (2.4) clearly holds also for the Hamiltonian (2.1) with free boundary conditions).

Given an inverse temperature $\beta > 0$, the grand canonical finite volume Gibbs measure $\nu_{\Lambda,\beta,\alpha}$ associated to the interaction (2.1) is defined by

$$\nu_{\Lambda,\beta,\alpha}(d\xi_\Lambda) \doteq \frac{1}{Z_\Lambda(\beta,\alpha)} \exp[-\beta H^{\mathrm{s}}_{\Lambda,\alpha}(\sigma_\Lambda)]\rho_{\Lambda,\beta,\alpha}(d\phi_\Lambda) \tag{2.6}$$

where $Z_\Lambda(\beta,\alpha)$ is the normalization factor and $\rho_{\Lambda,\beta,\alpha}$ is given by

$$\rho_{\Lambda,\beta,\alpha}(d\phi_\Lambda) \doteq \Big(\frac{\beta}{\pi}\Big)^{|\Lambda|/2} \exp[-\beta H^{\mathrm{c}}_{\Lambda,\alpha}(\phi_\Lambda)]d\phi_\Lambda \tag{2.7}$$

We define analogously, for any boundary condition $\bar{\xi} \in X$,

$$\nu_{\Lambda,\beta,\alpha}(d\xi_\Lambda|\bar{\xi}) \doteq \frac{1}{Z_{\Lambda,\bar{\xi}}(\beta,\alpha)} \exp[-\beta H^{\mathrm{s}}_{\Lambda,\alpha}(\sigma_\Lambda|\bar{\sigma})]\rho_{\Lambda,\beta,\alpha}(d\phi_\Lambda) \tag{2.8}$$

We also consider the finite volume canonical Gibbs measures $\nu^q_{\Lambda,\beta}$ and $\nu^q_{\Lambda,\beta,\bar{\xi}}$ which are obtained by conditioning the grand canonical ones $\nu_{\Lambda,\beta,0}$

and $\nu_{\Lambda,\beta,0}(\cdot|\bar{\xi})$ respectively, with respect to $m_\Lambda(\omega) = q$, having defined, for any function f on $\mathbb{Z}^d$,

$$m_\Lambda(f) \doteq \frac{1}{|\Lambda|} \sum_{x \in \Lambda} f(x) \tag{2.9}$$

We finally introduce the family of shift invariant infinite volume Gibbs states. For any $\alpha \in \mathbb{R}$, they are defined by

$$\mathcal{G}(\alpha) \doteq \{\mu \in \mathcal{M}_\tau(X) : \mu(\cdot|\xi_{\Lambda^c} = \bar{\xi}_{\Lambda^c}) = \nu_{\Lambda,\beta,\alpha}(\cdot|\bar{\xi}) \ \forall \Lambda \subset\subset \mathbb{Z}^d \, \bar{\xi} \ \mu\text{-a.s. } \}$$

while the canonical Gibbs states are defined by

$$\begin{aligned} \mathcal{G}^c \doteq \{\mu \in \mathcal{M}_\tau(X) \ : \ \mu(\cdot|\xi_{\Lambda^c} = \bar{\xi}_{\Lambda^c};\, m_\Lambda(\omega) = \\ = m_\Lambda(\bar{\omega}) = q) = \nu^q_{\Lambda,\beta,\bar{\xi}} \ \ \forall \Lambda \subset\subset \mathbb{Z}^d \ \ \bar{\xi}, q \ \mu\text{-a.s. } \} \end{aligned}$$

The connection with the thermodynamics is done by means of the partition function. We define the pressure of the system as

$$p(\beta,\alpha) \doteq \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{\beta|\Lambda|} \log Z_{\Lambda,\bar{\xi}}(\beta,\alpha) \tag{2.10}$$

where Λ tends to $\mathbb{Z}^d$ in the sense of van Hove. The limit is uniform in $\bar{\xi}$ and define a convex function of α and β^{-1}. It coincides with the analogous limit obtained by replacing $Z_{\Lambda,\bar{\xi}}(\beta,\alpha)$ with $Z_\Lambda(\beta,\alpha)$ and also with the definition (1.19). By an explicit computation we get

$$Z_{\Lambda,\bar{\xi}}(\beta,\alpha) = Z^{\mathrm{s}}_{\Lambda,\bar{\sigma}}(\beta,\alpha) \exp[\beta|\Lambda|\left(\alpha\lambda^{-1}\right)^2] \tag{2.11}$$

where $Z^{\mathrm{s}}_{\Lambda,\bar{\sigma}}(\beta,\alpha)$ is the partition function for the Ising Model. Therefore

$$p(\beta,\alpha) = p_0(\beta,\alpha) + \left(\frac{\alpha}{\lambda}\right)^2 \tag{2.12}$$

with $p_0(\beta,\alpha)$ the pressure of the Ising Model with external magnetic field α and inverse temperature β.

It is well known that the pressure $p_0(\beta,\alpha)$ of the Ising Model is a convex and continuous function of α for any $\beta > 0$. Moreover it is differentiable except that in the transition region $\{(\beta, 0) : \beta > \beta_c\}$, where β_c is the inverse critical temperature. Finally,

$$\lim_{\alpha \to 0^\pm} \frac{\partial p_0}{\partial \alpha} = \pm m_\beta, \qquad m_\beta \geq 0 \tag{2.13}$$

where $m_\beta > 0$ if and only if $\beta > \beta_c$ (spontaneous magnetization).
Let $h(\beta, q)$ be the convex conjugate of the pressure as defined in (1.20). From what stated before and by standard properties on the Legendre transform we easily get that $h(\beta, q)$ is a convex and differentiable function of q for any $\beta > 0$. Moreover, its derivative with respect to q is a non decreasing function of q, strictly increasing for $\beta < \beta_c$. More precisely, with m_β as in (2.13),

$$\frac{\partial h}{\partial q}(\beta, q) = 0 \qquad \text{if and only if } q \in [-m_\beta, m_\beta] \tag{2.14}$$

Now we can state the main result of this Section.

THEOREM 2.1. *Let*

$$\Lambda_n \doteq \{x \in \mathbb{Z}^d \,:\, 0 \leq x_j \leq n,\; x = \{x_j \,:\, j = 1, ..., d\}\} \tag{2.15}$$

Then, for any compact set $K \subset \mathbb{R}$,

$$\lim_{n \to \infty} \sup_{q \in K} \sup_{\bar{\xi} \in X} \int \nu^q_{\Lambda_n, \beta, \bar{\xi}}(d\xi_\Lambda) \left| 2\lambda^{-1} m_{\Lambda_n}(\phi) - \frac{2}{\lambda^2} \frac{\partial h}{\partial q}(\beta, q) \right| = 0 \tag{2.16}$$

We prove first that, uniformly in $\bar{\xi} \in X$ and $q \in K$,

$$\lim_{n \to \infty} \nu^q_{\Lambda_n, \beta, \bar{\xi}} \Big(\left| 2\lambda^{-1} m_{\Lambda_n}(\phi) - \frac{2}{\lambda^2} \frac{\partial h}{\partial q}(\beta, q) \right| > \delta \Big) = 0, \qquad \forall\, \delta > 0 \tag{2.17}$$

and then that the sequence $\{2\lambda^{-1} m_{\Lambda_n}(\phi)\}_{n \geq 1}$ is uniformly integrable with respect to the measures $\nu^q_{\Lambda_n, \beta, \bar{\xi}}$.

First of all, by an explicit computation, for any $\Lambda \subset\subset \mathbb{Z}^d$,

$$\begin{aligned} &\int \rho_{\Lambda, \beta, 0}(d\phi_\Lambda)\, \delta(2\lambda^{-1} m_\Lambda(\phi) + m_\Lambda(\sigma) - q) = \\ &= \frac{\lambda |\Lambda|}{2} \sqrt{\frac{\beta}{\pi}} \exp\Big[-\beta |\Lambda| \frac{\lambda^2}{4} (q - m_\Lambda(\sigma))^2 \Big] \end{aligned} \tag{2.18}$$

so that

$$\begin{aligned}\nu^q_{\Lambda,\beta,\bar\xi}(d\xi_\Lambda) &= Z_{\Lambda,\beta,\bar\sigma}(q)^{-1}\times\\ &\times \exp\Big[-\beta H^{\rm s}_{\Lambda,0}(\sigma_\Lambda|\bar\sigma) - \beta|\Lambda|\frac{\lambda^2}{4}(q-m_\Lambda(\sigma))^2\Big]\rho^{q-m_\Lambda(\sigma)}_{\Lambda,\beta}(d\phi_\Lambda)\end{aligned} \tag{2.19}$$

where $Z_{\Lambda,\beta,\bar\sigma}(q)$ is the normalization constant and $\rho^y_{\Lambda,\beta}$ denotes the conditional distribution of ϕ_Λ on the hyperplane $2\lambda^{-1}m_\Lambda(\phi) = y$, $y \in \mathbb{R}$. Clearly, for any Borel set $A \subseteq \mathbb{R}$,

$$\nu^q_{\Lambda,\beta,\bar\xi}(2\lambda^{-1}m_\Lambda(\phi)\in A) = \nu^q_{\Lambda,\beta,\bar\xi}(q-m_\Lambda(\sigma)\in A) \tag{2.20}$$

and, from (2.19), for any Borel set $\Gamma \subseteq \mathbb{R}$,

$$\begin{aligned}&\nu^q_{\Lambda,\beta,\bar\xi}(m_\Lambda(\sigma)\in\Gamma) =\\ &= Z_{\Lambda,\beta,\bar\sigma}(q)^{-1}\sum_{\substack{\{\sigma_\Lambda\}\\ m_\Lambda(\sigma)\in\Gamma}}\exp\Big[-\beta H^{\rm s}_{\Lambda,0}(\sigma_\Lambda|\bar\sigma)-\beta|\Lambda|\frac{\lambda^2}{4}(q-m_\Lambda(\sigma))^2\Big]\end{aligned} \tag{2.21}$$

Let $h_0(\beta,m)$ be the convex conjugate of the pressure $p_0(\beta,\alpha)$ and

$$G(\beta,q) \doteq \inf_{|m|\le 1}\Big\{\frac{\lambda^2}{4}(q-m)^2 + h_0(\beta,m)\Big\} \tag{2.22}$$

In order to prove (2.17) we need the following lemmata.

Lemma 2.2. *Let* $h(\beta,q)$ *as in* (1.20) *and* $G(\beta,q)$ *as in* (2.22). *Then*

$$h(\beta,q) = G(\beta,q) \tag{2.23}$$

Proof. We prove separately the two inequalities $h(\beta,q)\le G(\beta,q)$ and $h(\beta,q)\ge G(\beta,q)$. From (2.12) and the definition of $h_0(\beta,m)$, for any α, $q\in\mathbb{R}$ and $m\in[-1,1]$,

$$\begin{aligned}\alpha q - p(\beta,\alpha) &\le \alpha q - \alpha m + h_0(\beta,m) - \Big(\frac{\alpha}{\lambda}\Big)^2 \le\\ &\le \frac{\lambda^2}{4}(q-m)^2 + h_0(\beta,m) - \Big(\frac{\lambda}{2}(q-m)-\frac{\alpha}{\lambda}\Big)^2 \le\\ &\le \frac{\lambda^2}{4}(q-m)^2 + h_0(\beta,m)\end{aligned} \tag{2.24}$$

From (2.24) it follows immediately that $h(\beta, q) \le G(\beta, q)$. On the other hand, from the definition of $h_0(\beta, m)$, for any $\epsilon > 0$ there is $\bar{m} \in [-1, 1]$ such that

$$p_0(\beta, \alpha) \le \alpha\bar{m} - h_0(\beta, \bar{m}) + \epsilon$$

Then, by reasoning as in (2.24), for any α, $q \in \mathbb{R}$,

$$\alpha q - p(\beta, \alpha) \ge \frac{\lambda^2}{4}(q - \bar{m})^2 + h_0(\beta, \bar{m}) - \Big(\frac{\lambda}{2}(q - \bar{m}) - \frac{\alpha}{\lambda}\Big)^2 - \epsilon \tag{2.25}$$

By choosing $\bar{\alpha} = \lambda^2(q - \bar{m})/2$, (2.25) becomes

$$\bar{\alpha} q - p(\beta, \bar{\alpha}) \ge \frac{\lambda^2}{4}(q - \bar{m})^2 + h_0(\beta, \bar{m}) - \epsilon \tag{2.26}$$

From (2.26) we get $h(\beta, q) \ge G(\beta, q) - \epsilon$ for any $\epsilon > 0$, that is $h(\beta, q) \ge G(\beta, q)$. ∎

LEMMA 2.3. *Let*

$$P^q_{n,\beta,\bar{\sigma}}(dm) \doteq \nu^q_{\Lambda_n,\beta,\bar{\xi}}(m_{\Lambda_n}(\sigma) \in dm) \tag{2.27}$$

Then $\{(P^q_{n,\beta,\bar{\sigma}}, \beta|\Lambda_n|);\, n \in \mathbb{N}\}$ *has the large deviation property with rate function*

$$I_{\beta,q}(m) \doteq \frac{\lambda^2}{4}(q - m)^2 + h_0(\beta, m) - h(\beta, q) \tag{2.28}$$

PROOF. From (2.21) and definition (2.27) we have

$$\begin{aligned} P^q_{n,\beta,\bar{\sigma}}(dm) = \Big(\int Q_{n,\beta,\bar{\sigma}}(dm') \exp\Big[-\beta|\Lambda_n|\frac{\lambda^2}{4}(q - m')^2\Big]\Big)^{-1} \times \\ \times \exp\Big[-\beta|\Lambda_n|\frac{\lambda^2}{4}(q - m)^2\Big] Q_{n,\beta,\bar{\sigma}}(dm) \end{aligned} \tag{2.29}$$

where $Q_{n,\beta,\bar{\sigma}}$ is the probability distribution of $m_{\Lambda_n}(\sigma)$ with respect to the grand canonical Gibbs measure of the Ising model in the volume Λ_n, inverse temperature β, with 0 external magnetic field and boundary condition $\bar{\sigma}$.

It is known, see [5], that $\{(Q_{n,\beta,\bar\sigma}, \beta|\Lambda_n|);\ n \in \mathbb{N}\}$ has a large deviation property with rate function

$$I^{s}_{\beta}(m) \doteq h_0(\beta, m) + p_0(\beta, 0) \tag{2.30}$$

By (2.29) and (2.30), from Theorem 2.7.2 of [5] we get that $\{(P^{q}_{n,\beta,\bar\sigma}, \beta|\Lambda_n|);\ n \in \mathbb{N}\}$ has a large deviation property with rate function

$$I_{\beta,q}(m) = I^{s}_{\beta}(m) + \frac{\lambda^2}{4}(q-m)^2 - \inf_{|w|\le 1}\left\{I^{s}_{\beta}(w) + \frac{\lambda^2}{4}(q-w)^2\right\} \tag{2.31}$$

From Lemma 2.2 and the definition (2.30) the Lemma follows. □

Lemma 2.4. *Let $I_{\beta,q}(m)$ be as in* (2.28). *Then, for any $\beta > 0$ and $q \in \mathbb{R}$, $I_{\beta,q}(m) = 0$ if and only if*

$$q - m = \frac{2}{\lambda^2}\frac{\partial h}{\partial q}(\beta, q) \tag{2.32}$$

Proof. From standard results on the Ising Model, it is known that, for any $\beta > 0$, $h_0(\beta, m)$ is a convex and continuous function for $m \in [-1, 1]$ and differentiable for $m \in (-1, 1)$. Moreover

$$\frac{\partial h_0}{\partial m}(\beta, m) = 0 \quad \forall\, m \in [-m_\beta, m_\beta], \qquad \lim_{m\to\pm 1}\frac{\partial h_0}{\partial m}(\beta, m) = \pm\infty \tag{2.33}$$

where m_β was introduced in (2.13). From (2.33) we get that, for any $\beta > 0$ and $q \in \mathbb{R}$, $I_{\beta,q}(m)$ achieves its minimum value in the (unique) solution $\bar m = \bar m(\beta, q)$ of the equation

$$q - \bar m = \frac{2}{\lambda^2}\frac{\partial h_0}{\partial m}(\beta, \bar m) \tag{2.34}$$

It is easy to see that, for any $\beta > 0$, $\bar m(\beta, q)$ is a non decreasing function of q, differentiable for $|q| \ne m_\beta$ and such that

$$\bar m(\beta, q) = q \qquad \forall\, q \in [-m_\beta, m_\beta] \tag{2.35}$$

From Lemma 2.2 and the definition (2.28) we have

$$h(\beta, q) = \frac{\lambda^2}{4}(q - \bar{m}(\beta, q))^2 + h_0(\beta, \bar{m}(\beta, q)) \tag{2.36}$$

We first prove that $\bar{m}$ solves (2.32). For $|q| \leq m_\beta$ it is obvious because of (2.14) and (2.35). For $|q| > m_\beta$ we compute

$$\frac{\partial h}{\partial q}(\beta, q) = \Big[\frac{\lambda^2}{2}(\bar{m} - q) + \frac{\partial h_0}{\partial m}(\beta, \bar{m})\Big]\frac{\partial \bar{m}}{\partial q} + \frac{\lambda^2}{2}(q - \bar{m}) = \frac{\lambda^2}{2}(q - \bar{m}) \tag{2.37}$$

where, in the second equality, we have used (2.34).

Conversely, let m be such that (2.32) holds. Using (2.14) and (2.37) we have

$$m = q - \frac{2}{\lambda^2}\frac{\partial h}{\partial q} = \begin{cases} q & \text{if } |q| \leq m_\beta \\ \bar{m} & \text{if } |q| > m_\beta \end{cases} \tag{2.38}$$

Then, from (2.35), $m = \bar{m}$ for any $q \in \mathbb{R}$ and $\beta > 0$. □

We next conclude the proof of the ergodic theorem.

Proof of Theorem 2.1. Since for any $\delta > 0$,

$$\begin{aligned} &\nu^q_{\Lambda_n,\beta,\bar{\xi}}\Big(\Big|2\lambda^{-1} m_{\Lambda_n}(\phi) - \frac{2}{\lambda^2}\frac{\partial h}{\partial q}(\beta, q)\Big| > \delta\Big) = \\ &= P^q_{n,\beta,\bar{\sigma}}\Big(\Big|m_\Lambda(\sigma) - \Big[q - \frac{2}{\lambda^2}\frac{\partial h}{\partial q}(\beta, q)\Big]\Big| > \delta\Big) \end{aligned} \tag{2.39}$$

the property (2.17) follows immediately from Lemmata 2.3 and 2.4. Note in fact the uniformity in $\bar{\xi} \in X$, $q \in K$ follows from the uniformity of the limit in (2.10) w.r.t. $\bar{\xi} \in X$ and α in compacts.

We are left with the proof of the uniform integrability of the sequence $\{2\lambda^{-1} m_{\Lambda_n}(\phi)\}_{n\geq 1}$ with respect to the measures $\nu^q_{\Lambda_n,\beta,\bar{\xi}}$. This follows by (2.39) noticing that $m_\Lambda(\sigma)$ is uniformly bounded and the family $\{P^q_{n,\beta,\bar{\sigma}}(dm)\}$ is tight. The Theorem is thus proven. □

3 – Local gibbs states.

In this Section we consider a distribution density f_N^t with respect to ν_N (see Section 1 for the notation) which solves the forward eq. (1.13) with an initial datum f_N^0 satisfying the entropy bound (1.16). We shall prove that, for N large and any $t > 0$, the measure $f_N^t d\nu_N$ behaves as a "local Gibbs state". Roughly speaking, a local Gibbs state is any measure μ on X_N such that

$$\mu(d\xi) \sim \exp\Big[\sum_{x\in S_N^d} \alpha(x)\omega(x)\Big]\nu_N(d\xi) \tag{3.1}$$

where $\alpha(x) = \bar\alpha(x/N)$ with $\bar\alpha$ a smooth function on the d-dimensional torus. In this case we say that μ corresponds to the macroscopic charge density $q(\theta) = \partial_\alpha p(\beta, \bar\alpha(\theta))$.

For large N, the measure $f_N^t d\nu_N$ locally looks as a grand canonical Gibbs state with an external field $\alpha = \partial_q h(\beta, q)$ where q equals the average of the conserved quantity ω in a "macroscopically small" but "microscopically large" subset of S_N^d. We state this rigorously in the next theorem, which is the main result of this Section.

THEOREM 3.1. *Let f_N^t be solution of* (1.13) *with an initial condition f_N^0 for which* (1.16) *holds. For any $x \in S_N^d$ and $n \le N$ let $\Lambda_n(x) = x + \Lambda_n$ with Λ_n as in* (2.15)*. Finally, for any ϵ, $k > 0$, $n \le N$ and $t \ge 0$, define*

$$\begin{aligned} A_{n,N}^{\epsilon,k}(t) \doteq \frac{1}{N^d} \sum_{x\in S_N^d} \int_0^t ds \int \nu_N(d\xi)\, f_N^s(\xi) \big|2\lambda^{-1} m_{\Lambda_n(x)}(\phi) + \\ - \psi_k \circ F\big(m_{\Lambda_{N\epsilon}}(\omega)\big)\big| \end{aligned} \tag{3.2}$$

where F is defined in (1.24) *and*

$$\psi_k(r) \doteq \begin{cases} r & \textit{if } |r| \le k \\ \mathrm{sign}(r) k & \textit{if } |r| > k \end{cases} \tag{3.3}$$

Then, for any $t \ge 0$,

$$\lim_{k\to\infty} \limsup_{\epsilon\to 0} \limsup_{n\to\infty} \limsup_{N\to\infty} A_{n,N}^{\epsilon,k}(t) = 0 \tag{3.4}$$

We fix $t > 0$ and define (by omitting the explicit dependence on t)

$$f_N(\xi) \doteq \frac{1}{N^d} \sum_{x \in S_N^d} \frac{1}{t} \int_0^t ds\, f_N^s(\tau_x \xi) \tag{3.5}$$

where, for any $x \in S_N^d$, τ_x is the shift operator of X_N.

Let P^N be the measure $f_N d\nu_N$. Clearly P^N is a shift invariant measure on X_N and

$$A_{n,N}^{\epsilon,k}(t) = t\mathbb{E}^{P^N} |2\lambda^{-1} m_{\Lambda_n}(\phi) - \psi_k \circ F(m_{\Lambda_{N\epsilon}}(\omega))|$$

where $\mathbb{E}^\mu$ denotes the expectation with respect to the measure μ. Then (3.4) can be rewritten as

$$\lim_{k\to\infty} \limsup_{\epsilon\to 0} \limsup_{n\to\infty} \limsup_{N\to\infty} \mathbb{E}^{P^N} |2\lambda^{-1} m_{\Lambda_n}(\phi) - \psi_k \circ F(m_{\Lambda_{N\epsilon}}(\omega))| = 0. \tag{3.6}$$

We split the proof of (3.6) into several steps.

THEOREM 3.2. *The 1-block estimate.*
For any $\delta > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} P^N \Big(|2\lambda^{-1} m_{\Lambda_n}(\phi) - F(m_{\Lambda_n}(\omega))| > \delta \Big) = 0\,. \tag{3.7}$$

The proof of this Theorem is based on a characterization of the limit points of the sequence $\{P^N\}$, which will be deduce from the entropy bounds. We first need some preliminaries.

We denote by $\mathcal{M}(X_N)$ the space of probability measures on X_N and by $\mathcal{M}_\tau(X_N)$ the space of the shift invariant ones. For any $\mu \in \mathcal{M}(X_N)$ which has a density f with respect to ν_N we define the entropy functional

$$\mathcal{H}_N(\mu) \doteq \int \nu_N(d\xi)\, f(\xi) \log f(\xi) \tag{3.8}$$

The following variational characterization of $\mathcal{H}_N$ is well known, see e.g. [4],

$$\mathcal{H}_N(\mu) = \sup \Big\{ \int \mu(d\xi) U(\xi) - \log \int \nu_N(d\xi) \exp[U(\xi)] : U \in C_b(X_N) \Big\} \tag{3.9}$$

where $C_b(X_N)$ denotes the set of continuous and bounded functions on X_N. Note that (3.9) implies that $\mathcal{H}_N$ is positive. Let

$$\sigma_N(\mu) \doteq -\frac{1}{N^2}\frac{d}{dt}\Big|_{t=0}\mathcal{H}_N(\mu e^{L_N t}) = = -\frac{1}{N^2}\int \nu_N(d\xi)\,[L_N f(\xi)]\,\log f(\xi) \tag{3.10}$$

(in the last equality we used the fact that f is a probability density). By an explicit computation one easily gets

$$\sigma_N(\mu) = \frac{1}{2}\int \nu_N(d\xi) \sum_{\substack{x,y\in S_N^d\\ |x-y|=1}} \frac{1}{f(\xi)}\Big(\frac{\partial f}{\partial\phi(x)} - \frac{\partial f}{\partial\phi(y)}\Big)^2(\xi) + - \int \nu_N(d\xi) \sum_{x\in S_N^d} c_\beta(x,\xi)[f(\delta_x\xi) - f(\xi)]\log f(\xi) \tag{3.11}$$

Notice that, by symmetry,

$$- \int \nu_N(d\xi) \sum_{x\in S_N^d} c_\beta(x,\xi)[f(\delta_x\xi) - f(\xi)]\log f(\xi) = = \frac{1}{2}\int \nu_N(d\xi) \sum_{x\in S_N^d} c_\beta(x,\xi)[f(\delta_x\xi) - f(\xi)][\log f(\delta_x\xi) - \log f(\xi)] \tag{3.12}$$

Using the basic inequality

$$2(\sqrt{u} - \sqrt{v})^2 \le (u-v)(\log u - \log v) \qquad \forall\, u,v \ge 0 \tag{3.13}$$

we have the estimate

$$\sigma_N(\mu) \ge D_N(\sqrt{f}) \tag{3.14}$$

where

$$D_N(f) \doteq \frac{1}{2}\int \nu_N(d\xi) \sum_{\substack{x,y\in S_N^d\\ |x-y|=1}} \Big(\frac{\partial f}{\partial\phi(x)} - \frac{\partial f}{\partial\phi(y)}\Big)^2(\xi) + + \int \nu_N(d\xi) \sum_{x\in S_N^d} c_\beta(x,\xi)[f(\delta_x\xi) - f(\xi)]^2 \tag{3.15}$$

is the Dirichlet form associated to the generator L_N in (1.9).

Then we have the following result.

LEMMA 3.3. *Let f_N be as in* (3.5) *and C be the constant that appears in* (1.16). *Then, for any $t \geq 0$,*

$$\frac{1}{N^d}\int \nu_N(d\xi)\, f_N(\xi)\log f_N(\xi) \leq C \tag{3.16}$$

and

$$\frac{1}{N^d} D_N(\sqrt{f_N}) \leq \frac{C}{N^2 t} \tag{3.17}$$

PROOF. Let $\mu_N^t \doteq f_N^t d\nu_N$. From definition (3.10) we get

$$\mathcal{H}_N(\mu_N^0) = \mathcal{H}_N(\mu_N^t) + N^2 \int_0^t ds\, \sigma_N(\mu_N^s) \tag{3.18}$$

Since both $\mathcal{H}_N$ and σ_N are positive, from (1.16) and (3.18), for any $t \geq 0$,

$$\mathcal{H}_N(\mu_N^t) \leq C N^d, \qquad \int_0^t ds\, \sigma_N(\mu_N^s) \leq C N^{d-2} \tag{3.19}$$

Finally, since $\mathcal{H}_N$ and σ_N are convex and shift invariant functionals, the Lemma follows from the definitions (3.5), (3.8) and the bounds (3.14), (3.19). □

Now we can study the sequence $\{P^N\}$.

LEMMA 3.4. *Let Π_n be the projection on X_{Λ_n}. For any $n \in \mathbb{N}$ the sequence $\{\Pi_n(P^N)\}$ of probabilities on X_{Λ_n} is tight. Let also*

$$\mathcal{F} \doteq \{\mu \in \mathcal{M}_\tau(X) \,:\, \Pi_n(\mu) \textit{ is a w-limit point of } \{\Pi_n(P^N)\}\ \forall\, n \in \mathbb{N}\}$$

Then

$$\sup_{\mu\in\mathcal{F}} \sup_{n\in\mathbb{N}} \mathbb{E}^\mu \frac{1}{|\Lambda_n|} \sum_{x\in\Lambda_n} |\phi(x)|^\gamma < \infty \qquad \forall\, \gamma \in [0,2) \tag{3.20}$$

and

$$\mathcal{F} \subseteq \mathcal{G}^c \tag{3.21}$$

Proof. We recall the "basic entropy estimate": if μ, ν are two probability measures such that $\mu \ll \nu$, for any $g \in L^1(d\nu)$,

$$\int d\mu\, g \leq \log \int d\nu \exp[g] + \int d\mu \log \frac{d\mu}{d\nu} \tag{3.22}$$

so that

$$\begin{aligned} &\mathbb{E}^{P^N} \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} |\phi(x)|^\gamma = \mathbb{E}^{P^N} \frac{1}{N^d} \sum_{x \in S_d^N} |\phi(x)|^\gamma \leq \\ &\leq \frac{1}{N^d} \log \int \nu_N(d\xi)\, \exp\Big[\sum_{x \in S_d^N} |\phi(x)|^\gamma \Big] + \\ &\quad + \frac{1}{N^d} \int \nu_N(d\xi)\, f_N(\xi) \log f_N(\xi) \end{aligned} \tag{3.23}$$

From (3.16), (3.23) and the definition (1.10) of the reference measure ν_N, there is a constant C_γ so that, for any $n \leq N$,

$$\mathbb{E}^{P^N} \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} |\phi(x)|^\gamma \leq C_\gamma < \infty \qquad \forall\, \gamma \in [0, 2) \tag{3.24}$$

This proves both the tightness of the family $\{\Pi_n(P^N)\}_{N \geq n}$ and the estimate (3.20).

We are left with the proof of (3.21). Consider the generators

$$\begin{aligned} L_{x,y}^{(0)} &= \frac{1}{2}\Big(\frac{\partial}{\partial\phi(x)} - \frac{\partial}{\partial\phi(y)}\Big)^2 - \beta(\phi(x) - \phi(y))\Big(\frac{\partial}{\partial\phi(x)} - \frac{\partial}{\partial\phi(y)}\Big) \\ L_x^{(1)} &= c_\beta(x, \xi)[\delta_x - 1] \end{aligned}$$

and define, for any $\Lambda \subset\subset \mathbb{Z}^d$

$$L_\Lambda^{(0)} \doteq \sum_{\substack{x,y \in \Lambda \\ |x-y|=1}} L_{x,y}^{(0)}, \qquad L_\Lambda^{(1)} \doteq \sum_{x \in \Lambda} L_x^{(1)}$$

We introduce also the following forms

$$I_\Lambda^{(i)}(\mu) = \sup\Big\{ \int d\mu \, \frac{-L_\Lambda^{(i)} u}{u} \;:\; u \in \mathcal{D}(L_\Lambda^{(i)}),\ u \geq 1 \Big\}, \quad i = 0, 1$$

defined for any $\mu \in \mathcal{M}(X_V)$ for some V such that $\Lambda \subset V$ (if $i = 1$ we require also that $\mathrm{dist}(\Lambda, \partial V) \geq 1$). $\mathcal{D}(L_\Lambda^{(i)})$ is the domain of the Feller generator $L_\Lambda^{(i)}$. When $\mu \in \mathcal{M}(X_V)$, the supremum is over $\mathcal{D}(L_\Lambda^{(i)}|_V) \subset C_b(X_V)$. We refer to [4] for general properties of these forms. Clearly

$$I_\Lambda^{(0)}(\mu) = \sum_{\substack{x,y\in\Lambda \\ |x-y|=1}} I_{x,y}^{(0)}(\mu), \quad I_\Lambda^{(1)}(\mu) = \sum_{x\in\Lambda} I_x^{(1)}(\mu) \tag{3.25}$$

where $I_{x,y}^{(0)}$ and $I_x^{(1)}$ are defined analogously. It can be proven that

$$I_{x,y}^{(0)}(P^N) \leq D_{x,y}^{(0)}(\sqrt{f_N}), \quad I_x^{(1)}(P^N) \leq D_x^{(1)}(\sqrt{f_N}) \tag{3.26}$$

where

$$\begin{aligned} D_{x,y}^{(0)}(f) &\doteq \int \nu_N(d\xi) \Big(\frac{\partial f}{\partial \phi(x)} - \frac{\partial f}{\partial \phi(y)}\Big)^2(\xi), \quad D_x^{(1)}(f) \doteq \\ &\doteq \int \nu_N(d\xi)\, c_\beta(x,\xi)[f(\delta_x\xi) - f(\xi)]^2 \end{aligned} \tag{3.27}$$

(see [4, Chapter 4] for general relations between the previous Dirichlet forms and the forms on measures defined above). From the definitions (3.15) and (3.27), the estimates (3.17) and (3.26), since P^N is shift invariant, we have, for any x, $y \in S_N^d$, $|x - y| = 1$,

$$I_{x,y}^{(0)}(P^N) \leq \frac{C}{N^2 td}, \quad I_x^{(1)}(P^N) \leq \frac{C}{N^2 t} \tag{3.28}$$

Notice that

$$I_{x,y}^{(0)}(P^N) \geq I_{x,y}^{(0)}(\Pi_n(P^N)) \qquad \forall\, x,y \in \Lambda_n,\ |x-y| = 1 \tag{3.29}$$

$$I_x^{(1)}(P^N) \geq I_x^{(1)}(\Pi_n(P^N)) \qquad \forall\, x \in \Lambda_{n-1} \tag{3.30}$$

Using the lower semicontinuity of the functionals $I_\Lambda^{(i)}$, from (3.28), (3.29) and (3.30), for any $\mu \in \mathcal{F}$,

$$I_\Lambda^{(0)}(\Pi_n(\mu)) = 0 \qquad \forall\, n \in \mathbb{N} : \Lambda \subseteq \Lambda_n, \quad \forall\, \Lambda \subset\subset \mathbb{Z}^d \tag{3.31}$$

$$I_\Lambda^{(1)}(\Pi_n(\mu)) = 0 \qquad \forall\, n \in \mathbb{N} : \Lambda \subseteq \Lambda_{n-1}, \quad \forall\, \Lambda \subset\subset \mathbb{Z}^d \tag{3.32}$$

We conclude the proof of (3.21) by showing that any measure $\mu \in \mathcal{M}_\tau(X)$ satisfying (3.31) and (3.32) belongs to $\mathcal{G}^c$.

From (3.31), (3.32) and the definition of $I_\Lambda^{(i)}$ we have

$$\int d\mu \, \frac{L_\Lambda^{(i)} u}{u} \geq 0, \qquad i = 0, 1 \tag{3.33}$$

for any cylindrical function $u \geq 1$ which belongs to the domain of $L_\Lambda^{(i)}$. Among these we can choose $u = \exp[sv]$ where $s \geq 0$ and $v \in C^\infty_{0,\mathrm{loc}}(X)$, the class of cylinder functions, which are smooth and with compact support in their variables. By varying s we get, from (3.33),

$$\int d\mu \, L_\Lambda^{(i)} v = 0 \qquad \forall\, v \in C^\infty_{0,\mathrm{loc}}(X), \qquad i = 0, 1 \tag{3.34}$$

If we choose $v(\xi) = f(\xi_\Lambda) g(\xi)$, with $f \in C_0^\infty(X_\Lambda)$ and g a cylinder function depending only on ξ_{Λ^c}, then $L_\Lambda^{(i)} v = g L_\Lambda^{(i)} f$ so that

$$\int \mu(d\bar{\xi}) \, g(\bar{\xi}) \int \mu^q_{\Lambda,\bar{\xi}}(d\xi_\Lambda) L_\Lambda^{(i)} f(\xi_\Lambda) = 0, \qquad i = 0, 1 \tag{3.35}$$

where

$$\mu^q_{\Lambda,\bar{\xi}} \doteq \mu(\cdot | \xi_{\Lambda^c} = \bar{\xi}_{\Lambda^c},\, m_\Lambda(\omega) = m_\Lambda(\bar{\omega}) = q)$$

Varying g, from (3.35) we get

$$\int \mu^q_{\Lambda,\bar{\xi}}(d\xi_\Lambda) L_\Lambda^{(i)} f(\xi_\Lambda) = 0 \quad \forall\, f \in C_0^\infty(X_\Lambda), \qquad i = 0, 1 \tag{3.36}$$

for almost all $\bar{\xi}$ with respect to μ (note that in (3.36) we can add any constant to f). We rewrite (3.36) as

$$\sum_{\{\sigma_\Lambda\}} \mu^q_{\Lambda,\bar{\xi}}(\sigma_\Lambda) \int \mu^q_{\Lambda,\bar{\xi}}(d\phi_\Lambda | \sigma_\Lambda) L_\Lambda^{(i)} f(\xi_\Lambda) = 0, \qquad i = 0, 1 \tag{3.37}$$

where $\mu^q_{\Lambda,\bar{\xi}}(\sigma_\Lambda)$ denotes the marginal distribution of the spin component of ξ_Λ.

We consider (3.37) with $i = 0$ for the class of functions $f(\xi_\Lambda)$ of the form $\mathbf{1}_{\sigma_\Lambda} f_1(\phi_\Lambda)$, with $\mathbf{1}_{\sigma_\Lambda}$ the characteristic function of the spin configuration σ_Λ and $f_1 \in C_0^\infty(\mathbb{R}^\Lambda)$. For such functions we have

$$\int \mu^q_{\Lambda,\bar{\xi}}(d\xi_\Lambda) L_\Lambda^{(0)} f(\xi_\Lambda) = \mu^q_{\Lambda,\bar{\xi}}(\sigma_\Lambda) \int \mu^q_{\Lambda,\bar{\xi}}(d\phi_\Lambda | \sigma_\Lambda) L_\Lambda^{(0)} f_1(\phi_\Lambda) = 0 \tag{3.38}$$

Since the $L_\Lambda^{(0)}$ is elliptic on the hyperplanes $2\lambda^{-1}m_\Lambda(\phi) = q - m_\Lambda(\sigma)$ from (3.38) we get that, for almost all $\bar\xi$,

$$\mu^q_{\Lambda,\bar\xi}(d\phi_\Lambda|\sigma_\Lambda) = \rho^{q-m_\Lambda(\sigma)}_{\Lambda,\beta}(d\phi_\Lambda)\ \forall\,\sigma_\Lambda \in \{-1,1\}^\Lambda : \mu^q_{\Lambda,\bar\xi}(\sigma_\Lambda) > 0 \tag{3.39}$$

where $\rho^{q-m_\Lambda(\sigma)}_{\Lambda,\beta}$ was defined just after (2.19).

Now we consider (3.37) with $i = 1$ for the class of functions f which depends only on the spin component of ξ_Λ. Using (3.39) and the definition of $L_\Lambda^{(0)}$ we get

$$\sum_{\{\sigma_\Lambda\}} \mu^q_{\Lambda,\bar\xi}(\sigma_\Lambda) \sum_{x\in\Lambda} \hat c_\beta(x,\sigma_\Lambda)[f(\sigma^x_\Lambda) - f(\sigma_\Lambda)] = 0 \tag{3.40}$$

where σ^x_Λ is the spin configuration obtained from σ_Λ by flipping the spin at x, while

$$\hat c_\beta(x,\sigma_\Lambda) = \int \rho^{q-m_\Lambda(\sigma)}_{\Lambda,\beta}(d\phi_\Lambda)\, c_\beta(x,\xi_\Lambda)$$

From definition (1.7) and an explicit computation it is not difficult to verify that

$$\hat c_\beta(x,\sigma_\Lambda) \propto \exp\Big[-\frac{\beta}{2}\Big(H^{(q)}_\Lambda(\sigma^x_\Lambda|\bar\sigma) - H^{(q)}_\Lambda(\sigma_\Lambda|\bar\sigma)\Big)\Big] \tag{3.41}$$

where (see (2.4))

$$H^{(q)}_\Lambda(\sigma_\Lambda|\bar\sigma) \doteq H^{\mathrm{s}}_{\Lambda,0}(\sigma_\Lambda|\bar\sigma) + |\Lambda|\frac{\lambda^2}{4}(q - m_\Lambda(\sigma))^2 \tag{3.42}$$

Since the state space $\{-1,1\}^\Lambda$ is finite, from (3.40), (3.41) and (3.42) it follows immediately that

$$\mu^q_{\Lambda,\bar\xi}(\sigma_\Lambda) \propto \exp\Big[-\beta H^{\mathrm{s}}_{\Lambda,0}(\sigma_\Lambda|\bar\sigma) - \beta|\Lambda|\frac{\lambda^2}{4}(q - m_\Lambda(\sigma))^2\Big] \tag{3.43}$$

Comparing (3.39) and (3.43) with (2.19) we finally get, for almost all $\bar\xi$ with respect to μ,

$$\mu^q_{\Lambda,\bar\xi}(d\xi_\Lambda) = \nu^q_{\Lambda,\beta,\bar\xi}(d\xi_\Lambda) \qquad \forall\,\Lambda \subset\subset \mathbb{Z}^d$$

and thus $\mu \in \mathcal{G}^c$. □

Proof of Theorem 3.2. From the tightness of the families $\{\Pi_n (P^N)\}$ and the definition of $\mathcal{F}$ we have

$$\begin{aligned}
&\limsup_{N\to\infty} P^N\Big(\big|2\lambda^{-1}m_{\Lambda_n}(\phi) - F\big(m_{\Lambda_n}(\omega)\big)\big| > \delta\Big) \le \\
&\le \sup_{\mu\in\mathcal{F}} \mu\Big(\big|2\lambda^{-1}m_{\Lambda_n}(\phi) - F\big(m_{\Lambda_n}(\omega)\big)\big| > \delta\Big) \qquad \forall\, n \in \mathbb{N}
\end{aligned} \tag{3.44}$$

and, by using (3.21),

$$\begin{aligned}
&\mu(|2\lambda^{-1}m_{\Lambda_n}(\phi) - F(m_{\Lambda_n}(\omega))| > \delta) = \\
&= \int \mu(d\bar\xi)\, \nu^q_{\Lambda,\beta,\bar\xi}(|2\lambda^{-1}m_{\Lambda_n}(\phi) - F(q)| > \delta)|_{q=m_{\Lambda_n}(\bar\omega)}
\end{aligned} \tag{3.45}$$

Now we observe that, for any $\mu\in\mathcal{F}$, $n\in\mathbb{N}$ and $p>0$,

$$\begin{aligned}
\mu(|m_{\Lambda_n}(\bar\omega)| > p) &\le \mu\Big(|m_{\Lambda_n}(\bar\phi)| > \frac{\lambda}{2}(p-1)\Big) \le \\
&\le \frac{2}{\lambda(p-1)}\mathbb{E}^\mu|m_{\Lambda_n}(\bar\phi)| \le \frac{2}{\lambda(p-1)}\mathbb{E}^\mu \frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}|\bar\phi(x)|
\end{aligned}$$

Then, by (3.20) for $\gamma = 1$,

$$\limsup_{p\to\infty}\limsup_{n\to\infty}\sup_{\mu\in\mathcal{F}} \mu\Big(|m_{\Lambda_n}(\bar\omega)| > p\Big) = 0 \tag{3.46}$$

From (3.44), (3.45) and (3.46) we have

$$\begin{aligned}
&\lim_{n\to\infty}\limsup_{N\to\infty} P^N(|2\lambda^{-1}m_{\Lambda_n}(\phi) - F(m_{\Lambda_n}(\omega))| > \delta) \le \\
&\le \limsup_{p\to\infty}\limsup_{n\to\infty}\sup_{\mu\in\mathcal{F}} \int\limits_{|m_{\Lambda_n}(\bar\omega)|\le p} \mu(d\bar\xi)\,\nu^q_{\Lambda,\beta,\bar\xi}(|2\lambda^{-1}m_{\Lambda_n}(\phi) - \\
&\quad - F(q)| > \delta)|_{q=m_{\Lambda_n}(\bar\omega)}
\end{aligned} \tag{3.47}$$

Recalling the definition (1.24) of F, by Chebyshev's inequality, Theorem 2.1 and (3.47) we finally get (3.7). The theorem is proven. $\square$

THEOREM 3.5. *The two blocks estimate.*

Let $T_{\epsilon,n,N} = \{x \in S_N^d \,:\, \Lambda_n(x) \subseteq \Lambda_{N\epsilon},\ \Lambda_{n+1} \cap \Lambda_{n+1}(x) = \emptyset\}$. *Then, for any* $\delta > 0$,

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\limsup_{N\to\infty}\sup_{x\in T_{\epsilon,n,N}} P^N(|F(m_{\Lambda_n}(\omega)) - F(m_{\Lambda_n(x)}(\omega))| > \delta) = 0. \tag{3.48}$$

PROOF. For any $x \in T_{\epsilon,n,N}$ let $\Pi_{n,x}$ be the projection on $X_{\Lambda_n} \times X_{\Lambda_n(x)}$. Fix a point x_0 such that $\Lambda_{n+1} \cap \Lambda_{n+1}(x_0) = \emptyset$. By changing variables we can view any projection $\Pi_{n,x}(P^N)$ as a probability on $X_{\Lambda_n} \times X_{\Lambda_n(x_0)}$. Call $\mathcal{F}_{\epsilon,n}^N$ the collection of such probabilities associated to $\{\Pi_{n,x}(P^N);\, x \in T_{\epsilon,n,N}\}$. By arguing as in (3.23) we have an estimate like (3.24) with Λ_n replaced by $\Lambda_n \cup \Lambda_n(x)$. From this we get the tightness of the family of probabilities $\{\mathcal{F}_{\epsilon,n}^N;\, N \in \mathbb{N}\}$. We denote by $\mathcal{F}_{\epsilon,n}$ the set of all its limit points. Let $\mu_{\epsilon,n} \in \mathcal{F}_{\epsilon,n}$. By using (3.28) and reasoning as in the proof of (3.34) we get

$$\begin{gathered}\int d\mu_{\epsilon,n}\, L_{\Lambda_n}^{(i)} v = \int d\mu_{\epsilon,n}\, L_{\Lambda_n(x_0)}^{(i)} v = 0 \\ \forall\, v \in C_0^\infty(X_{\Lambda_n} \times X_{\Lambda_n(x_0)}), \qquad i = 0, 1\end{gathered} \tag{3.49}$$

Then, by arguing as in the proof of (3.21), we get

$$\begin{aligned}&\mu_{\epsilon,n}(d\xi_{\Lambda_n}, d\xi_{\Lambda_n(x_0)}) = \\ &= \int \gamma_{\epsilon,n}(dq_1, dq_2, d\bar\xi)\, \nu_{\Lambda_n,\beta,\bar\xi}^{q_1}(d\xi_{\Lambda_n}) \nu_{\Lambda_n(x_0),\beta,\bar\xi}^{q_2}(d\xi_{\Lambda_n(x_0)})\end{aligned} \tag{3.50}$$

for some probability measure $\gamma_{\epsilon,n}$ on $\mathbb{R}^2 \times X$.

Consider now the diffusion generator $L_{0,x}^{(0)}$ and the associated forms $I_{0,x}^{(0)}$ and $D_{0,x}^{(0)}$. Let C be the constant that appears in (1.16). From the proof of Lemma 3.2 in [6] we have,

$$\sup_{x\in T_{\epsilon,n,N}} D_{0,x}^{(0)}(\sqrt{f_N}) \le \frac{C\epsilon^2 d}{4t} \tag{3.51}$$

and, by (3.26) and the analogous of (3.29),

$$I_{0,x}^{(0)}(\Pi_{n,x}(P^N)) \le I_{0,x}^{(0)}(P^N) \le D_{0,x}^{(0)}(\sqrt{f_N}) \qquad \forall\, x \in S_N^d \tag{3.52}$$

Recalling the definition of $\mathcal{F}^N_{\epsilon,n}$, from (3.51), (3.52) and the lower semi-continuity of $I^{(0)}_{n,x}$ (and of $I^{(0)}_{n,x_0}$), for any limit point $\mu_{\epsilon,n}$,

$$I^{(0)}_{0,x_0}(\mu_{\epsilon,n}) \leq \frac{C\epsilon^2 d}{4t} \tag{3.53}$$

From (3.53) and the definition of $I^{(0)}_{n,x_0}$ it follows that

$$\int d\mu_{\epsilon,n} \frac{L^{(0)}_{0,x_0} u}{u} \geq -\frac{C\epsilon^2 d}{4t} \qquad \forall\, u \in \mathcal{D}(L^{(0)}_{0,x_0}),\;\; u \geq 1 \tag{3.54}$$

From the definition of $L^{(0)}_{0,x_0}$, we can choose

$$u(\xi_{\Lambda_n}, \xi_{\Lambda_n(x_0)}) = \exp[|\Lambda_n| v(m_{\Lambda_n}(\omega), m_{\Lambda_n(x_0)}(\omega))], \qquad v \in C^2_0(\mathbb{R}^2),\; v \geq 0$$

Using the representation (3.50) and the explicit form (2.19) of the canonical Gibbs measures, from the definition of $L^{(0)}_{0,x_0}$ it is easy to verify that (3.54) gives

$$\begin{aligned} &\int \gamma_{\epsilon,n}(dq_1, dq_2, d\bar{\xi}) \nu^{q_1}_{\Lambda_n,\beta,\bar{\xi}}(d\xi_{\Lambda_n}) \nu^{q_2}_{\Lambda_n(x_0),\beta,\bar{\xi}}(d\xi_{\Lambda_n(x_0)}) \Big\{ \frac{1}{|\Lambda_n|} \Big(\frac{\partial}{\partial q_1} - \frac{\partial}{\partial q_2} \Big)^2 v + \\ &\quad + \Big(\frac{\partial v}{\partial q_1} - \frac{\partial v}{\partial q_2} \Big)^2 - 2\beta (m_{\Lambda_n}(\phi) - m_{\Lambda_n(x_0)}(\phi)) \Big(\frac{\partial v}{\partial q_1} - \frac{\partial v}{\partial q_2} \Big) \Big\} \geq \\ &\geq -\frac{C\epsilon^2 d}{2t} \qquad \forall\, v \in C^2_0(\mathbb{R}^2),\; v \geq 0 \end{aligned} \tag{3.55}$$

By the estimate like (3.24) with Λ_n replaced by $\Lambda_n \cup \Lambda_n(x)$, since $|\omega(x')| \leq 1 + 2\lambda^{-1}|\phi(x')|$, for any $x' \in \mathbb{Z}^d$,

$$\sup_{n\in\mathbb{N}} \int \gamma_{\epsilon,n}(dq_1, dq_2, d\bar{\xi})\, (|q_1| + |q_2|) < \infty \tag{3.56}$$

so that the family of measures $\{\gamma_{\epsilon,n};\, n \in \mathbb{N}\}$ is tight. We let $n \to \infty$ in (3.55) and, since v has compact support, we can use Theorem 2.1. Then, recalling the definition (1.24), we get

$$\begin{aligned} &\int \gamma_\epsilon(dq_1, dq_2) \Big\{ \Big(\frac{\partial v}{\partial q_1} - \frac{\partial v}{\partial q_2} \Big)^2 - \beta\lambda (F(q_1) - F(q_2)) \Big(\frac{\partial v}{\partial q_1} - \frac{\partial v}{\partial q_2} \Big) \Big\} \geq \\ &\geq -\frac{C\epsilon^2 d}{2t} \qquad \forall\, v \in C^2_0(\mathbb{R}^2),\; v \geq 0 \end{aligned} \tag{3.57}$$

where γ_ϵ is any limit point of $\{\gamma_{\epsilon,n};\, n \in \mathbb{N}\}$. By density arguments the inequality (3.57) extends to any positive $v \in C_0^1(\mathbb{R}^2)$. Then we can choose

$$v_\tau(q_1, q_2) = \frac{\beta\lambda}{2}[h(\beta, q_1) + h(\beta, q_2) - 2 \inf_q h(\beta, q)]\chi_\tau(q_1, q_2)$$

where χ_τ is a mollification of the characteristic function of the set $[0, \tau] \times [0, \tau]$. Putting $v = v_\tau$ in (3.57) and letting $\tau \to \infty$ we have

$$\int \gamma_\epsilon(dq_1, dq_2)\,(F(q_1) - F(q_2))^2 \leq \frac{2C\epsilon^2 d}{\beta^2\lambda^2 t} \tag{3.58}$$

From the definition of γ_ϵ and the Chebyshev's inequality,

$$\begin{aligned} &\limsup_{n\to\infty} \limsup_{N\to\infty} \sup_{x\in T_{\epsilon,n,N}} P^N(|F(m_{\Lambda_n}(\omega)) - F(m_{\Lambda_n(x)}(\omega))| > \delta) \leq \\ &\leq \frac{1}{\delta^2} \int \gamma_\epsilon(dq_1, dq_2)\,(F(q_1) - F(q_2))^2 \end{aligned} \tag{3.59}$$

Then, letting $\epsilon \to 0$ in (3.59) and using (3.58), we get (3.48). □

From Theorem 3.5, Corollary 3.6 below follows. We state it without proof, since it is exactly the same as in [10, Thm. 3.6].

COROLLARY 3.6. *For any $\delta > 0$,*

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \limsup_{N\to\infty} P^N(|F(m_{\Lambda_n}(\omega)) - F(m_{\Lambda_{N\epsilon}}(\omega))| > \delta) = 0 \tag{3.60}$$

PROOF OF THEOREM 3.1. From Theorem 3.2, Corollary 3.6 and the definition 3.3, we get, for any $k > 0$ and $\delta > 0$,

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \limsup_{N\to\infty} P^N(|\psi_k \circ 2\lambda^{-1} m_{\Lambda_n}(\omega) - \psi_k \circ F(m_{\Lambda_{N\epsilon}}(\omega))| > \delta) = 0 \tag{3.61}$$

Since ψ_k is a bounded function from (3.61) we get also

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \limsup_{N\to\infty} \mathbb{E}^{P^N} |\psi_k \circ 2\lambda^{-1} m_{\Lambda_n}(\omega) - \psi_k \circ F(m_{\Lambda_{N\epsilon}}(\omega))| = 0 \tag{3.62}$$

To prove (3.6) and then Theorem 3.1 we only need the uniform integrability of $\{2\lambda^{-1} m_{\Lambda_n}(\phi)\}$ with respect to P^N. But this comes from the estimate (3.24) for $\gamma \in (1, 2)$. □

4 – Hydrodynamic limit

In this Section we prove a large deviation result for the empirical measure of the conserved quantity in equilibrium and complete the proof of the hydrodynamic scaling limit. Since the arguments are rather standard we will just sketch the proofs and refer to [6,10] for the details.

Let $\omega \doteq \{\omega(x) = \sigma(x) + 2\lambda^{-1}\phi(x), x \in S_N^d\}$ and denote by $\mu_N \doteq N^{-d} \sum_{x \in S_N^d} \omega(x)\delta_{x/N}$ the empirical measure associated to it (recall (1.12) and (1.14)). As ω is random, μ_N is a random element in $\mathcal{M}_s(S^d)$, the space of signed measure on the d-dimensional torus S^d. We consider $\mathcal{M}_s(S^d)$ equipped with the weak* topology as the dual of $C(S^d)$, the continuous function on S^d and denote the duality paring by $\langle \cdot, \cdot \rangle$. We first state an upper bound large deviation property in equilibrium.

THEOREM 4.1. *Let $\mathcal{Q}_N$ be the distribution of μ_N when ω is distributed according to the equilibrium measure $\nu_N(d\xi)$ defined in* (1.10). *The family $\{\mathcal{Q}_N\}$ has the upper bound large deviation property with the rate function*

$$K(\gamma) = \begin{cases} \beta \int d\theta \; h\Big(\beta, \frac{d\gamma}{d\theta}\Big) & \text{if } \gamma \ll d\theta \\ +\infty & \text{otherwise} \end{cases} \tag{4.1}$$

here $d\theta$ is the Lebesgue measure on S^d and $h(\beta, q)$ is defined in (1.20).

We remark the statement in the above Theorem means that for every closed subset $G \subset \mathcal{M}_s(S^d)$

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \mathcal{Q}_N\{G\} \le - \inf_{\gamma \in G} K(\gamma) \tag{4.2}$$

This equilibrium result has the following consequence when the distribution of ω satisfies the entropy bound with respect to ν_N. The proof of the implication is given in [6, Lemma 6.3].

COROLLARY 4.2. *Let $\widetilde{\mathcal{Q}}_N$ be the distribution of μ_N when ω is distributed according to a probability measure $f_N \nu_N(d\xi)$. Assume f_N satisfies the entropy bound* (1.16). *Then any weak limit $\mathcal{Q}$ of $\{\widetilde{\mathcal{Q}}_N\}$ satisfies $\mathcal{Q}\{\gamma : \gamma \ll d\theta\} = 1$ and*

$$\beta \, \mathbb{E}^{\mathcal{Q}}\Big(\int d\theta \; h\Big(\beta, \frac{d\gamma}{d\theta}\Big)\Big) \le C \tag{4.3}$$

where C is the constant in (1.16).

LEMMA 4.3. *For any continuous function J on S^d*

$$\lim_{N\to\infty}\frac{1}{N^d}\log\mathbb{E}^{\nu_N}(\exp\{N^d\langle\mu_N,J\rangle\})=\beta\int d\theta p(\beta,\beta^{-1}J(\theta))\doteq\Psi(J) \tag{4.4}$$

where $p(\beta,\alpha)$ has been defined in (1.19).

SKETCH OF THE PROOF. Recalling the definition (1.10) of ν_N and (2.11), (2.12), it is enough to show

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N^d}\Big[\log\sum_{\sigma\in\{-1,1\}^{S_N^d}}e^{-\beta H_N^s(\sigma)+\sum_{x\in S_N^d}\sigma(x)J(x/N)}-\log Z_N\Big]=\\&=\beta\int d\theta\ p_0(\beta,\beta^{-1}J(\theta))\end{aligned} \tag{4.5}$$

In order to prove (4.5) it is enough to partition S_N^d into smaller cubes in which $J(x/N)$ is almost constant and then use the definition of the pressure p_0. The details are as in [10, Thm. 4.3] and we omit them. □

PROOF OF THEOREM 4.1. Let us first prove that the family $\mathcal{Q}_N$ is *exponentially tight*, i.e. for each $L>0$ there exists a compact set C_L with the property that

$$\lim_{L\to\infty}\limsup_{N\to\infty}\frac{1}{N^d}\log\mathcal{Q}_N\{C_L^c\}=-\infty \tag{4.6}$$

where C^c denotes the complementar of C. We take $C_L\doteq\{\gamma:\|\gamma\|\le L\}$ in which $\|\gamma\|$ is the total variation of γ. It is a compact subset of $\mathcal{M}_s(S^d)$ and

$$\mathcal{Q}_N\{C_L^c\}=\nu_N\Big\{\frac{1}{N^d}\sum_{x\in S_N^d}|\omega(x)|>L\Big\}\le e^{-LN^d}\mathbb{E}^{\nu_N}\Big(e^{\sum_{x\in S_N^d}|\omega(x)|}\Big) \tag{4.7}$$

so that (4.6) follows.

From the exponential tightness, Lemma 4.3 and [4, Thm. 2.2.4] it follows $\mathcal{Q}_N$ satisfies the upper bound large deviation principle with rate function

$$K'(\gamma)\doteq\sup_J\{\gamma(J)-\Psi(J)\} \tag{4.48}$$

The proof that $K' = K$ is given in [10, Lemma 4.4]. It is based on the fact that we can, by applying Luzin's Theorem, extend the supremum in (4.8) to all J which are bounded and measurable. Since $h(\beta, q)$ is a convex and differentiable function of q, it is then not too difficult to prove the inequalities $K(\gamma) \le K'(\gamma)$ and $K'(\gamma) \le K(\gamma)$. □

We finally show how Corollary 4.2 and Theorem 3.1 imply our main result.

Proof of Theorem 1.1.

Step 1. Tightness. We start the microscopic process ξ_t from a distribution $f_N^0 d\nu_N$ where f_N^0 satisfies the entropy bound (1.16). To the process ξ_t it is associated a probability $\widetilde{\mathcal{P}}_N$ on the Skorohod space $D(\mathbb{R}^+; X_N)$. We shall denote by $\mathcal{P}_N$ the distribution of the empirical measure μ_N under $\widetilde{\mathcal{P}}_N$. Therefore $\mathcal{P}_N$ is a probability on $D(\mathbb{R}^+; \mathcal{M}_s(S^d))$. We want to show the tightness of the family $\{\mathcal{P}_N\}$.

The tightness follows from the following two estimates. For each $T > 0$

$$\lim_{L\to\infty} \limsup_{N\to\infty} \mathcal{P}_N\Big(\sup_{t\le T} \|\mu_N(t)\| > L \Big) = 0 \tag{4.9}$$

and, for each $\varepsilon > 0$ and any smooth function J on S^d

$$\lim_{\delta\downarrow 0} \limsup_{N\to\infty} \mathcal{P}_N\Big(\sup_{\substack{t,s\in[0,T]\\ |t-s|\le\delta}} |\langle \mu_N(t), J\rangle - \langle \mu_N(s), J\rangle| > \varepsilon \Big) = 0 \tag{4.10}$$

We also note the above estimates also imply any limit point of $\{\mathcal{P}_N\}$ is supported by $C(\mathbb{R}^+; \mathcal{M}_s(S^d))$, see [1, Thm. 15.5].

The bound (4.9) is proven in [6, Lemma 6.1] by using the entropy bound and an estimate from the Dirichlet form. The proof of (4.10) requires a martingale computation. We have that

$$\begin{aligned} M_t^N(J) &\doteq \langle \mu_N(t), J\rangle - \langle \mu_N(0), J\rangle - \int_0^t ds\ L_N \langle \mu_N(s), J\rangle = \\ &= \langle \mu_N(t), J\rangle - \langle \mu_N(0), J\rangle - \frac{4\beta}{\lambda} \int_0^t ds \frac{1}{N^d} \sum_{x\in S_N^d} \Delta_N J(x/N)\phi_s(x) \end{aligned} \tag{4.11}$$

where

$$\Delta_N J(x/N) \doteq N^2 \sum_{y\in S_N^d : |y-x|=1} [J(y/N) - J(x/N)] \tag{4.12}$$

is a $\widetilde{\mathcal{P}}_N$ martingale with bracket

$$\begin{aligned}\langle M^N(J)\rangle_t =& \int_0^t ds [L_N(\langle \mu_N(s), J\rangle)^2 - 2\langle \mu_N(s), J\rangle L_N \langle \mu_N(s), J\rangle] = \\ =& \, t \frac{2N^2}{\lambda^2 N^{2d}} \sum_{\substack{x,y\in S_N^d \\ |x-y|=1}} [J(y/N) - J(x/N)]^2\end{aligned} \tag{4.13}$$

Formula (4.10) follows then by using the basic entropy estimate (3.22) to control the drift term, see [6, Lemma 6.2], and Doob's inequality to control the martingale part.

STEP 2. Identification of the limit. Let $\mathcal{P}$ be any limit point of $\{\mathcal{P}_N\}$. Since the entropy in non increasing we have

$$\frac{1}{N^d} \int \nu_N(d\xi)\, f_N^t(\xi) \log f_N^t(\xi) \le C \tag{4.14}$$

where f_N^t is the density of the marginal at time t of $\mathcal{P}_N$. We can therefore apply Corollary 4.2 and obtain that, for any $t \ge 0$, $\mu(t) = q(t,\theta)d\theta$ $\mathcal{P}$-a.s. for some density $q(t,\theta)$. Furthermore

$$\beta \int d\theta\ h\Big(\beta, q(t,\theta)\Big) \le C \tag{4.15}$$

We want to show that for each $t > 0$ and any smooth J

$$\begin{aligned}&\int d\theta q(t,\theta) J(\theta) - \int d\theta, q_0(\theta) J(\theta) = \\ &= 2\beta \int_0^t ds \int d\theta F(q(s,\theta)) \Delta J(\theta)\ \mathcal{P} - a.s.\end{aligned} \tag{4.16}$$

where q_0 is defined in (1.18).

Let us introduce

$$V_N^{\varepsilon,k}(t) \doteq \langle \mu_N(t), J\rangle - \langle \mu_N(0), J\rangle + \\ - 2\beta \int_0^t ds \frac{1}{N^d} \sum_{x\in S_N^d} \Delta_N J(x/N)\, \psi_k \circ F(m_{\Lambda_{N\epsilon}}(\omega)) \tag{4.17}$$

where ψ_k has been defined in (3.3). By the martingale computation in the previous Step (see (4.11)), we have that

$$V_N^{\varepsilon,k}(t) = 2\beta \int_0^t ds \frac{1}{N^d} \sum_{x\in S_N^d} \Delta_N J(x/N)[2\lambda^{-1}\phi_s(x) + \\ - \psi_k \circ F(m_{\Lambda_{N\epsilon}}(\omega)\ )] + M_t^N(J) \tag{4.18}$$

Since J is smooth and the bracket of $M_t^N(J)$ is easily controlled (see (4.13)), from Theorem 3.1 we get

$$\lim_{k\to\infty} \limsup_{\varepsilon\to 0} \limsup_{N\to\infty} \mathbb{E}^{\mathcal{P}_N} |V_N^{\varepsilon,k}(t)| = 0 \tag{4.19}$$

Let now

$$V^k(t) \doteq \int d\theta\ q(t,\theta)\, J(\theta) - \int d\theta\ q_0(\theta)\, J(\theta) + \\ - 2\beta \int_0^t ds \int d\theta\ \Delta J(\theta)\, \psi_k \circ F(q(s,\theta)) \tag{4.20}$$

From (4.19) and (1.17) we deduce that, for some subsequence $\mathcal{P}_N$ weakly convergent to $\mathcal{P}$, (we know it exists by tightness, Step 1)

$$\lim_{k\to\infty} \mathbb{E}^{\mathcal{P}} |V^k(t)| \le \lim_{k\to\infty} \limsup_{\varepsilon\to 0} \limsup_{N\to\infty} \mathbb{E}^{\mathcal{P}_N} \Big(|V_N^{\varepsilon,k}(t)| + \\ + \Big| \int d\theta q_0(\theta) J(\theta) - \langle \mu_N(0), J\rangle \Big| \Big) = 0 \tag{4.21}$$

In order to take also the limit $k \to \infty$ inside the expectation on the l.h.s. of the above inequality, we need the following estimate, which also follows from the basic entropy inequality, see [10, Lemma 2.21]. There exists a constant C_1 such that $|F(\beta, q)| \le C_1 + h(\beta, q)$. By using the bound (4.3) we can then justify also this last step and conclude the proof of (4.16).

STEP 3. Uniqueness of the limit. By (4.16), in order to conclude the proof of Theorem 1.1, we only need the uniqueness of weak solutions to the non-linear diffusion eq. (1.23). Let us state the appropriate result [10, Thm. 5.2].

PROPOSITION 4.4. *There exists a unique weak solution of the eq.* (1.24) *in the class of functions satisfying* (4.15) *and*

$$\int_0^T dt \int d\theta \left[\nabla F(q(t,\theta))\right]^2 \leq C'(T) < \infty \tag{4.22}$$

for any $T > 0$.

We already verified (4.15) holds $\mathcal{P}$-a.s. for any limit point $\mathcal{P}$ of $\{\mathcal{P}_N\}$. The proof that the same holds for (4.22) follows, see [6, Lemma 6.6], from the estimate (3.58). □

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REFERENCES

[1] P. BILLINGSLEY: *Convergence of probability measures*, Wiley, New York, 1968.

[2] G. CAGINALP – X. CHEN: *Convergence of the phase field model to its sharp interface limits*, European J. Appl. Math. **9** (1998), 417-445.

[3] A. DE MASI – E. ORLANDI – E. PRESUTTI – L. TRIOLO: *Glauber evolution with Kac potential I. Mesoscopic and macroscopic limits, interface dynamics*, Nonlinearity, **7** (1994), 633-696.

[4] J.D. DEUSCHEL – D.J. STROOCK: *Large Deviations*, Academic Press, San Diego, 1989.

[5] R.S. ELLIS: *Entropy, Large Deviations and Statistical Mechanics*, Springer-Verlag, Berlin, 1985.

[6] M.Z. GUO – G.C. PAPANICOLAOU – S.R.S. VARADHAN: *Nonlinear Diffusion Limit for a System with Nearest Neighbor Interactions*, Commun. Math. Phys., **118** (1988), 31-59.

[7] A.M. Meirmanov: *The Stefan problem*, De Gruyter Exposition in Mathematics 3. de Gruyter, Berlin, 1992.

[8] O. Penrose – P.C. Fife: *Thermodynamically consistent models of phase-field type for the kinetics of phase transition*, Physica D., **43** (1990), 44-62.

[9] E. Presutti – L. Tubaro – A. Visintin: *Private Communication*, .

[10] F. Rezakhanlou: *Hydrodynamic Limit for a System with Finite Range Interactions*, Commun. Math. Phys., **129** (1990), 445-480.

[11] H. Spohn: *Large scale dynamics of interacting particles*, Springer, Berlin, 1991.

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