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Existence of unbounded solutions for some quasilinear elliptic problems

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RIASSUNTO: *In questo lavoro si prova l'esistenza di soluzioni per una classe di equazioni quasilineari con crescita quadratica nel gradiente. Si suppone che esistano una sotto soluzione e una sopra soluzione non limitate. La soluzione che si trova non è una soluzione debole in senso classico, ma una soluzione "rinormalizzata". Si costruiscono problemi approssimanti e si ottengono stime sulle troncate delle rispettive soluzioni usando particolari funzioni test.*

ABSTRACT: *We study the existence of a solution of some quasilinear elliptic equation with quadratic growth in the gradient, assuming the existence of a pair of sub and super solutions which are not bounded. The solution we obtain is not a classical weak solution, but a "renormalized" solution. We define approximated problems, and we obtain estimates on the truncates of the corresponding solutions, by using appropriate test functions.*

1 – Introduction and hypotheses

Let Ω be a bounded open set of $\mathbb{R}^N$ with $N \geq 1$. We consider the following hypotheses:

(1.1) $a(x,s)$ is a Caratheodory function from $\Omega \times \mathbb{R} \to \mathbb{R}^{N\times N}$,

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$$\forall i,j,\ 1 \le i,j \le n,\ \exists\, c \ge 0,\ \text{such that,} \quad |a_{i,j}(x,s)| \le c \text{ a.e. } x \in \Omega,\ \forall\, s \in \mathbb{R}, \tag{1.2}$$

$$\sum_{i,j=1}^{n} a_{i,j}(x,s)\xi_i\xi_j \ge \alpha|\xi|^2 \ a.e.x \in \Omega, \forall\, s \in \mathbb{R}, \tag{1.3}$$

(1.4) $g(x,s,\xi)$ is a Caratheodory function from $\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$,

$$|g(x,s,\xi)| \le b(|s|)(|\xi|^2 + f(x)), \tag{1.5}$$

where f is a function of $L^1(\Omega)$, and b is a function which is defined everywhere in $\mathbb{R}^+$, and bounded on bounded intervals of $\mathbb{R}^+$.

We denote by $a(x,u)$ the matrix $(a_{i,j}(x,u))$, and we study the following problem:

$$\begin{cases} -\operatorname{div}[a(x,u)\nabla u] + g(x,u,\nabla u) = h, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.6}$$

with $h \in H^{-1}(\Omega)$.

We know that for this type of problems, the existence of a pair of bounded ordered sub and super solutions implies the existence of a bounded weak solution (see [3], [4]). Our goal in the present paper is to study the problem of the existence of a solution (in a sense which has to be precised) if we assume the existence of unbounded sub and super solutions. Some existence results for unbounded solutions are proved in [1], [2], [6], [8] for instance. In this work, the solutions we obtain are not classical, but “renormalized” solutions, so that we shall be interested in “renormalized” solutions studied in [7] and [8] and especially in the uniqueness of such solutions. That is why we shall be first interested in following basic problem:

$$\begin{cases} -\operatorname{div}[a(x)\nabla u] = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.7}$$

where f lies in $L^1(\Omega)$ and $a(x)$ lies in $L^\infty(\Omega)$ with $a(x) \ge \alpha > 0$. In [7], the following definition of a “renormalized” solution is given:

Definition 1.1. We say that u is a “renormalized” solution of (1.7) if:

$$
(1.8)\qquad \begin{cases} u \in L^0(\Omega) \text{ (espace des fonctions mesurables} \\ \qquad\qquad \text{finies presque partout sur } \Omega), \\ T_k(u) \in H_0^1(\Omega),\ \forall\, k \in \mathbb{N}, \\ \lim\limits_{n\to+\infty} \dfrac{1}{n} \displaystyle\int_{n\le|u|\le 2n} |\nabla u|^2 dx \to 0, \end{cases}
$$

and if:

$$
(1.9)\qquad \int_\Omega a(x)\nabla u \nabla v h(u) dx + \int_\Omega a(x)\nabla u \nabla u v h'(u) dx = \int_\Omega f(x) h(u) v dx \\ \forall\, v \in H_0^1(\Omega)\cap L^\infty(\Omega), \forall\, h \in W^{1,\infty}_{\text{comp}}(\mathbb{R}),
$$

where $W^{1,\infty}_{\text{comp}}(\mathbb{R})$ is the set of functions of $W^{1,\infty}(\mathbb{R})$ with compact support.

In [7], it is showed that f lies in $L^1(\Omega)$, then there exists a unique renormalized solution of (1.7). In the proof of the uniqueness, the assumption:

$$
(1.10)\qquad \lim_{n\to+\infty} \frac{1}{n} \int_{\{n\le|u|\le 2n\}} |\nabla u|^2 dx \to 0
$$

is essential, but it can be modified in the following way:

Theorem 1.1. *Let w_1 and w_2 be two functions of $L^0(\Omega)$ such that:*

$$
(1.11)\qquad w_1 \le w_2 \ \text{a.e. in } \Omega
$$

$$
(1.12)\qquad T_k(w_i) \in H^1(\Omega) \text{ for } i = 1,2 \text{ and } \forall\, k \in \mathbb{N}
$$

$$
(1.13)\qquad \lim_{n\to+\infty} \frac{1}{n} \int_{n\le|w_i|\le 2n} |\nabla w_i|^2 dx \to 0 \text{ for } i = 1,2.
$$

Then there exists no more than one solution of (1.7), which satisfies:

$$
(1.14)\qquad \begin{cases} u \in L^0(\Omega) \text{ with } w_1 \le u \le w_2 \text{ a.e. in } \Omega, \\ T_k(u) \in H_0^1(\Omega)\ \forall\, k \in \mathbb{N}, \\ \displaystyle\int_\Omega a(x)\nabla u \nabla v dx = \int_\Omega f(x) v dx, \end{cases}
$$

$\forall\, v \in H_0^1(\Omega)\cap L^\infty(\Omega)$ such that there exists M, with $v = 0$ in $\{|u| \ge M\}$.

Theorem 1.1 shows that the solutions we shall obtain are solutions in a suitable sense.

Proof of Theorem 1.1. We take $v = \theta_n(u)T_1(u^+)$ in (1.14), where $\theta_n(s)$ is the function defined as follows:

$$\theta_n(s) = \begin{cases} s \ \operatorname{sgn}(s) - n \text{ if } |s| \le n, \\ 0 \text{ if } |s| \ge n. \end{cases} \tag{1.15}$$

Then we obtain (C designates different constants which are independent of n),

$$\int_{\{0\le u\le 1\}} a(x)\nabla u \nabla u \theta_n(u) dx + \int_{\{0\le u\le n\}} a(x)\nabla u\nabla u T_1(u^+)dx = \\ = \int_\Omega f(x)T_1(u^+)\theta_n(u)dx.$$

As,
$$\int_{\{0\le u\le 1\}} a(x)\nabla u\nabla u T_1(u^+)dx \ge 0,$$

we have,

$$\alpha\int_{\{1\le u\le n\}} |\nabla u|^2 dx \le \int_\Omega f(x)T_1(u^+)\theta_n(u)dx + C\int_{\{0\le u\le 1\}} |\nabla u|^2|\theta_n(u)|dx,$$

that is to say, since $T_1(u)$ belongs to $H_0^1(\Omega)$,

$$\int_{\{1\le u\le n\}} |\nabla u|^2 dx \le Cn$$

and,
$$\int_{\{0\le u\le 1\}} |\nabla u|^2 dx = \int_\Omega |\nabla T_1(u)|^2 dx$$

and so,
$$\frac{1}{n}\int_{\{0\le u\le n\}} |\nabla u|^2 dx \le C.$$

We can show likewise:

$$\frac{1}{n}\int_{\{-n\le u\le 0\}} |\nabla u|^2 dx \le C$$

and so,

$$\frac{1}{n}\int_{\{0\le|u|\le n\}}|\nabla u|^2\le C. \tag{1.16}$$

We suppose now that there exist two solutions u_1 and u_2 of (1.14). In the equations corresponding to u_1 and u_2, we take $v=h_n(w_1)h_n(w_2)T_k(u_1-u_2)$, where $h_n(s)$ is the function defined as follows:

$$h_n(s)=\begin{cases}1 & \text{if } |s|\le n\\ -\dfrac{s}{n}\,\mathrm{sgn}(s)+2 & \text{if } n\le|s|\le 2n\\ 0 & \text{if } 2n\le|s|.\end{cases}$$

One can check, recalling that $w_1\le u_1\le w_2$, that v equals zero in the set $\{|u_1|\ge 2n\}$ and in the set $\{|u_2|\ge 2n\}$. We set $E_n=\{|u_1|\le 2n\}\cap\{|u_2|\le 2n\}$.

We subtract the equation corresponding to u_2 from the one corresponding to u_1. That leads to the following equality:

$$\begin{aligned}&\int_{\{|u_1-u_2|\le k\}\cap E_n}a(x)\nabla(u_1-u_2)\nabla(u_1-u_2)h_n(w_1)h_n(w_2)dx+\\&\quad+\int_{\{n\le|w_1|\le 2n\}\cap E_n}\frac{1}{n}a(x)\nabla u_1\nabla w_1T_k(u_1-u_2)h_n(w_2)dx+\\&\quad-\int_{\{n\le|w_1|\le 2n\}\cap E_n}\frac{1}{n}a(x)\nabla u_2\nabla w_1T_k(u_1-u_2)h_n(w_2)dx+\\&\quad+\int_{\{n\le|w_2|\le 2n\}\cap E_n}\frac{1}{n}a(x)\nabla u_1\nabla w_2T_k(u_1-u_2)h_n(w_1)dx+\\&\quad-\int_{\{n\le|w_2|\le 2n\}\cap E_n}\frac{1}{n}a(x)\nabla u_2\nabla w_2T_k(u_1-u_2)h_n(w_1)dx=0.\end{aligned}$$

The four lastest terms tend to zero when n tends to infinity, since for instance:

$$\begin{aligned}&\Big|\int_{\{n\le|w_1|\le 2n\}\cap E_n}\frac{1}{n}a(x)\nabla u_1\nabla w_1T_k(u_1-u_2)h_n(w_2)dx\Big|\le\\&\le Ck\Big(\frac{1}{n}\int_{\{0\le|u_1|\le 2n\}}|\nabla u_1|^2dx\Big)^{1/2}\Big(\frac{1}{n}\int_{\{n\le|w_1|\le 2n\}}|\nabla w_1|^2dx\Big)^{1/2}.\end{aligned}$$

Formula (1.16), together with the assumptions on w_1, imply that the right hand side tends to zero, as n tends to infinity. So, we have that:

$$0\leq\liminf\int_{\{|u_1-u_2|\leq k\}\cap E_n} a(x)\nabla(u_1-u_2)\nabla(u_1-u_2)h_n(w_1)h_n(w_2)dx\leq$$
$$\leq\limsup\int_{\{|u_1-u_2|\leq k\}\cap E_n} a(x)\nabla(u_1-u_2)\nabla(u_1-u_2)h_n(w_1)h_n(w_2)dx\leq 0$$

and thus,

$$\lim_{n\to+\infty}\int_{\{|u_1-u_2|\leq k\}\cap E_n} a(x)\nabla(u_1-u_2)\nabla(u_1-u_2)h_n(w_1)h_n(w_2)dx=0.$$

This proves Theorem 1.1.

We call "renormalized" sub solution of (1.7), a function φ such that:

$$\begin{cases} \varphi\in L^0(\Omega), \\ T_k(\varphi)\in H^1(\Omega),\ \forall\, k\in\mathbb{N}, \\ \varphi \text{ is bounded and nonpositive on } \partial\Omega, \\ \displaystyle\lim_{n\to\infty}\frac{1}{n}\int_{n\leq|\varphi|\leq 2n}|\nabla\varphi|^2dx=0, \\ \displaystyle\int_\Omega a(x,\varphi)\nabla\varphi\nabla v dx+\int_\Omega g(x,\varphi,\nabla\varphi)v dx\leq\langle h,v\rangle, \end{cases}$$

$\forall\, v\in H_0^1(\Omega)\cap L^\infty(\Omega)$, such that there exists M such that $v=0$ in $\{|\varphi|\geq M\}$. We define likewise a super solution, exchanging "$\leq$" with "$\geq$".

Notation.

$$T_k(s)=\begin{cases} s & \text{if } -k\leq s\leq k, \\ k & \text{if } s\geq k, \\ -k & \text{if } s\leq -k. \end{cases}$$

If φ and ψ are two functions such that $\varphi\leq\psi$, we define:

$$T_{\varphi\psi}(u)=\begin{cases} u & \text{in } \{\varphi\leq u\leq\psi\}, \\ \psi & \text{in } \{\psi\leq u\}, \\ \varphi & \text{in } \{u\leq\varphi\}, \end{cases}$$

$$g_k(x,u,\nabla u)=\begin{cases} g(x,u,\nabla u) & \text{in } \{T_k(\varphi)\leq u\leq T_k(\psi)\}, \\ g(x,T_k(\psi),\nabla T_k(\psi)) & \text{in } \{T_k(\psi)\leq u\}, \\ g(x,T_k(\varphi),\ \nabla T_k(\varphi)) & \text{in } \{u\leq T_k(\varphi)\}, \end{cases}$$

$$a_k(x,u) = \begin{cases} a(x,u) & \text{in } \{T_k(\varphi) \le u \le T_k(\psi)\}, \\ a(x,T_k(\psi)) & \text{in } \{T_k(\psi) \le u\}, \\ a(x,T_k(\varphi)) & \text{in } \{u \le T_k(\varphi)\}. \end{cases}$$

2 – Existence theorem

THEOREM 2.1. *We assume* (1.1), (1.2), (1.3), (1.4). *We assume in addition, that there exists a renormalized subsolution* φ *and a renormalized supersolution* ψ *such that:*

$$\varphi \le \psi \text{ a.e. in } \Omega, \tag{2.1}$$

$$\exists\, M_1, M_2 \ge 0 \text{ such that } \varphi \le M_1 \text{ and } \psi \ge -M_2 \text{ a.e. in } \Omega \tag{2.2}$$

$$\varphi \text{ and } \psi \text{ are bounded on } \partial\Omega, \tag{2.3}$$

$$\exists\, M_0 \ge 0 \text{ such that } \{\varphi \ge -M_0\} \cup \{\psi \le M_0\} = \Omega. \tag{2.4}$$

Then there exists $u \in L^1(\Omega)$ *such that:*

$$\begin{cases} \varphi \le u \le \psi \text{ a.e. in } \Omega \\ T_M(u) \in H_0^1(\Omega)\ \forall\, M \ge 0 \\ \displaystyle\int_\Omega a(x,u)\nabla u \nabla v dx + \int_\Omega g(x,u,\nabla u) v dx = \langle h, v\rangle \\ \text{for every } v \in H_0^1 \cap L^\infty(\Omega), \text{ such that there exists } M, \\ \quad \text{such that } v = 0 \text{ in } \{|u| \ge M\} \end{cases}$$

REMARKS 2.1.

- If either φ or ψ is a function of $L^\infty(\Omega)$, hypothesis (2.4)) is always verified. In fact, hypothesis (2.4) means that, if φ and ψ have singularities, they are not in the same place.
- If u is bounded, $\{|u| > \|u\|_\infty\} = \emptyset$ and then, for every $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we can consider that $v = 0$ in $\{|u| \ge \|u\|_\infty\}$, that is to say, u is an ordinary weak solution. Thus, the existence result of [4] in which the sub and the super solution are bounded, is a particular case of Theorem 2.1.

- If we suppose in addition that:

$$\lim_{n\to+\infty}\frac{1}{n}\int_{\{n\le|\varphi|\le 2n\}}|\nabla\varphi|^2dx=0,$$

and,

$$\lim_{n\to+\infty}\frac{1}{n}\int_{\{n\le|\psi|\le 2n\}}|\nabla\psi|^2dx=0,$$

we have seen in the first part that for linear problems the solution is unique.

Proof of Theorem 2.1.

1^{st} step: construction of a sequence of approaching solutions:

Let $k\ge\sup(M_1,M_2)$ where M_1,M_2 are given by (2.2), then there exists $u_k\in H_k=\{v\in H_0^1(\Omega),\, T_k(\varphi)-1\le v\le T_k(\psi)+1\}$ satisfying:

$$\int_\Omega a_{k+2}(x,u_k)\nabla u_k\nabla(v-u_k)dx+\int_\Omega g_{k+2}(x,u_k,\nabla u_k)(v-u_k)dx\ge \\ \ge\langle h,v-u_k\rangle\quad\forall\, v\in H_k. \tag{2.5}$$

That can be proved with standard arguments, approaching g_{k+2} by $g_{k+2}/(1+\epsilon|g_{k+2}|)$. Indeed there is no difficulty to pass to the limit, as H_k is bounded in $L^\infty(\Omega)$ (see for instance [5]).

Lemma 2.1. *Under assumptions* (1.1) *to* (1.5)*, let u_k be a solution of* (2.5)*. Then, we have:*

$$\varphi\le u_k\le\psi \text{ a.e. in } \Omega,$$

and it verifies in fact:

$$\int_\Omega a(x,u_k)\nabla u_k\nabla(v-u_k)dx+\int_\Omega g(x,u_k,\nabla u_k)(v-u_k)dx\ge \\ \ge\langle h,v-u_k\rangle\quad\forall\, v\in H_k. \tag{2.6}$$

Proof of Lemma 2.1. We know that $u_k \geq -k-1$ then $\varphi - u_k \leq M_1 + k + 1$ and so $(\varphi - u_k)^+ \in L^\infty(\Omega)$. One can verify also that $(\varphi - u_k)^+ \in H_0^1(\Omega)$. Moreover $(\varphi - u_k)^+ = 0$ in $\{\varphi \leq -k-2\}$ and as soon as $k+2 > M_1$, $\{\varphi \geq k+2\} = \emptyset$. Thus we can take $(\varphi - u_k)^+$ as test function in the inequality satisfied by φ, and so,

$$\tag{2.7} \begin{aligned} &\int_\Omega a(x,\varphi)\nabla\varphi\nabla(\varphi - u_k)^+ dx + \int_\Omega g(x,\varphi,\nabla\varphi)(\varphi - u_k)^+ dx \leq \\ &\leq \langle h, (\varphi - u_k)^+ \rangle \,. \end{aligned}$$

On the other hand, one can verify that $u_k + (\varphi - u_k)^+$ is in H_k and so, taking $u_k + (\varphi - u_k)^+$ as test function in (2.5):

$$\tag{2.8} \begin{aligned} &\int_\Omega a_{k+2}(x,u_k)\nabla u_k \nabla(\varphi - u_k)^+ dx + \\ &\quad + \int_\Omega g_{k+2}(x,u_k,\nabla u_k)(\varphi - u_k)^+ dx \geq \\ &\geq \langle h, (\varphi - u_k)^+ \rangle \,. \end{aligned}$$

If we subtract (2.8) from (2.7), we obtain:

$$\tag{2.9} \begin{aligned} &\int_\Omega a(x,\varphi)\nabla(\varphi - u_k)^+ \nabla(\varphi - u_k)^+ dx + \\ &+ \int_\Omega (a(x,\varphi) - a_{k+2}(x,u_k))\nabla u_k \nabla(\varphi - u_k)^+ dx + \\ &+ \int_\Omega (g(x,\varphi,\nabla\varphi) - g_{k+2}(x,u_k,\nabla u_k))(\varphi - u_k)^+ dx \leq 0. \end{aligned}$$

As anyway, $u_k \geq T_k(\varphi) - 1$, then, in the set $\{\varphi \geq u_k\}$, we have the following properties:

$$\varphi \geq -k-1$$

$$a_{k+2}(x,u_k) = a(x, T_{k+2}(\varphi)) = a(x,\varphi)$$

$$g_{k+2}(x,u_k,\nabla u_k) = g(x, T_{k+2}(\varphi), \nabla T_{k+2}(\varphi)) = g(x,\varphi,\nabla\varphi).$$

Thus the two last terms in (2.9) are equal to zero, and therefore:

$$\int_\Omega a(x,\varphi)\nabla(\varphi - u_k)^+ \nabla(\varphi - u_k)^+ dx \leq 0.$$

Consequently, from (1.2):

$$\|\nabla(\varphi - u_k)^+\|_2 \leq 0$$

that is to say, $\varphi \leq u_k$ a.e. in Ω. But, $u_k \geq -(k+2)$ so that, $u_k \geq T_{k+2}(\varphi)$. In the same way, we can show that:

$$u_k \leq T_{k+2}(\psi)$$

and then,

$$a_{k+2}(x, u_k) = a(x, u_k)$$
$$g_{k+2}(x, u_k, \nabla u_k) = g(x, u_k, \nabla u_k).$$

That proves Lemma 2.1.

2^{nd} step: estimate for the gradient.

LEMMA 2.2. *Under assumptions* (1.1) *to* (1.5), *let* u_k *be a solution of* (2.5). *Then, there exists a constant* C_M *which depends on M, but not on k, such that:*

$$\int_\Omega |\nabla(T_M(u_k))|^2 dx \leq C_M.$$

PROOF OF LEMMA 2.2. We denote by $\mathcal{T}r\varphi$ and $\mathcal{T}r\psi$ the value of φ and ψ on the boundary of Ω. Let M be such that, $M \geq \sup(M_1, M_2, \|\mathcal{T}r\varphi\|_{L^\infty(\partial\Omega)} + 2, \|\mathcal{T}r\psi\|_{L^\infty(\partial\Omega)} + 2)$ and $k \geq M$. We set $\theta = T_1((\varphi + M + 1)^+)$, then,

$$\begin{cases} 0 \leq \theta \leq 1 \\ \theta = 0 \text{ in } \{\varphi \leq -M - 1\} \\ \theta = 1 \text{ in } \{\varphi \geq -M\}. \end{cases}$$

We set $\tau = \sup(T_1(((\psi - \|\mathcal{T}r\psi\|_{L^\infty(\partial\Omega)})^+), T_1(((-\|\mathcal{T}r\varphi\|_{L^\infty(\partial\Omega)} - \varphi)^+))$, then $\tau \in H_0^1(\Omega)$ and $0 \leq \tau \leq 1$. Now we set, $w = \tau(\theta T_M(\psi) + (1-\theta)\varphi)$ and $z = u_k - \delta\varphi_\lambda(-(u_k - w)^-)$, where $\varphi_\lambda(s) = se^{\lambda s^2}$ and where δ and λ are constants which will be precised in the following. Remark: a priori, $\theta T_M(\psi) + (1-\theta)\varphi$ is not equal to zero on the boundary of Ω, that's why we use the function τ. If φ and ψ are equal to zero on the boundary, we can take $\tau = 1$.)

One can verify that:

$$-1 - M - \tau T_M(\psi) \leq -(u_k - w)^- \leq 0$$

so $(u_k - w)^- \in L^\infty(\Omega)$. Then, we can choose δ (independent of k) so that:

$$\delta \leq \frac{1}{\| \exp(\lambda(-(u_k - w)^-)^2)\|_\infty}$$

which implies $z \in H_k$, and $z \geq u_k \geq T_k(\varphi) - 1$.
On another hand,

$$w \leq \tau T_M(\psi) \leq T_k(\psi) + 1 \text{ if } k \geq M$$

thus, in the set $\{(u_k - w)^- \neq 0\}$, we have,

$$\begin{aligned} z = (1 - \delta \exp(\lambda(-(u_k - w)^-))^2)u_k + \delta \exp(\lambda(-(u_k - w)^-)^2)w \leq \\ \leq w \leq T_k(\psi) + 1 \end{aligned}$$

and in the set $\{\ (u_k - w)^- = 0\}$, we have,

$$z = u_k \leq T_k(\psi) + 1.$$

One can verify too, that $z \in H_0^1(\Omega)$, and so $z \in H_k$, and we can take z as test function in (2.6), that leads to:

$$\begin{aligned} \int_\Omega - a(x, u_k)\nabla u_k \nabla (u_k - w)^- \varphi'_\lambda(-(u_k - w)^-)dx + \\ + \int_\Omega g(x, u_k, \nabla u_k)\varphi_\lambda(-(u_k - w)^-)dx \leq \langle h, \varphi_\lambda(-(u_k - w)^-)\rangle. \end{aligned}$$

Then,

$$\begin{aligned} &\int_{\{u_k \leq w\}} a(x, u_k)\nabla u_k \nabla u_k \varphi'_\lambda(-(u_k - w)^-)dx + \\ &\qquad - \int_{\{u_k \leq w\}} a(x, u_k)\nabla u_k \nabla w \varphi'_\lambda(-(u_k - w)^-)dx \leq \\ &\qquad \leq \int_{\{u_k \leq w\}} b(|u_k|)|\nabla u_k|^2 |\varphi_\lambda(-(u_k - w)^-)|dx + \\ &\qquad + \int_{\{u_k \leq w\}} f(x)b(|u_k|)|\varphi_\lambda(-(u_k - w)^-)|dx + \\ &\qquad + C\|h\|_{H^{-1}}\Big(\int_{\{u_k \leq w\}} |\nabla \varphi_\lambda(-(u_k - w)^-)|^2 dx\Big)^{1/2}. \end{aligned}$$

In the set $\{u_k \le w\}$, we have, $u_k \le w \le \tau T_M(\psi) \le M$ and, $\varphi \ge -M-1$ and therefore, $u_k \ge -M-1$. So, $b(|u_k|) \le C_M$. For λ large enough (independent of k), we have, $\alpha\varphi'_\lambda - C_M|\varphi_\lambda| \ge \frac{\alpha}{2}$ and thus, we obtain,

$$\frac{\alpha}{2}\int_{\{u_k\le w\}} |\nabla u_k|^2 dx \le C_M\Big(\int_{\{u_k\le w\}} |\nabla u_k|^2 dx\Big)^{1/2} + C_M.$$

So,

$$\int_{\{u_k\le w\}} |\nabla u_k|^2 dx \le C_M$$

and then,

$$\int_{E_M} |\nabla u_k|^2 dx \le C_M$$

where $E_M = \{u_k \le w\} \cap \{\varphi \ge -M\} \cap \{\tau = 1\}$.

But, in the set $\{\varphi \ge -M\} \cap \{\tau = 1\}$, we have $w = T_M(\psi)$ and therefore $u_k \le w$ is equivalent to $u_k \le M$. That leads to,

$$\int_{E'_M} |\nabla u_k|^2 dx \le C_M$$

where $E'_M = \{u_k \le M\} \cap \{\varphi \ge -M\} \cap \{\tau = 1\}$.

In the same way, if we replace θ by $\theta' = T_1((M+1-\psi)^+)$, we can show that,

$$\int_{F'_M} |\nabla u_k|^2 dx \le C_M$$

where $F'_M = \{u_k \ge -M\} \cap \{\psi \le M\} \cap \{\tau = 1\}$.

Consequently,

$$\int_{E'_M\cup F'_M} |\nabla u_k|^2 dx \le C_M. \tag{2.10}$$

We are going to show that,

$$\int_{\{\tau<1\}} |\nabla u_k|^2 dx \le C.$$

Let $\omega \in H^1(\Omega) \cap L^\infty(\Omega)$ such that $0 \le \omega \le 1$, with $\omega = 1$ in the set $\{\tau < 1\}$, $\omega = 0$ in $\{\psi(x) \ge M\} \cup \{\varphi(x) \le -M\}$ (we can take for instance:

$\omega = 1 - \sup(T_1((\psi - \|\mathcal{T}r\psi\|_{L^\infty(\partial\Omega)} - 1)^+), T_1((\|\mathcal{T}r\varphi\|_{L^\infty(\partial\Omega)} - 1 - \varphi)^+))$. Let $v = u_k - \delta\omega\varphi_\lambda(u_k)$, then $v \in H_k$, because in the set $\{\psi \geq M\} \cup \{\varphi \leq -M\}$, we have, $-M \leq u_k \leq M$, and thus $T_k(\varphi) \leq T_M(\varphi) \leq u_k \leq T_M(\psi) \leq T_k(\psi)$. So , if we choose δ such that: $\delta\|\omega\varphi_\lambda(u_k)\|_\infty \leq 1$, then, $T_k(\varphi) - 1 \leq v \leq T_k(\psi) + 1$. With such a choice of δ, v is in H_k, and we can take v as test function in (2.6), and if we take (1.3) into account, that leads to:

$$\int_\Omega a(x,u_k)\nabla u_k \nabla u_k \omega\varphi'_\lambda(u_k)dx + \int_\Omega a(x,u_k)\nabla u_k \nabla\omega\varphi_\lambda(u_k)dx \leq$$
$$\leq \int_\Omega C_M|\varphi_\lambda(u_k)|\omega dx + C_M\int_\Omega f(x)|\varphi_\lambda(u_k)|\omega dx + \langle h, \varphi_\lambda(u_k)\omega\rangle.$$

For λ large enough, we obtain:

$$\frac{\alpha}{2}\int_\Omega |\nabla u_k|^2\omega dx \leq -\int_\Omega a(x,u_k)\nabla u_k\nabla\omega\varphi_\lambda(u_k)dx +$$
$$+ C_M + \int_\Omega h_2\nabla u_k\varphi'_\lambda(u_k)\omega dx.$$

Then,

$$\frac{\alpha}{2}\int_{\{\tau<1\}} |\nabla u_k|^2\omega dx \leq C_M - \int_{\{\tau<1\}} a(x,u_k)\nabla u_k\nabla\omega\varphi_\lambda(u_k)dx +$$
$$-\int_{\{\tau=1\}} a(x,u_k)\nabla u_k\nabla\omega\varphi_\lambda(u_k)dx + \int_{\{\tau<1\}} h_2\nabla u_k\varphi'_\lambda(u_k)\omega dx +$$
$$+\int_{\{\tau=1\}} h_2\nabla u_k\varphi'_\lambda(u_k)\omega dx.$$

From (2.10) and the properties of τ, we deduce that,

$$\frac{\alpha}{2}\int_{\{\tau<1\}} |\nabla u_k|^2\omega dx \leq C_M + C_M\Big(\int_{\{\tau<1\}} |\nabla u_k|^2 dx\Big)^{1/2}$$

and as $\omega = 1$ in the set $\{\tau < 1\}$,

$$\frac{\alpha}{2}\int_{\{\tau<1\}} |\nabla u_k|^2 dx \leq C_M + C_M\frac{\alpha}{2}\Big(\int_{\{\tau<1\}} |\nabla u_k|^2 dx\Big)^{1/2}$$

and so,

$$\frac{\alpha}{2}\int_{\{\tau<1\}} |\nabla u_k|^2 dx \leq C_M. \tag{2.11}$$

On another hand, if $M \geq M_0$ (where M_0 is given by (2.4)),

$$(\{u_k \geq -M\} \cap \{\psi \leq M\}) \cup (\{u_k \leq M\} \cap \{\varphi \geq -M\}) = \\ = \{-M \leq u_k \leq M\} \cup (\{\psi \leq M\} \cup \{\varphi \geq -M\}) = \{-M \leq u_k \leq M\}$$

because if $M \geq M_0$, then $\{\psi \leq M\} \cup \{\varphi \geq -M\} = \Omega$. Then finally, regrouping (2.10), (2.11) and (2.12), we obtain that,

$$\int_{\{-M \leq u_k \leq M\}} |\nabla u_k|^2 dx \leq C_M,$$

so that Lemma 2.2 is proved.

3^{rd} step: convergence of the sequence (u_k).

Lemma 2.3. *Under assumptions* (1.1) *to* (1.5)*, let u_k be a solution of* (2.5)*. Then there exists $u \in L^0(\Omega)$ such that:*

$$u_k \to u \text{ a.e. in } \Omega,$$
$$\nabla T_M(u_k) \to \nabla T_M(u) \text{ in } L^2(\Omega) \text{ weak}.$$

Proof of Lemma 2.3. After extraction of a diagonal subsequence relatively to k, we can suppose that,

$$T_M(u_k) \to u_M \text{ in } H^1_0(\Omega) \text{ weak}, L^2(\Omega) \text{ strong and a.e. in } \Omega.$$

Moreover, one can verify that there exists a function u of $L^0(\Omega)$ such that $T_M(u) = u_M$, so the lemma is proved.

Lemma 2.4. *Under the assumptions* (1.1) *to* (1.5)*, let u_k be a solution of* (2.5)*. We have,*

$$\nabla T_M(u_k) \to \nabla T_M(u) \text{ in } L^2(\Omega) \text{ strong}.$$

Proof of Lemma 2.4. Let $v_k = T_M(u_k) - T_M(\sup(u, u_k))$ and $w_k = \varphi_\lambda v_k (\varphi + M_0 + 1)^+$ then $w_k \in H^1_0(\Omega) \cap L^\infty(\Omega)$, and we can choose δ such that $\delta \|w_k\|_\infty \leq 1$. We are going to verify that $u_k - \delta w_k$ belongs to H_k: as w_k is negative , $u_k - \delta w_k \geq u_k$. Then we have only to show that $u_k - \delta w_k \leq T_k(\psi) + 1$. Let us choose $k \geq M$.

- In the set $\{u_k \geq M\}$, we have $w_k = 0$ so $u_k - \delta w_k = u_k \leq T_k(\psi) + 1$.

- In the set $\{u_k \leq M\}$, we have $u_k \leq \psi$ and $-\delta w_k \leq 1$, therefore $u_k - \delta w_k \leq \psi + 1$, and as $u_k \leq M$, then $u_k - \delta w \leq M + 1 \leq k + 1$, which gives, $u_k - \delta w_k \leq T_k(\psi) + 1$.

  We now take $u_k - \delta w_k$ as test function in (2.6), then from (1.3):

$$\int_\Omega a(x, u_k)\nabla u_k \nabla(T_M(u_k) - T_M(\sup(u, u_k)))\varphi'_\lambda(v_k)(\varphi + M_0 + 1)^+ dx+$$

$$+ \int_\Omega a(x, u_k)\nabla u_k \nabla(\varphi + M_0 + 1)^+ \varphi_\lambda(v_k) dx \leq$$

$$\leq \int_\Omega b(|u_k|)|\nabla u_k|^2 |\varphi_\lambda(v_k)|(\varphi + M_0 + 1)^+ dx+$$

$$+ \int_\Omega b(|u_k|) f(x)|\varphi_\lambda(v_k)|(\varphi + M_0 + 1)^+ dx + \langle h, \varphi_\lambda(v_k)(\varphi + M_0 + 1)^+\rangle$$

We choose $M > M_0 + 1$. Then,

- in the set $\{u_k \leq -M\}$, $w_k = 0$.
- in the set $\{u_k \geq M\}$ we have $w_k = 0$ too.
- in the set $\{u_k \geq u\}$, we have $w_k = 0$ too.

  Then,

$$\int_{\{u_k \leq u\}} a(x, u_k)\nabla(T_M(u_k) - T_M(u))\nabla(T_M(u_k)+$$

$$- T_M(u))\varphi'_\lambda(v_k)(\varphi + M_0 + 1)^+ dx+$$

$$+ \int_{\{u_k \leq u\}} a(x, u_k)\nabla T_M(u)\nabla(T_M(u_k) - T_M(u))\varphi'_\lambda(v_k)(\varphi + M_0 + 1)^+ dx \leq$$

$$\leq C_M \int_{\{u_k \leq u\}} |\nabla(T_M(u_k) - T_M(u))|^2 |\varphi_\lambda(v_k)|(\varphi + M_0 + 1)^+ dx+$$

$$+ C_M \int_{\{u_k \leq u\}} |\nabla T_M(u)|^2 |\varphi_\lambda(v_k)|(\varphi + M_0 + 1)^+ dx+$$

$$+ C_M \int_\Omega f(x)|\varphi_\lambda(v_k)|(\varphi + M_0 + 1)^+ dx + \langle h, \varphi_\lambda(v_k)(\varphi + M_0 + 1)^+\rangle+$$

$$- \int_\Omega a(x, u_k)\nabla T_M(u_k)\nabla(\varphi + M_0 + 1)^+ \varphi_\lambda(v_k) dx.$$

For λ large enough, we obtain,

$$\begin{aligned}
&\frac{\alpha}{2}\int_{\{u_k\le u\}}\nabla(T_M(u_k)-T_M(u))\nabla(T_M(u_k)-T_M(u))(\varphi+M_0+1)^+dx\le\\
&\le-\int_{\{u_k\le u\}}a(x,u_k)\nabla T_M(u)\nabla(T_M(u_k)-T_M(u))\varphi'_\lambda(v_k)(\varphi+M_0+1)^+dx+\\
&\quad-\int_{\{u_k\le u\}}a(x,u_k)\nabla T_M(u_k)\nabla(\varphi+M_0+1)^+\varphi_\lambda(v_k)dx+\\
&\quad+C_M\int_{\{u_k\le u\}}|\nabla T_M(u)|^2|\varphi_\lambda(v_k)|(\varphi+M_0+1)^+dx+\\
&\quad+C_M\int_{\{u_k\le u\}}f(x)|\varphi_\lambda(v_k)|(\varphi+M_0+1)^+dx+\langle h,\varphi_\lambda(v_k)(\varphi+M_0+1)^+\rangle.
\end{aligned}\tag{2.13}$$

But,

$$\begin{aligned}
&\frac{\alpha}{2}\int_{\{u_k\le u\}}\nabla(T_M(u_k)-T_M(u))\nabla(T_M(u_k)-T_M(u))(\varphi+M_0+1)^+dx\ge\\
&\ge\frac{\alpha}{2}\int_{\{u_k\le u\}\cap\{\varphi\ge -M_0\}}\nabla(T_M(u_k)-T_M(u))\nabla(T_M(u_k)-T_M(u))(\varphi+M_0+1)^+dx\ge\\
&\ge\frac{\alpha}{2}\int_{\{u_k\le u\}\cap\{\varphi\ge -M_0\}}\nabla(T_M(u_k)-T_M(u))\nabla(T_M(u_k)-T_M(u))dx.
\end{aligned}\tag{2.14}$$

Moreover, as,

- $T_M(u_k)\to T_M(u)$ weakly in $H_0^1(\Omega)$
- $T_M(u_k)\to T_M(u)$ in $L^\infty(\Omega)$ weak$\star$ and a.e. in Ω
- $w_k\to 0$ in $H_0^1(\Omega)$ weak,$L^\infty(\Omega)$ weak $\star$ and a.e. in Ω,

we can show that the right hand side of (2.13), tends to zero as k tends to infinity. From (2.13) and (2.14), we can deduce that,

$$\lim_{k\to+\infty}\int_{\{u_k\le u\}\cap\{\varphi\ge -M_0\}}\nabla(T_M(u_k)-T_M(u))\nabla(T_M(u_k)-T_M(u))dx=0.\tag{2.15}$$

We now choose as test functions: $v_{k,l}=T_M(u_l)-T_M(\sup(u_k,u_l))$ and $w_{k,l}=\varphi_\lambda(v_{k,l})(\varphi+M_0+1)^+$, where u_k and u_l are two elements of the sequence $(u_k)_{k\in\mathbb{N}}$. Choosing $l\ge M$, we can take $u_l-\delta w_{k,l}$, as test function in the equation (2.6) corresponding to l, and we obtain:

$$\int_{\{u_l \le u_k\}} a(x,u_l)\nabla u_l \nabla (T_M(u_l) - T_M(\sup(u_k,u_l)))\varphi'_\lambda(v_{k,l})(\varphi + M_0 + 1)^+ dx +$$

$$+ \int_{\{u_l \le u_k\}} a(x,u_l)\nabla u_l \nabla (\varphi + M_0 + 1)^+ \varphi_\lambda(v_{k,l}) dx \le$$

$$\le \int_{\{u_l \le u_k\}} b(|u_l|)|\nabla u_l|^2 |\varphi_\lambda(v_{k,l})|(\varphi + M_0 + 1)^+ dx +$$

$$+ \int_{\{u_l \le u_k\}} !b(|u_l|) f(x)|\varphi_\lambda(v_{k,l})|(\varphi + M_0 + 1)^+ dx + \langle h, \varphi_\lambda(v_{k,l})(\varphi + M_0 + 1)^+ \rangle.$$

So,

$$\int_{\{u_l \le u_k\}} a(x,u_l)\nabla (T_M(u_l) - T_M(u_k))\nabla (T_M(u_l) +$$

$$- T_M(u_k))\varphi'_\lambda(v_{k,l})(\varphi + M_0 + 1)^+ dx +$$

$$+ \int_{\{u_l \le u_k\}} a(x,u_l)\nabla T_M(u_k)\nabla (T_M(u_l) +$$

$$- T_M(u_k))\varphi'_\lambda(v_{k,l})(\varphi + M_0 + 1)^+ dx \le$$

$$\le C_M \int_{\{u_l \le u_k\}} |\nabla (T_M(u_l) - T_M(u_k))|^2 |\varphi_\lambda(v_{k,l})|(\varphi + M_0 + 1)^+ dx +$$

$$+ C_M \int_{\{u_l \le u_k\}} |\nabla T_M(u_k)|^2 |\varphi_\lambda(v_{k,l})|(\varphi + M_0 + 1)^+ dx +$$

$$+ C_M \int_{\{u_l \le u_k\}} f(x)|\varphi_\lambda(v_{k,l})|(\varphi + M_0 + 1)^+ dx +$$

$$- \int_{\{u_l \le u_k\}} a(x,u_l)\nabla T_M(u_l)\nabla (\varphi + M_0 + 1)^+ \varphi_\lambda(v_{k,l}) dx +$$

$$+ \langle h, \varphi_\lambda(v_{k,l})(\varphi + M_0 + 1)^+ \rangle.$$

For λ large enough, we have,

$$\frac{\alpha}{2}\int_{\{u_l\le u_k\}}\nabla(T_M(u_l)-T_M(u_k))\nabla(T_M(u_l)-T_M(u_k))(\varphi+M_0+1)^+dx\le$$

$$\le-\int_{\{u_l\le u_k\}}a(x,u_l)\nabla T_M(u_k)\nabla(T_M(u_l)+$$

$$-T_M(u_k))\varphi'_\lambda(v_{k,l})(\varphi+M_0+1)^+dx+$$

$$+C_M\int_{\{u_l\le u_k\}}|\nabla T_M(u_k)|^2|\varphi_\lambda(v_{k,l})|(\varphi+M_0+1)^+dx+ \tag{2.16}$$

$$+C_M\int_{\{u_l\le u_k\}}f(x)|\varphi_\lambda(v_{k,l}|(\varphi+M_0+1)^+dx+$$

$$-\int_{\{u_l\le u_k\}}a(x,u_l)\nabla T_M(u_l)\nabla(\varphi+M_0+1)^+\varphi_\lambda(v_{k,l})dx+$$

$$+\langle h,\varphi_\lambda(v_{k,l})(\varphi+M_0+1)^+\rangle$$

but,

(2.17)

$$\frac{\alpha}{2}\int_{\{u_l\le u_k\}}\nabla(T_M(u_l)-T_M(u_k))\nabla(T_M(u_l)-T_M(u_k))(\varphi+M_0+1)^+dx\le$$

$$\ge\frac{\alpha}{2}\int_{\{u_l\le u_k\}\cap\{\varphi\ge-M_0\}}\nabla(T_M(u)-T_M(u_k))\nabla(T_M(u)-T_M(u_k))dx+$$

$$+\alpha\int_{\{u_l\le u_k\}\cap\{\varphi\ge-M_0\}}\nabla(T_M(u)-T_M(u_k))\nabla(T_M(u_l)-T_M(u))dx\,.$$

Letting l tend to $+\infty$, we have

$$T_M(u_l)\to T_M(u)\text{ weakly in }H_0^1(\Omega)$$

$$\chi_{\{u_l \le u_k\}} \to \chi_{\{u \le u_k\}} \text{ in } L^\infty(\Omega) \text{ weak }_\star, \text{ and a.e. in } \Omega,$$

thus the right hand side of (2.17) tends to,

$$\frac{\alpha}{2}\int_{\{u\le u_k\}\cap\{\varphi\ge -M_0\}} \nabla(T_M(u)-T_M(u_k))\nabla(T_M(u)-T_M(u_k))dx.$$

Moreover,

$$a(x,u_l)\nabla T_M u_k \varphi'_\lambda(v_{k,l})\chi_{\{u_l\le u_k\}}(\varphi+M_0+1)^+$$

$$\to a(x,u)\nabla T_M u_k \varphi'_\lambda(T_M(u)-T_M(u_k))\chi_{\{u\le u_k\}}(\varphi+M_0+1)^+$$

$$\text{in } L^2(\Omega) \text{ strong}$$

$$(\varphi+M_0+1)^+|\varphi_\lambda(v_{k,l})|\chi_{\{u_l\le u_k\}} \to$$

$$\to (\varphi+M_0+1)^+|\varphi_\lambda(T_M u - T_M u_k)|\chi_{\{u\le u_k\}}$$

$$\text{in } L^\infty(\Omega) \text{ weak} \star .$$

Assuming $h = h_1 - \mathrm{div} h_2$, where h_1 and h_2 are in $L^2(\Omega)$, one can also verify that,

$$\langle h, \varphi_\lambda(v_{k,l})(\varphi+M_0+1)^+\rangle \to$$

$$\to \int_{\{u\le u_k\}} h_1\varphi_\lambda(T_M(u)-T_M(u_k))(\varphi+M_0+1)^+dx+$$

$$+\int_{\{u\le u_k\}} h_2\nabla(T_M(u)-T_M(u_k))\varphi'_\lambda(T_M(u)-T_M(u_k))(\varphi+M_0+1)^+dx+$$

$$+\int_{\{u\le u_k\}} h_2\nabla(\varphi+M_0+1)^+\varphi_\lambda(T_M(u)-T_M(u_k))dx.$$

Regrouping (2.16) and (2.17), and if l tends to infinity, we obtain that,

(2.18)

$$\frac{\alpha}{2}\int_{\{u\le u_k\}\cap\{\varphi\ge -M_0\}} \nabla(T_M(u)-T_M(u_k))\nabla(T_M(u)-T_M(u_k))dx$$

$$-\int_{\{u\le u_k\}} a(x,u)\nabla T_M(u_k)\nabla(T_M(u)+$$

$$-T_M(u_k))\varphi'_\lambda(T_M(u)-T_M(u_k))(\varphi+M_0+1)^+dx+$$

$$+C_M\int_{\{u\le u_k\}} |\nabla T_M(u_k)|^2|\varphi_\lambda(T_M(u)-T_M(u_k))|(\varphi+M_0+1)^+dx+$$

$$+C_M\int_{\{u\le u_k\}} f(x)|\varphi_\lambda(T_M(u)-T_M(u_k))|(\varphi+M_0+1)^+dx+$$

$$+\int_{\{u\le u_k\}} h_1\varphi_\lambda(T_M(u)-T_M(u_k))(\varphi+M_0+1)^+dx+$$

$$+\int_{\{u\le u_k\}} h_2\nabla(T_M(u)-T_M(u_k))\varphi'_\lambda(T_M(u)-T_M(u_k))(\varphi+M_0+1)^+dx+$$

$$+\int_{\{u\le u_k\}} h_2\nabla(\varphi+M_0+1)^+\varphi_\lambda(T_M(u)-T_M(u_k))dx.$$

But from Lemma 2.2,

$$-\int_{\{u\le u_k\}} a(x,u_k)\nabla T_M(u_k)\nabla(T_M(u)-T_M(u_k))\varphi'_\lambda(T_M(u)+$$

$$-T_M(u_k))(\varphi+M_0+1)^+dx=$$

$$=\int_{\{u\le u_k\}} a(x,u_k)\nabla T_M(u_k)\nabla T_M(u_k)\varphi'_\lambda(T_M(u)+$$

$$-T_M(u_k))(\varphi+M_0+1)^+dx+$$

$$-\int_{\{u\le u_k\}} a(x,u_k)\nabla T_M(u_k)\nabla T_M(u_k)\varphi'_\lambda(T_M(u)+$$

$$-T_M(u_k))(\varphi+M_0+1)^+dx\le$$

$$\le C_M\int_{\{u\le u_k\}} |\varphi'_\lambda(T_M(u)-T_M(u_k))(\varphi+M_0+1)^+|dx+$$

$$-\int_{\{u\le u_k\}} a(x,u)\nabla T_M(u_k)\nabla T_M(u)\varphi'_\lambda(T_M(u)-T_M(u_k))(\varphi+M_0+1)^+dx$$

similarly,

$$\int_{\{u\leq u_k\}} |\nabla T_M(u_k)|^2|\varphi_\lambda(T_M(u)-T_M(u_k))|(\varphi+m_0+1)^+dx \leq$$
$$\leq C_M\int_{\{u\leq u_k\}} |\varphi_\lambda(T_M(u)-T_M(u_k))|(\varphi+M_0+1)^+dx$$

so, we can show as before, that the second member of (2.18) tends to zero as k tends to infinity and finally,

$$\lim_{k\to+\infty}\int_{\{u\leq u_k\}\cap\{\varphi\geq -M_0\}} \nabla(T_M(u)+ - T_M(u_k))\nabla(T_M(u)-T_M(u_k))dx = 0. \tag{2.19}$$

From (2.15) and (2.19), we deduce that,

$$\lim_{k\to+\infty}\int_{\{\varphi\geq -M_0\}} \nabla(T_M(u)-T_M(u_k))\nabla(T_M(u)-T_M(u_k))dx = 0,$$

and similarly, if we replace $(\varphi+M_0+1)^+$ by $(M_0+1-\psi)^+$, we can prove that,

$$\lim_{k\to+\infty}\int_{\{\psi\leq M_0\}} \nabla(T_M(u)-T_M(u_k))\nabla(T_M(u)-T_M(u_k))dx = 0$$

and then finally,

$$\lim_{k\to+\infty}\int_{\Omega} \nabla(T_M(u)-T_M(u_k))\nabla(T_M(u)-T_M(u_k)dx = 0.$$

This is the end of the proof of Lemma 2.4.

4^{th} step: passage to the limit in the equation.

We consider the function β defined by:

$$\begin{cases} 1 & \text{if } -M\leq s\leq M \\ 0 & \text{if } s\geq M+1 \text{ or } s\leq -M-1 \\ -s+M+1 & \text{if } M\leq s\leq M+1 \\ s+M+1 & \text{if } -M-1\leq s\leq -M. \end{cases}$$

Let $\Phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be such that: $\exists\, M \geq 0$ such that $\Phi = 0$ in $\{|u| \geq M\}$. We consider $v = u_k + \delta\Phi\beta(u_k)$, then $v \in H_k$ if $k \geq M+1$ and $0 \leq \delta \leq \frac{1}{\|\Phi\|_\infty}$. Thus we can take v as test function in (2.6), similarly we can take $w = u_k - \delta\Phi\beta(u_k)$, which finally gives,

$$\begin{aligned}(2.20)\qquad &\int_\Omega a(x,u_k)\nabla u_k \nabla(\Phi\beta(u_k))dx+\\ &+\int_\Omega g(x,u_k,\nabla u_k)\Phi\beta(u_k)dx = \langle h, \Phi\beta(u_k)\rangle.\end{aligned}$$

We now study the convergence of the term $\Phi\beta(u_k)$. We have

$$\begin{aligned}&\int_\Omega |\nabla(\Phi\beta(u_k)) - \nabla\Phi|^2 dx = \int_\Omega |\nabla\Phi|^2(1-\beta(u_k))^2 dx+\\ &\quad -2\int_\Omega \nabla\Phi\nabla u_k \beta'(u_k)\Phi(1-\beta(u_k))dx + \int_\Omega |\nabla u_k|^2|\Phi|^2(\beta'(u_k))^2 dx =\\ &= \int_\Omega |\nabla\Phi|^2(1-\beta(u_k))^2 dx - 2\int_{\{M\leq|u_k|\leq M+1\}} \nabla\Phi\nabla u_k\Phi(1-\beta(u_k))dx+\\ &\quad +\int_{\{M\leq|u_k|\leq M+1\}} |\nabla u_k|^2|\Phi|^2 dx.\end{aligned}$$

Observe that

$$\int_\Omega |\nabla\Phi|^2(1-\beta(u_k))^2 dx \to \int_\Omega |\nabla\Phi|^2(1-\beta(u))^2 dx = 0\,,$$

$$\begin{aligned}&-2\int_{\{M\leq|u_k|\leq M+1\}} \nabla\Phi\nabla u_k\Phi(1-\beta(u_k))dx \to\\ &\qquad \to -2\int_{\{M\leq|u|\leq M+1\}} \nabla\Phi\nabla\Phi(1-\beta(u))dx = 0\,,\end{aligned}$$

$$\int_{\{M\leq|u_k|\leq M+1\}} |\nabla u_k|^2|\Phi|^2 dx \to \int_{\{M\leq|u|\leq M+1\}} |\nabla u|^2|\Phi|^2 dx = 0.$$

Therefore $\Phi\beta(u_k) \to \Phi$ in $H_0^1(\Omega)$, and thus,

$$\begin{aligned}&\int_\Omega a(x,u_k)\nabla u_k\nabla(\Phi\beta(u_k))dx =\\ &\qquad = \int_\Omega a(x,u_k)\nabla(T_M u_k)\nabla(\Phi\beta(u_k))dx \to \int_\Omega a(x,u)\nabla u\nabla\Phi dx\end{aligned}$$

and, as $g(u_k, \nabla u_k)\Phi\beta(u_k)$ is equiintegrable,

$$\int_\Omega g(u_k, \nabla u_k)\Phi\beta(u_k)dx \to \int_\Omega g(x, u, \nabla u)\Phi dx.$$

On another hand,

$$\langle h, \Phi\beta(u_k)\rangle \to \langle h, \Phi\rangle$$

then, we can pass to the limit in (2.20), and we obtain,

$$\int_\Omega a(x, u)\nabla u\nabla\Phi dx + \int_\Omega g(x, u, \nabla u)\Phi dx = \langle h, \Phi\rangle$$

and Theorem 2.1 is proved.

3 – An application

We suppose that,

$$g(x, s, \xi) = g_1(x, s, \xi) - f$$

where,

$$g_1(x, s, \xi) \geq 0 \tag{3.1}$$

$$g_1(x, 0, 0) = 0 \tag{3.2}$$

$$f \in L^1(\Omega),\ \ f \geq 0. \tag{3.3}$$

We consider the problem:

$$\begin{cases} -\operatorname{div}(a(x, u)\nabla u) + g_1(x, u, \nabla u) = f \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{3.4}$$

This type of problem is studied in [1]. From hypothesis (3.2) and because f is positive, we verify that $\varphi = 0$ is a subsolution of (3.4). On another hand (see [2]), we know that there exists a function ψ such that:

$$\psi \in W_0^{1,q}(\Omega) \text{ for every } q < \frac{N}{N-1}$$

$$T_k(\psi) \in H_0^1(\Omega) \text{ for every } k \in \mathbb{N}$$

which is a solution in the sense of distribution of,

$$- \operatorname{div}(a(x, \psi)\nabla\psi) = f.$$

Moreover, if we multipy (3.4) by $-T_k(\psi)$, we easily show that $\psi \geq 0$, so that ψ is a renormalized super solution of (3.4). Then we can apply Theorem 2.1 to problem (3.4), thus obtaining the existence of a solution.

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