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# A remark on optimal temporal decay estimates for systems of multidimensional parabolic conservation laws

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*In memory of a good friend and a fine mathematician*

Riassunto: *Il lavoro concerne le stime di decadimento temporale per le soluzioni del problema di Cauchy di un sistema di leggi di conservazione paraboliche a più dimensioni. I risultati ottenuti migliorano quelli ottenuti dagli autori in un lavoro precedente.*

Abstract: *This paper is concerned with optimal temporal decay estimates for solutions of the Cauchy problem of a systems of multidimensional parabolic conservation laws. The results obtained in this paper supplement the results obtained in our previous paper.*

## 1 – Introduction and statement of the main results

This paper is concerned with optimal temporal decay estimates for solutions of the following multidimensional parabolic conservation laws

$$u_t + \sum_{j=1}^{N} f_j(u)_{x_j} = D\Delta u, \quad x \in R^N, \quad t > 0\,, \tag{1.1}$$

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with initial data

$$u(t,x)|_{t=0} = u_0(x), \quad x \in R^N, \quad N > 1, \quad n > 1\,. \tag{1.2}$$

Here $u(t,x) = (u_1(t,x), \cdots, u_n(t,x))^t$ is the unknown vector, $f_j(u) = \big(f_{j1}(u), \cdots, f_{jn}(u)\big)^t (j = 1, 2, \cdots, N)$ are arbitrary $n \times 1$ smooth vector-valued flux functions defined in $\overline{B}_r(0)$, a closed ball of radius $r$ centered at $u = 0$, and $D$ is a constant, diagonalizable matrix with positive eigenvalues (Without loss of generality, we can always assume $D = \mathrm{diag}(d_{11}, \cdots, d_{nn})$ with $d_{ii} > 0, i = 1, 2, \cdots, n$, in what follows.)

The Cauchy problem (1.1), (1.2) has been studied by many authors so in this paper we will not review the former results one by one, but only state the results which are closely related to the theme of this paper. The interested reader is referred to [2]-[4] and the references cited therein for more information, For the corresponding work on the multidimensional diffusion wave for the Navier-Stokes equations of compressible flow, see D. Hoff and K. Zumbrun [11], [12] and the references cited therein.

For general $N > 1, n > 1$, the results obtained in [3] show that if for each $r > 0$, the smooth nonlinear flux functions $f_j(u)(j = 1, 2, \cdots, N)$ satisfy

$$\frac{f_j(u)}{|u|^s} \in L^\infty(\overline{B}(0), R^n), \quad j = 1, 2, \cdots, N\,, \tag{1.3}$$

then for $u_0(x) \in L^1 \cap L^\infty(R^N, R^n)$ with $\|u_0(x)\|_{L^\infty(R^N,R^n)} < r$, $\|u_0(x)\|_{L^1(R^N,R^n)}$ sufficiently small, one can get the follwing results

i) If $s \geq 1 + \frac{1}{N}$, the Cauchy problem (1.1), (1.2) admits a unique globally smooth solution $u(t,x)$. Furthermore there exists a constant $\tau \in (0,1)$ such that for all $k \in Z^+$, $u(t,x)$ satisfies

$$\|u(t,x)\|_{L^\infty(R^+ \times R^N, R^n)} \leq r\,, \tag{1.4}$$

$$\begin{aligned} &\sup_{[0,\infty)} \{\|u(t,x)\|_{L^1(R^N,R^n)} + t^{\frac{N}{2(N+1)}} \|u(t,x)\|_{L^{\frac{N+1}{N}}(R^N,R^n)}\} \leq \\ &\leq C(r)\|u_0(x)\|_{L^1(R^N,R^n)}\,, \end{aligned} \tag{1.5}$$

and

$$
(1.6)\qquad \begin{cases} \|\Delta^{\frac{k}{2}}u(t,x)\|_{L^\infty([\tau,\infty)\times R^N,R^n)} \leq M_k(r,\tau)\,, \\ \sup\limits_{[\tau,\infty)}\{\|\Delta^{\frac{k}{2}}u(t,x)\|_{L^2(R^N,R^n)}\} \leq \overline{M}_k(r,\tau)\,, \\ \|\Delta^{\frac{k}{2}}u(\tau,x)\|_{L^1(R^N,R^n)} \leq \overline{N}_k(r,\tau)\,. \end{cases}
$$

Here $M_k, \overline{M}_k, \overline{N}_k$ are positive constants independent of $t$, and throughout this paper the symbol $C(\cdot)$ will be used to denote a generic positive constant which depends only on the quantities stated in the parenthesis but may vary from time to time.

ii) If $s > 2 + \frac{1}{N}$, the globally smooth solution $u(t,x)$ obtained in i) satisfies the following optimal temporal decay estimates: For each $k \in Z^+, 0 < \tau \leq t$, one can deduce that

$$
(1.7)\qquad \|\Delta^{\frac{k}{2}}u(t,x)\|_{L^2(R^N,R^n)} \leq C(r,\tau)(1+t)^{-\frac{N+2k}{4}}\,.
$$

From the above results, one can easily deduce that to obtain the optimal temporal decay estimates (1.7), the sufficient conditions that must be imposed on the nonlinear flux functions $f_j(u)(j=1,2,\cdots,N)$ are $f_j(u) = O(|u|^3)(j=1,2,\cdots,N)$ as $|u| \to 0$, which are stronger than the corresponding sufficient conditions to guarantee the global existence results. Such a fact is also noticed by the authors in [3], and in fact the authors point out in the second remark following immediately after Theorem 1 of [3] that if the system (1.1) admits a strictly convex quadratic entropy function $\eta(u)$ which is strongly consistent with the viscous matrix $D$, then the optimal temporal decay estimate (1.7) can still be obtained, but only under the conditions $f_j(u) = O(|u|^2)(j=1,2,\cdots,N)$ as $|u| \to 0$. However, as pointed out in [3], for $n > 2$, the corresponding entropy equation is overdetermined and the existence of a nontrivial entropy maybe attributed only to a fortunate coincidence. Hence for general systems of type (1.1), it is of some interests to see if the sufficient conditions imposed on the nonlinear flux functions $f_j(u)(j=1,2,\cdots,N)$ to guarantee the global existence results are also sufficient to deduce the optimal temporal decay estimate (1.7). One of our main contributions of this paper is to give a positive answer to the above problem and, in this sense, our present paper is a supplement to our previous one [3].

The above mentioned result can be stated as follows.

THEOREM 1.1. *Suppose that for each fixed* $r > 0$*, the smooth nonlinear flux functions* $f_j(u)(j = 1, 2, \cdots, N)$ *satisfy*

$$\frac{f_j(u)}{|u|^2} \in L^\infty(\overline{B}(0), R^n)\,, \quad j = 1, 2, \cdots, N\,, \tag{1.8}$$

*then for* $u_0(x) \in L^1 \cap L^\infty(R^N, R^n)$ *with* $\|u_0(x)\|_{L^\infty(R^N,R^n)} < r, \|u_0(x)\|_{L^1(R^N,R^n)}$ *sufficiently small, the unique globally smooth solution* $u(t, x)$ *obtained in Theorem* 1 *of* [3] *satisfies the optimal temporal decay estimates* (1.7)*.*

From Theorem 1.1 and the standard interpolation technique, we can deduce that for each multiindex $\alpha = (\alpha_1, \cdots, \alpha_N)$ with $|\alpha| = \sum_{i=1}^N \alpha_i = k \in Z^+, 2 \le p \le \infty, t \ge \tau > 0$, the following estimates hold

$$\Big\|\frac{\partial^k}{\partial x^\alpha} u(t, x)\Big\|_{L^p(R^N,R^n)} \le C(r, \tau)(1 + t)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{k}{2}}\,. \tag{1.9}$$

Our next result in this paper is concerned with the case $1 \le p < 2$. For result in this direction, we have

THEOREM 1.2. *Under the conditions of Theorem* 1.1*, we have that the estimates* (1.9) *hold for each multiindex* $\alpha = (\alpha_1, \cdots, \alpha_N)$ *with* $|\alpha| = \sum_{i=1}^N \alpha_i = k \in Z^+, 1 \le p \le \infty, t \ge \tau > 0$*.*

Our final goal in this paper is concerned with the asymptotic profile of the globally smooth solution $u(t, x)$ obtained in [3]. To state our result in this direction precisely, we first introduce some notations.

First, we use $\phi(t, x)$ to denote the unique globally smooth solution to the following Cauchy problem

$$\begin{cases} \phi_t = D\Delta\phi, \quad x \in R^N, \quad N > 1, \quad t > 0\,, \\ \phi(t, x)|_{t=0} = u_0(x), \quad x \in R^N\,. \end{cases} \tag{1.10}$$

It is easy to see that $\phi(t, x)$ has the following explicit expression

$$\begin{cases} \phi(t, x) = (\phi_1(t, x), \cdots, \phi_n(t, x))^t\,, \\ \phi_j(t, x) := \displaystyle\int_{R^N} (4\pi d_{jj} t)^{-\frac{N}{2}} \exp\Big(-\frac{|x - y|^2}{4d_{jj}t}\Big) u_0(y)dy, \\ \qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \cdots, n\,, \end{cases} \tag{1.11}$$

and for each multiindex $\alpha = (\alpha_1, \cdots, \alpha_N), |\alpha| = k, 1 \leq p \leq \infty$, $\phi(t,x)$ satisfies

$$\Big\| \frac{\partial^k}{\partial x^\alpha}(\phi(t,x) - \overline{\phi}(t,x)) \Big\|_{L^p(R^N, R^n)} \leq C(r,\tau)(1+t)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{k+1}{2}} \tag{1.12}$$

provided that $u_0(x)$ satisfies the additional assumption

$$\int_{R^N} |x||u_0(x)|dx < \infty\,.$$

Here $\overline{\phi}(t,x)$ solves the following Cauchy problem

$$\begin{cases} \overline{\phi}_t = D\Delta\overline{\phi}\,, \\ \overline{\phi}(t,x)|_{t=0} = (\overline{\phi}_{01}(x), \cdots, \overline{\phi}_{0n}(x))^t\,, \end{cases} \tag{1.13}$$

with

$$\begin{cases} \overline{\phi}_{0j}(x) = \delta_j(4\pi d_{jj})^{-\frac{N}{2}} \exp\Big( - \dfrac{|x|^2}{4d_{jj}}), \\ \delta_j := \displaystyle\int_{R^N} u_{0j}(x)dx, \quad j = 1, 2, \cdots, n\,. \end{cases}$$

Consequently

$$\begin{cases} \overline{\phi}(t,x) = (\overline{\phi}_1(t,x), \cdots, \overline{\phi}_n(t,x))\,, \\ \overline{\phi}_j(t,x) := \delta_j(4\pi d_{jj}(t+1))^{-\frac{N}{2}} \exp\Big( - \dfrac{|x|^2}{4d_{jj}(t+1)}\Big), \\ \qquad\qquad\qquad\qquad\qquad j = 1, \cdots, n\,. \end{cases} \tag{1.14}$$

Under the above notations and based on the results obtained in Theorem 1.1 and Theorem 1.2, we have the following result on the asymptotic profile of the globally smooth solution $u(t,x)$ obtained in [3]

Theorem 1.3 (Asymptotic profile). *Let the assumptions stated in Theorem 1.1 are satisfied, then for each multiindex $\alpha = (\alpha_1, \cdots, \alpha_N)$, $|\alpha| = k$, $1 \leq p \leq \infty$, $k \geq 0$, $t \geq \tau > 0$, we have that*

$$\Big\| \frac{\partial^k}{\partial x^\alpha}(u(t,x) - \overline{\phi}(t,x)) \Big\|_{L^p(R^N, R^n)} \leq C(r,\tau)p(t)(1+t)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{k+1}{2}}. \tag{1.15}$$

*Here*

$$p(t) = \begin{cases} \ln\,(1+t), & \text{if} \quad N = 2\,, \\ 1, & \text{if} \quad N > 2\,. \end{cases} \tag{1.16}$$

Remark 1.1. Since when $\delta_j \neq 0, j \in \{1, \cdots, n\}$, there exists a positive constant $C > 0$ such that for each multiindex $\alpha = (\alpha_1, \cdots, \alpha_N), |\alpha| = k, k \geq 0, 1 \leq p \leq \infty$

$$(1.17)\quad \frac{1}{C}(1+t)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{k}{2}} \leq \Big\|\frac{\partial^k}{\partial x^\alpha}\overline{\phi}(t,x)\Big\|_{L^p(R^N,R^n)} \leq C(1+t)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{k}{2}},$$

we can easily deduce from (1.9), (1.15), and (1.17) that in the case of $\delta_j \neq 0, j \in \{1, \cdots, n\}$, the decay estimates (1.9) are optimal and that $\overline{\phi}(t,x)$ is precisely the asymptotic profile of $u(t,x)$.

When $\delta_j = 0, j = 1, \cdots, n$, from (1.15) we can get the following improved decay estimates for $u(t,x)$

$$(1.18)\qquad \Big\|\frac{\partial^{|\alpha|}}{\partial x^\alpha}u(t,x)\Big\|_{L^p(R^N,R^n)} \leq C(r,\tau)p(t)(1+t)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}}\,.$$

Such an estimate is not optimal and in fact we can get the following improved decay estimates

Theorem 1.4. *Let the assumptions stated in Theorem* 1.1 *are satisfied, if* $\delta_j = 0, j = 1, 2, \cdots, n$, *then the globally smooth solution* $u(t,x)$ *obtained in* [3] *satisfies the following decay estimates*

$$(1.19)\qquad \begin{aligned}&\Big\|\frac{\partial^{|\alpha|}}{\partial x^\alpha}u(t,x)\Big\|_{L^p(R^N,R^n)} \leq \\ &\qquad \leq C(r,\tau)(1+t)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}},\ 1 \leq p \leq \infty,\ t \geq \tau\,.\end{aligned}$$

Generally speaking, the decay estimates (1.19) can not be improved even if $\int_{R^N} x_j^m u_0(x)dx = 0$ for $j = 1, 2, \cdots, n$ and some $m \geq 1$. Such a result is the content of our final theorem in this paper

Theorem 1.5. *In addition to the assumptions stated in Theorem* 1.4, *we assume further that*

$$(1.20)\qquad \int_{R^N}|x||u_0(x)|dx < \infty, \quad \int_{R^N}|x|^2|u_0(x)|dx < \infty\,,$$

*and that there exists a time-independent constant* $L_1 > 0$ *such that for* *some* $\bar{j} \in \{1, 2, \cdots, N\}, \bar{k} \in \{1, 2, \cdots, n\}, t \geq T_1 > 1$

$$\Big| \int_{R^N} x_{\bar{j}} u_{0\bar{k}}(x) dx + \int_0^t \int_{R^N} f_{\bar{j}\bar{k}}(u(s,x)) dx\, ds \Big| \geq L_1 > 0 \,, \tag{1.21}$$

*where* $T_1 > 1$ *is an arbitrarily fixed constant.*

*Then we can find a positive constant, which depends only on* $T_1$ *and* $\tau$*, such that for* $t \geq \tau > 0$

$$\Big\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} u(t,x) \Big\|_{L^2(R^N, R^n)} \geq L_2(T_1, \tau)(1+t)^{-\frac{N+2(|\alpha|+1)}{4}} \,. \tag{1.22}$$

Remark 1.2. Theorem 1.5 means that even if the initial data $u_0(x)$ satisfies

$$\int_{R^N} x_j^m u_0(x) dx = 0 \quad \text{for} \quad j = 1, 2, \cdots, N \quad \text{andsome} \quad m \geq 1 \,,$$

the estimates (1.19) can not be improved any longer. Such a result is quite different to the corresponding decay estimates to the Cauchy problem (1.10) with its corresponding initial data $u_0(x)$ satisfying the assumptions stated in Theorem 1.5. Such a phenomenon, which was first studied by M. Schonbek and S. Rajopadhye in [5] for the multidimensional KdV-Burgers system, shows that long waves, which are due to the nonlinear flux functions $f_j(u) (j = 1, 2, \cdots, N)$, can be created also for the solutions to the multidimensional parabolic conservation laws (1.1).

Remark 1.3. When $\int_{R^N} x_{\bar{j}} u_{0\bar{k}}(x) dx = 0$, if we assume that $f_{\bar{j}\bar{k}}(u) \geq 0$ and $f_{\bar{j}\bar{k}}(u) = 0$ if and only if $u = 0$, then the assumption (1.21) is easily seen to be satisfied and in this case the constant $L_1$ can be chosen as $\int_0^1 \int_{R^N} f_{\bar{j}\bar{k}}(u(s,x)) dx ds > 0$.

Remark 1.4. From the proof of Theorem 1.5, one can easily give other similar assumptions to replace (1.21) while the same result of Theorem 1.5 still holds.

Remark 1.5. In Theorem 1.5, we can only get the lower bounds of the decay rates for solutions to multidimensional parabolic conservation laws in the $L^2(R^N, R^n)$ setting. It would be of some interest to get the corresponding results in the $L^p(R^N, R^n) (1 \leq p \leq \infty)$ setting and we hope that we can come back to tackle such a problem in the near future.

REMARK 1.6. Although when $\delta_j = 0 (j = 1, 2, \cdots, n)$, we can get the optimal decay estimates for $u(t,x)$ but how to describe its asymptotic profile remains an open question. Such a problem will be our research in the future.

REMARK 1.7. It is our pleasure to point out that our Theorem 1.5 is motivated by the work of M. SCHONBEK and S. RAJOPADHYE [5] on the large time behaviour of solutions to the multidimensional KdV-Burgers system. Compared with that of [5], the assumptions we imposed on the initial data and the nonlinear flux functions $f_j(u) (j = 1, 2, \cdots, N)$ are weaker. Such an improvement is due to our sharp decay estimates obtained in Theorem 1.1-Theorem 1.4 which are our main contributions in the study of the decay estimates of solutions to multidimensional parabolic conservation laws.

This paper is arranged as follows: After this introduction and the statement of the main results, which constitutes Section 1, we prove Theorem 1.1, Theorem 1.2, and Theorem 1.3-Theorem 1.5 in Section 2, Section 3, and Section 4 respectively. Some remarks concerning the large time behaviour of solutions to multidimensional parabolic conservation laws with large initial data will be given in Section 5.

## 2 – The proof of Theorem 1.1

This section is devoted to proving Theorem 1.1.

Let $K(t,x)$ be the fundamental solution associated with the operator $\frac{\partial}{\partial t} - D\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$. That is, $K(t,x)$ is an $n$-vector whose $j$-th component is

$$k_j(t,x) = t(4\pi d_{jj} t)^{-\frac{N}{2}} \exp\Big( - \frac{|x|^2}{4 d_{jj} t}\Big) . \tag{2.1}$$

Then, the solution $u(t,x)$ of the Cauchy problem (1.1), (1.2) satisfies the integral representation

$$u(t,x) = K(t,x) * u_0(x) + \sum_{j=1}^{N} \int_0^t K_{x_j}(t-s,x) * f_j(u(s,x)) ds \, , \tag{2.2}$$

where $*$ denotes convolution in space, taken componentwise.

For later use, we first record the following fundamental results

Lemma 2.1 (Gronwall's Inequality). *Suppose that the nonnegative continuous functions $g(t), h(t)$ satisfy*

$$g(t) \leq N_1(1+t-\tau)^{-\alpha} + N_2 \int_\tau^t (1+t-s)^{-\alpha} g(s)h(s)ds, \quad t \geq s \geq \tau > 0\,, \tag{2.3}$$

*and*

$$\int_\tau^\infty h(s)ds < \infty\,, \tag{2.4}$$

*where $N_1, N_2$, and $\alpha$ are nonnegative constants.*

*Then*

$$g(t) \leq N_1(1+t-\tau)^{-\alpha} \exp\Big(N_3(\tau)N_2 \int_\tau^t h(s)ds\Big) \leq \leq C(\tau)(1+t)^{-\alpha}, \quad t \geq \tau > 0\,, \tag{2.5}$$

*where*

$$N_3(\tau) = \sup_{\tau \leq s \leq t} \Big\{\Big[\frac{1+t-\tau}{(1+t-s)(1+s-\tau)}\Big]^\alpha\Big\}. \tag{2.6}$$

The proof is trivial and so will be omitted.

Lemma 2.2. *For each $t > s \geq 0$, we have*

$$\|\nabla K(t-s,x) * u(s,x)\|_{L^2(R^N,R^n)} \leq C(1+t-s)^{-\frac{N+2}{4}} \times \times \left(\|u(s,x)\|_{L^1(R^N,R^n)} + \|\nabla u(s,x)\|_{L^2(R^N,R^n)}\right), \tag{2.7}$$

*and*

$$\|K(t-s,x) * u(s,x)\|_{L^2(R^N,R^n)} \leq C(1+t-s)^{-\frac{N}{4}} \times \times \left(|u(s,x)\|_{L^1(R^N,R^n)} + \|u(s,x)\|_{L^2(R^N,R^n)}\right). \tag{2.8}$$

Proof. We only prove (2.7). The proof of (2.8) is similar and so will be omitted

First, from the Hausdorff-Young's inequality, we have

$$\begin{aligned}
&\|\nabla K(t-s,x)*u(s,x)\|_{L^2(R^N,R^n)}\le\\
&\quad\le C\|\nabla K(t-s,x)\|_{L^2(R^N,R^n)}\|u(s,x)\|_{L^1(R^N,R^n)}\le\\
&\quad\le C(t-s)^{-\frac{N+2}{4}}\|u(s,x)\|_{L^1(R^N,R^n)}\,.
\end{aligned}\tag{2.9}$$

On the other hand

$$\begin{aligned}
&\|\nabla K(t-s,x)*u(s,x)\|_{L^2(R^N,R^n)}=\\
&\quad=\|K(t-s,x)*\nabla u(s,x)\|_{L^2(R^N,R^n)}\le\\
&\quad\le C\|K(t-s,x)\|_{L^1(R^N,R^n)}\|\nabla u(s,x)\|_{L^2(R^N,R^n)}\le\\
&\quad\le C\|\nabla u(s,x)\|_{L^2(R^N,R^n)}\,.
\end{aligned}\tag{2.10}$$

Combining (2.9) with (2.10), we can immediately get (2.7). This completes the proof of Lemma 2.2.

Lemma 2.3. *Suppose that* $u(t,x)$ *is the globally smooth solution obtained in* [3] *and that the* $f_j(u)(j=1,2,\cdots,N)$ *satisfy the assumption* (1.8)*, then*

$$\begin{aligned}
&\|u(t,x)\|_{L^2(R^N,R^n)}\le C(r,\tau)\Big\{(1+t-\tau)^{-\frac{N}{4}}+\\
&\quad+\int_\tau^t(1+t-s)^{-\frac{N+2}{4}}(\|u(s,x)\|_{L^2(R^N,R^n)}+\\
&\quad+\|\nabla u(s,x)\|_{L^\infty(R^N,R^n)})\|u(s,x)\|_{L^2(R^N,R^n)}ds\Big\},\quad t\ge\tau>0\,.
\end{aligned}\tag{2.11}$$

Proof. From (2.2), we have

$$\begin{aligned}
&u(t,x)=K(t-\tau,x)*u(\tau,x)+\\
&\qquad+\sum_{j=1}^N\int_\tau^t K_{x_j}(t-s,x)*f_j(u(s,x))ds,\quad t\ge\tau>0\,.
\end{aligned}\tag{2.12}$$

Thus we deduce from (1.5) and Lemma 2.2 that

$$
\begin{aligned}
\|u(t,x)\|_{L^2(R^N,R^n)} &\le C(1+t-\tau)^{-\frac{N}{4}}\times \\
&\times(\|u(\tau,x)\|_{L^1(R^N,R^n)}+\|u(\tau,x)\|_{L^2(R^N,R^n)})+ \\
&+C\sum_{j=1}^{N}\int_\tau^t(1+t-s)^{-\frac{N+2}{4}}\times \\
&\times(\|f_j(u(s,x))\|_{L^1(R^N,R^n)}+\|f_j(u(s,x))_{x_j}\|_{L^2(R^N,R^n)})ds\le \\
&\le C(r,\tau)\Big\{(1+t-\tau)^{-\frac{N}{4}}+ \\
&+\int_\tau^t(1+t-s)^{-\frac{N+2}{4}}(\|u(s,x)\|_{L^2(R^N,R^n)}+ \\
&+\|\nabla u(s,x)\|_{L^\infty(R^N,R^n)})\|u(s,x)\|_{L^2(R^N,R^n)}ds\Big\}, \\
&\qquad t\ge\tau>0\,.
\end{aligned}
\tag{2.13}
$$

This is (2.11) and consequently the proof of Lemma 2.3 is complete.

Lemma 2.4 (Nirenberg's inequality). *If we assume thatu* $\in L^q(R^N,R^n)$ *and* $\Delta^{\frac{m}{2}}u\in L^r(R^N,R^n)$ *with* $1\le q,r\le+\infty$ *then, for any integer* $j$ *such that* $0\le j\le m$*, we have*

$$
\|\Delta^{\frac{j}{2}}u\|_{L^p(R^N,R^n)}\le C\|\Delta^{\frac{m}{2}}u\|^{\alpha}_{L^r(R^N,R^n)}\|u\|^{1-\alpha}_{L^q(R^N,R^n)}\,,
\tag{2.14}
$$

*where* $p$ *is determined by*

$$
\frac{1}{p}=\frac{j}{N}+\alpha\Big(\frac{1}{r}-\frac{m}{N}\Big)+(1-\alpha)\frac{1}{q},\quad \frac{j}{m}\le\alpha\le1\,.
\tag{2.15}
$$

A direct corollary of Lemma 2.4 is

Corollary 2.1. *If* $u(t,x)$ *satisfies* (1.4), (1.6) *and*

$$
\|u(t,x)\|_{L^2(R^N,R^n)}\le C(r,\tau)(1+t-\tau)^{-\beta+\epsilon},\quad t\ge\tau>0\,,
\tag{2.16}
$$

*then for each* $j\in Z^+$*, we have*

$$
\|\Delta^{\frac{j}{2}}u(t,x)\|_{L^\infty(R^N,R^n)}\le C(r,\tau)(1+t-\tau)^{-\beta+\epsilon},\quad t\ge\tau>0\,.
\tag{2.17}
$$

*Here and in what follows* $\epsilon$ *will always be used to denote a sufficiently small generic positive constant which may vary from time to time.*

Lemma 2.5. *Suppose that* $0 < \tau < 1$*, then*

$$\int_\tau^t (1+t-s)^{r-1}(1+s-\tau)^{-r} ds \le B(1-r,r), \quad 0<r<1\,, \tag{2.18}$$

*and*

$$\int_\tau^t (1+s-\tau)^{-r} ds \le \frac{1}{r-1}, \quad r>1\,. \tag{2.19}$$

Using the above results, we have:

Theorem 2.1. *Under the assumptions of Theroem* 1.1*, the results of Theorem* 1.1 *are true for* $N > 2$*.*

Proof. From the remark after Corollary 3.9 of [3], we only need to get the $L^2\ (R^N, R^n)$-norm optimal temporal decay estimate, i.e., we need to show that (1.7) with $k = 0$ is true.

From (1.5), (1.6), and Corollary 2.1, we deduce that

$$\begin{cases} \|u(t,x)\|_{L^2(R^N,R^n)} \le C(r,\tau)(1+t-\tau)^{-\frac{N}{2(N+1)}+\epsilon}, & t\ge\tau>0\,, \\ \|\nabla u(t,x)\|_{L^\infty(R^N,R^n)} \le C(r,\tau)(1+t-\tau)^{-\frac{N}{2(N+1)}+\epsilon}, & t\ge\tau>0\,. \end{cases} \tag{2.20}$$

Substituting (2.20) into (2.11) gives

$$\begin{aligned} \|u(t,x)\|_{L^2(R^N,R^n)} &\le C(r,\tau)(1+t-\tau)^{-\frac{N}{4}} + \\ &\quad + C(r,\tau)\int_\tau^t (1+t-s)^{-\frac{N+2}{4}}(1+s-\tau)^{-\frac{N}{2(N+1)}+\epsilon} \times \\ &\quad \times \|u(s,x)\|_{L^2(R^N,R^n)} ds \le \\ &\le C(r,\tau)(1+t-\tau)^{-\frac{N}{4}} + C(r,\tau)\int_\tau^t (1+t-s)^{-\frac{N^2+N-2}{4(N+1)}+\epsilon} \times \\ &\quad \times (1+t-s)^{-\frac{N+2}{2(N+1)}-\epsilon}(1+s-\tau)^{-\frac{N}{2(N+1)}+\epsilon} \times \\ &\quad \times \|u(s,x)\|_{L^2(R^N,R^n)} ds\,. \end{aligned} \tag{2.21}$$

Let $h(s) = (1+t-s)^{-\frac{N+2}{2(N+1)}-\epsilon}(1+s-\tau)^{-\frac{N}{2(N+1)}+\epsilon}$, $\alpha = \frac{N^2+N-2}{4(N+1)} - \epsilon \in (\frac{N}{2(N+1)}, \frac{N}{4})$, we have from Gronwall's inequality that

$$\|u(t,x)\|_{L^2(R^N,R^n)} \le C(r,\tau)(1+t-\tau)^{-\frac{N^2+N-2}{4(N+1)}+\epsilon}, \quad t\ge\tau>0\,. \tag{2.22}$$

From (2.22) and Corollary 2.1, we arrive at

$$\|\nabla u(t,x)\|_{L^\infty(R^N,R^n)} \le C(r,\tau)(1+t-\tau)^{-\frac{N^2+N-2}{4(N+1)}+\epsilon}, \quad t \ge \tau > 0\,. \tag{2.23}$$

Substituting (2.22), (2.23) into (2.11) again, we have

$$\begin{aligned} \|u(t,x)\|_{L^2(R^N,R^n)} &\le C(r,\tau)(1+t-\tau)^{-\frac{N}{4}}+ \\ &+ C(r,\tau)\int_\tau^t (1+t-s)^{-\frac{N+2}{4}}(1+s-\tau)^{-\frac{N^2+N-2}{4(N+1)}+\epsilon}\times \\ &\times \|u(s,x)\|_{L^2(R^N,R^n)}ds\,. \end{aligned} \tag{2.24}$$

If $N \ge 5$, we can choose $\epsilon > 0$ sufficiently small such that

$$\frac{N^2+N-2}{4(N+1)} - \epsilon > 1\,, \tag{2.25}$$

and consequently if we let $h(s) = (1+s-\tau)^{-\frac{N^2+N-2}{4(N+1)}+\epsilon}$, then from Gronwall's inequality, we have

$$\begin{aligned} \|u(t,x)\|_{L^2(R^N,R^n)} \le C(r,\tau)(1+t-\tau)^{-\frac{N}{4}} &\le C(r,\tau)(1+t)^{-\frac{N}{4}}\,, \\ &t \ge \tau > 0\,. \end{aligned} \tag{2.26}$$

Thus Theorem 2.1 follows immediately.

Otherwise, if $2 < N < 5$, we have

$$0 < \frac{N^2+N-2}{4(N+1)} - \epsilon < 1\,, \tag{2.27}$$

and in this case (2.24) can be rewritten as

$$\begin{aligned} \|u(t,x)\|_{L^2(R^N,R^n)} &\le C(r,\tau)(1+t-\tau)^{-\frac{N}{4}}+ \\ &+ C(r,\tau)\int_\tau^t (1+t-s)^{-\frac{N^2-2}{2(N+1)}+\epsilon}\times \\ &\times (1+s-\tau)^{-\frac{N^2+N-2}{4(N+1)}+\epsilon}(1+t-s)^{\frac{N^2-3N-6}{4(N+1)}-\epsilon}\times \\ &\times \|u(s,x)\|_{L^2(R^N,R^n)}ds\,. \end{aligned} \tag{2.28}$$

Setting $h(s) = (1+t-s)^{\frac{N^2-3N-6}{4(N+1)}-\epsilon}(1+s-\tau)^{-\frac{N^2+N-2}{4(N+1)}+\epsilon}$, and noticing that

$$\frac{N^2-2}{2(N+1)} > \frac{N}{4}, \quad N = 3, 4\,, \tag{2.29}$$

we find from Gronwall's inequality that (2.26) also holds. Hence the proof of Theorem 2.1 is complete.

To complete the proof of Theorem 1.1, we only need to treat the case $N = 2$. For results in this direction, we have:

THEOREM 2.2. *Under the assumptions of Theorem* 1.1*, the results of Theorem* 1.1 *also hold for* $N = 2$.

Before proving Theorem 2.2, we give the following preliminary results:

First, from (1.5), (1.6) and Corollary 2.1, we have

LEMMA 2.6. *When* $N = 2$*, the globally smooth solution* $u(t,x)$ *obtained in* [3] *satisfies*

$$\begin{cases} \|u(t,x)\|_{L^2(R^2,R^n)} \le C(r,\tau)(1+t)^{-\frac{1}{3}+\epsilon}, & t \ge \tau > 0\,, \\ \|u(t,x)\|_{L^\infty(R^2,R^n)} \le C(r,\tau)(1+t)^{-\frac{1}{3}+\epsilon}, & t \ge \tau > 0\,. \end{cases} \tag{2.30}$$

As $N = 2$, from Nirenberg's inequality we can deduce:

LEMMA 2.7. *Suppose that* $u(t,x)$ *is the globally smooth solution obtained in* [3]*, then we have*

$$\begin{aligned} \|\Delta^{\frac{j}{2}}u(t,x)\|_{L^\infty(R^2,R^n)} \le C\|\Delta^{\frac{j+2}{2}}u(t,x)\|^{\frac{1}{2}}_{L^2(R^2,R^n)}\times \\ \times\,\|\Delta^{\frac{j}{2}}u(t,x)\|^{\frac{1}{2}}_{L^2(R^2,R^n)}, \quad j \ge 0\,, \end{aligned} \tag{2.31}$$

*and*

$$\begin{aligned} \|\Delta^{\frac{j}{2}}u(t,x)\|_{L^2(R^2,R^n)} \le C\|\Delta^{\frac{j+1}{2}}u(t,x)\|^{\frac{1}{2}}_{L^2(R^2,R^n)}\times \\ \times\,\|\Delta^{\frac{j-1}{2}}u(t,x)\|^{\frac{1}{2}}_{L^2(R^2,R^n)}, \quad j \ge 1\,. \end{aligned} \tag{2.32}$$

The following lemma, essentially due to M.E. SCHONBEK [4], plays an important role in the following analysis.

LEMMA 2.8. *If $u(t,x)$ satisfies the following differential integral inequality*

$$\frac{d}{dt}\int_{R^2}|\Delta^{\frac{k}{2}}u(t,x)|^2dx+N_4\int_{R^2}|\Delta^{\frac{k+1}{2}}u(t,x)|^2dx\leq N_5(r,\tau)(1+t)^{-q}, \quad t\geq\tau>0\,, \tag{2.33}$$

*and*

$$|\hat{u}(t,\xi)|\leq C\,, \tag{2.34}$$

*then we have*

$$\int_{R^2}|\Delta^{\frac{k}{2}}u(t,x)|^2dx\leq C(r,\tau)(1+t)^{-\min\{1+k,q-1\}}, \quad t\geq\tau>0\,. \tag{2.35}$$

*Here $N_4>0, N_5(r,\tau)\geq 0, q>1$.*

The above results can be proved by employing M.E. Schonbek's Fourier splitting method [4], the details of which are omitted.

From the above results, we have:

LEMMA 2.9. *If $N=2$ and $u(t,x)$ is the globally smooth solution obtained in* [3]*, then under the assumption* (1.8)*, we have*

$$\int_{R^2}|\nabla u(t,x)|^2dx\leq C(r,\tau)(1+t)^{-1+\epsilon}, \quad t\geq\tau>0\,, \tag{2.36}$$

*and*

$$\int_{R^2}|\Delta u(t,x)|^2dx\leq C(r,\tau)(1+t)^{-2+\epsilon}, \quad t\geq\tau>0\,. \tag{2.37}$$

*Consequently*

$$\|u(t,x)\|_{L^\infty(R^2,R^n)}\leq C(r,\tau)(1+t)^{-\frac{2}{3}+\epsilon}, \quad t\geq\tau>0\,. \tag{2.38}$$

PROOF. We first prove (2.36). To this end, multiplying (1.1) by $2\Delta u(t,x)^t$ and integrating the results over $R^2$, after some integrations by parts, we obtain from Lemma 2.7 that

$$
\begin{aligned}
&\frac{d}{dt}\int_{R^2}|\nabla u|^2dx+2d\int_{R^2}|\Delta u|^2dx\le\\
&\le\sum_{j=1}^{2}\int_{R^2}|\Delta u||f_j(u)_{x_j}|dx\le\\
&\le C(r)\|u\|_{L^\infty(R^2,R^n)}\|\Delta u\|_{L^2(R^2,R^n)}\|\nabla u\|_{L^2(R^2,R^n)}\le\\
&\le C(r)\|u\|_{L^\infty(R^2,R^n)}\|u\|^{\frac12}_{L^2(R^2,R^n)}\|\Delta u\|^{\frac32}_{L^2(R^2,R^n)}\le\\
&\le d\int_{R^2}|\Delta u|^2dx+C(r)\|u\|^4_{L^\infty(R^2,R^n)}\|u\|^2_{L^2(R^2,R^n)}\,,
\end{aligned}
\tag{2.39}
$$

thus

$$
\frac{d}{dt}\int_{R^2}|\nabla u|^2dx+d\int_{R^2}|\Delta u|^2dx\le C(r)\|u\|^4_{L^\infty(R^2,R^n)}\|u\|^2_{L^2(R^2,R^n)}\,. \tag{2.40}
$$

Here $d=\min\{d_{11},d_{22},\cdots,d_{nn}\}>0$.

Substituting (2.30) into (2.40), we deduce that

$$
\frac{d}{dt}\int_{R^2}|\nabla u|^2dx+d\int_{R^2}|\Delta u|^2dx\le C(r,\tau)(1+t)^{-2+\epsilon},\quad t\ge\tau>0\,. \tag{2.41}
$$

Thus Lemma 2.8 and (2.41) implies that (2.36) is true.

Now we prove (2.37). Similar to the proof of (2.39), we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{R^2}|\Delta u|^2dx+d\int_{R^2}|\Delta^{\frac32}u|^2dx\le\\
&\le C(r,\tau)\|u\|^2_{L^\infty(R^2,R^n)}\int_{R^2}|\Delta u|^2dx+\\
&\quad+C(r)\|\nabla u\|^2_{L^\infty(R^2,R^n)}\int_{R^2}|\nabla u|^2dx\le\\
&\le C(r,\tau)\|u\|_{L^2(R^2,R^n)}\|\Delta u\|^3_{L^2(R^2,R^n)}+\\
&\quad+C(r,\tau)\|\Delta^{\frac32}u\|_{L^2(R^2,R^n)}\|\nabla u\|^3_{L^2(R^2,R^n)}\le\\
&\le C(r,\tau)\|\Delta^{\frac32}u\|^{\frac32}_{L^2(R^2,R^n)}\|\nabla u\|^{\frac32}_{L^2(R^2,R^n)}(\|u\|_{L^2(R^2,R^n)}+1)\le\\
&\le\frac{d}{2}\int_{R^2}|\Delta^{\frac32}u|^2dx+C(r,\tau)(1+t)^{-3+\epsilon},\quad t\ge\tau>0\,.
\end{aligned}
\tag{2.42}
$$

Having obtained (2.42), by employing Lemma 2.8 again, we immediately obtain (2.37). (2.38) is a direct corollary of $(2.30)_1$ and (2.37). Thus the proof of Lemma 2.9 is complete.

Now we prove Theorem 2.2.

First, we have the following differential integral inequality

$$\frac{d}{dt}\int_{R^2}|u|^2dx + d\int_{R^2}|\nabla u|^2dx \le C(r,\tau)\|u\|^2_{L^\infty(R^2,R^n)}\|u\|^2_{L^2(R^2,R^n)}\,. \tag{2.43}$$

Inserting $(2.30)_1$, (2.38) into (2.43), we deduce that

$$\frac{d}{dt}\int_{R^2}|u|^2dx + d\int_{R^2}|\nabla u|^2dx \le C(r,\tau)(1+t)^{-2+\epsilon}, \quad t\ge\tau>0\,, \tag{2.44}$$

and consequently that

$$\int_{R^2}|u|^2dx \le C(r,\tau)(1+t)^{-1+\epsilon}, \quad t\ge\tau>0\,. \tag{2.45}$$

Substituting once more (2.45), (2.38) into (2.43), we get

$$\frac{d}{dt}\int_{R^2}|u|^2dx + d\int_{R^2}|\nabla u|^2dx \le C(r,\tau)(1+t)^{-\frac{7}{3}+\epsilon}, \quad t\ge\tau>0\,. \tag{2.46}$$

Thus

$$\int_{R^2}|u|^2dx \le C(r,\tau)(1+t)^{-1}, \quad t\ge\tau>0\,. \tag{2.47}$$

This completes the proof of Theorem 2.2.

Having obtained Theorem 2.1, Theorem 2.2, Theorem 1.1 follows immediately.

## 3 – The proof of Theorem 1.2

Before proving Theorem 1.2, we first give the following results:

LEMMA 3.1. *Under the conditions of Theorem* 1.1, *we have for each* $k\in Z^+, j=1,\cdots,N$ *that*

$$\|D^k f_j(u)\|_{L^1(R^N,R^n)} \le C(r,\tau)(1+t)^{-\frac{N+k}{2}}, \quad t\ge\tau>0\,. \tag{3.1}$$

*Here we have used the following notation*

$$\|D^k u(t,x)\|_{L^1(R^N,R^n)} := \sum_{|\alpha|=k} \Big\| \frac{\partial^k}{\partial x^\alpha} u(t,x) \Big\|_{L^1(R^N,R^n)} . \tag{3.2}$$

PROOF. Since for each multiindex $\alpha = (\alpha_1, \cdots, \alpha_N)$ with $|\alpha| = \sum_{i=1}^N \alpha_i = k \in Z^+$, we have

$$\frac{\partial^k}{\partial x^\alpha} f_j(u) = f_j'(u) \frac{\partial^k u}{\partial x^\alpha} + \sum_{l=2}^k C_\beta \frac{\partial^l f_j(u)}{\partial u^\beta} \Big(\frac{\partial u}{\partial x^{\gamma_1}}\Big)^{l_1} \cdots \Big(\frac{\partial^k u}{\partial x^{\gamma_k}}\Big)^{l_k} . \tag{3.3}$$

Here

$$\begin{cases} |\gamma_s| = s, \quad s = 1, \cdots, k\,, \\ l_1 + l_2 + \cdots + l_k = l, \quad l_1 \neq 0\,, \\ 1l_1 + 2l_2 + \cdots + kl_k = k\,. \end{cases} \tag{3.4}$$

Thus from the assumption (1.8) and the estimate (1.7), we have

$$\begin{aligned} &\|D^k f_j(u)\|_{L^1(R^N,R^n)} \le C(r,\tau)\|u\|_{L^2(R^N,R^n)}\|D^k u\|_{L^2(R^N,R^n)} + \\ &\quad + C(r,\tau)\sum_{l=2}^k \prod_{i=3}^k \|D^i u\|^{l_i}_{L^\infty(R^N,R^n)} \|Du\|^{l_1-1}_{L^\infty(R^N,R^n)} \|D^2 u\|^{l_2-1}_{L^\infty(R^N,R^n)} \times \\ &\quad \times \|Du\|_{L^2(R^N,R^n)}\|D^2 u\|_{L^2(R^N.R^n)} \le \\ &\le C(r,\tau)(1+t)^{-\frac{N+k}{2}} + C(r,\tau)\sum_{l=2}^k (1+t)^{-\frac{N}{2}(l-1)-\frac{k}{2}} \le \\ &\le C(r,\tau)(1+t)^{-\frac{N+k}{2}} . \end{aligned}$$

This is (3.1) and the proof of Lemma 3.1 is complete.

LEMMA 3.2. *Let $\alpha, \beta$, and $\gamma$ be positive numbers, $0 < \tau < 1, t \ge 2\tau$. Then*

$$\int_\tau^{\frac{t}{2}} (1+t-s)^{-\beta}(1+s)^{-\gamma} ds = O(1+t)^{-\alpha} \tag{3.5}$$

*if* $\alpha \le \beta, \alpha \le \beta + \gamma - 1, \gamma \ne 1,$ *or if* $\alpha < \beta, \alpha \le \beta + \gamma - 1, \gamma = 1,$

$$\int_{\frac{t}{2}}^{t} (1+t-s)^{-\beta}(1+s)^{-\gamma} ds = O(1+t)^{-\alpha} \tag{3.6}$$

*if* $\alpha \le \gamma, \alpha \le \beta + \gamma - 1, \beta \ne 1,$ *or if* $\alpha < \gamma, \alpha \le \beta + \gamma - 1, \beta = 1.$

Now we prove Theorem 1.2. To this end, we first have the following claim:

Claim. Under the conditions of Theorem 1.1, for each $k \in Z^+$, we have that (1.9) with $p = 1$ is true.
Toward a proof of the above claim, we have similar to that of Lemma 2.3 that

$$\begin{aligned} \|D^k u(t)\|_{L^1(R^N,R^n)} \le{}& C(r,\tau)(1+t)^{-\frac{k}{2}}(\|u(\tau)\|_{L^1(R^N,R^n)} + \\ &+ \|D^k u(\tau)\|_{L^1(R^N,R^n)}) + \\ &+ C(r,\tau)\sum_{j=1}^{N}\int_{\tau}^{\frac{t}{2}} (1+t-s)^{-\frac{k+1}{2}}(\|f_j(u)\|_{L^1(R^N,R^n)} + \\ &+ \|D^{k+1} f_j(u)\|_{L^1(R^N,R^n)}) ds + \\ &+ C(r,\tau)\sum_{j=1}^{N}\int_{\frac{t}{2}}^{t} (1+t-s)^{-\frac{1}{2}}(\|D^k f_j(u)\|_{L^1(R^N,R^n)} + \\ &+ \|D^{k+1} f_j(u)\|_{L^1(R^N,R^n)}) ds\,. \end{aligned} \tag{3.7}$$

Substituting (3.1) into (3.7), we have from (1.6), (3.5), (3.6) that

$$\begin{aligned} \|D^k u(t,x)\|_{L^1(R^N,R^n)} \le{}& C(r,\tau)(1+t)^{-\frac{k}{2}} + \\ &+ C(r,\tau)\int_{\tau}^{\frac{t}{2}} (1+t-s)^{-\frac{k+1}{2}}(1+s)^{-\frac{N}{2}} ds + \\ &+ C(r,\tau)\int_{\frac{t}{2}}^{t} (1+t-s)^{-\frac{1}{2}}(1+s)^{-\frac{N+k}{2}} ds \le \\ \le{}& C(r,\tau)(1+t)^{-\frac{k}{2}}\,. \end{aligned} \tag{3.8}$$

This proves the claim.

Having obtained the above result, Theorem 1.2 can be easily obtained by employing the standard interpolation technique and so we omit the details.

Remark 3.1. From the above proofs and the results obtained in [2], we can easily deduce that (1.9) also holds for $N=1$ provided that the corresponding inviscid system is hyperbolic at $u=0$ and the initial data belongs to $L^1\cap L^\infty(R^N,R^n)$ with its $L^1(R^N,R^n)$-norm sufficiently small.

## 4 – The proof of Theorem 1.3, Theorem 1.4, and Theorem 1.5

In this section, we prove Theorem 1.3, Theorem 1.4, and Theorem 1.5.

First, we prove Theorem 1.3. To this end, from (1.12), we only need to prove the following estimates: For each multiindex $\alpha=(\alpha_1,\cdots,\alpha_n)$, $|\alpha|=k$, $1\le p\le\infty$

$$\Big\|\frac{\partial^k}{\partial x^\alpha}(u(t,x)-\phi(t,x))\Big\|_{L^p(R^N,R^n)}\le C(r,\tau)p(t)(1+t)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{k+1}{2}}\,. \tag{4.1}$$

To show this, we first get from the integral representation (2.2) that

$$\begin{aligned}\frac{\partial^k}{\partial x^\alpha}(u(t,x)-\phi(t,x))&=\sum_{j=1}^N\int_0^{\frac{t}{2}}\frac{\partial^k}{\partial x^\alpha}K_{x_j}(t-s,x)*f_j(u(s,x))ds+\\&+\sum_{j=1}^N\int_{\frac{t}{2}}^t K_{x_j}(t-s,x)*\frac{\partial^k}{\partial x^\alpha}f_j(u(s,x))ds\,.\end{aligned}\tag{4.2}$$

Consequently

$$\begin{aligned}&\Big\|\frac{\partial^k}{\partial x^\alpha}(u(t,x)-\phi(t,x))\Big\|_{L^p(R^N,R^n)}\le\\&\le\sum_{j=1}^N\int_0^{\frac{t}{2}}\Big\|\frac{\partial^k}{\partial x^\alpha}K_{x_j}(t-s,x)*f_j(u(s,x))\Big\|_{L^p(R^N,R^n)}ds+\\&\quad+\sum_{j=1}^N\int_{\frac{t}{2}}^t\|K_{x_j}(t-s,x)\|_{L^1(R^N,R^n)}\times\\&\quad\times\Big\|\frac{\partial^k}{\partial x^\alpha}f_j(u(s,x))\Big\|_{L^p(R^N,R^n)}ds:=\\&:=I_1+I_2\,.\end{aligned}\tag{4.3}$$

Similar to the proof of Lemma 3.1, for $t \geq \tau > 0$ we have from the assumption (1.8) and the decay estimates (1.9) that

$$\Big\| \frac{\partial^k}{\partial x^\alpha} f_j(u(s,x)) \Big\|_{L^p(R^N,R^n)} \leq C(r,\tau)(1+s)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{N+k}{2}}\,, \quad j = 1,2,\cdots,N\,. \tag{4.4}$$

Consequently we have

$$\begin{aligned} I_2 &\leq C(r,\tau) \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} (1+s)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{N+k}{2}} ds \leq \\ &\leq C(r,\tau)(1+t)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{k+1}{2}} \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} (1+s)^{-\frac{N-1}{2}} ds \leq \\ &\leq C(r,\tau)p(t)(1+t)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{k+1}{2}}\,. \end{aligned} \tag{4.5}$$

As to $I_1$, we have from Hausdorff-Young's inequality that

$$\begin{aligned} |I_1| &\leq C(r) \sum_{j=1}^{N} \int_0^{\frac{t}{2}} (t-s)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{k+1}{2}} \|f_j(u(s,x))\|_{L^1(R^N,R^n)} ds \leq \\ &\leq C(r) t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{k+1}{2}} \int_0^{\frac{t}{2}} \|u(s,x)\|^2_{L^2(R^N,R^n)} ds \leq \\ &\leq C(r,\tau)(1+t)^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{k+1}{2}} \int_0^{\frac{t}{2}} \|u(s,x)\|^2_{L^2(R^N,R^n)} ds\,. \end{aligned} \tag{4.6}$$

Since

$$\|u(t,x)\|_{L^1 \cap L^\infty(R^N,R^n)} \leq C(r)\,,$$

we have from (4.4) that

$$\begin{aligned} \int_0^{\frac{t}{2}} \|u(s,x)\|^2_{L^2(R^N,R^n)} ds &\leq \Big( \int_0^{\frac{\tau}{2}} + \int_{\frac{\tau}{2}}^{\frac{t}{2}} \Big) \|u(s,x)\|^2_{L^2(R^N,R^n)} ds \leq \\ &\leq C(r) \int_0^{\frac{\tau}{2}} \|u(s,x)\|_{L^1(R^N,R^n)} ds + C(r,\tau) \int_{\frac{\tau}{2}}^{\frac{t}{2}} (1+s)^{-\frac{N}{2}} ds \leq \\ &\leq C(r,\tau)p(t)\,. \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7), we arrive at

$$I_1 \leq C(r,\tau)p(t)(1+t)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{k+1}{2}},\ t \geq \tau > 0\,, \tag{4.8}$$

and (1.15) follows from (4.6) and (4.8). This completes the proof of Theorem 1.3.

The proof of Theorem 1.4 follows essentially the same way as in the proof of Theorem 1.3. Thus we will only give a sketch in the following.

First notice that in the case of $\delta_j = 0 (j = 1, 2, \cdots, n)$, we can get for $t \geq \tau > 0$ that

$$\Big\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \phi(t, x) \Big\|_{L^p(R^N, R^n)} \leq C(r, \tau)(1+t)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}} \,. \tag{4.9}$$

Combining (4.9) with (4.1) deduce that for $t \geq \tau > 0$

$$\Big\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} u(t, x) \Big\|_{L^p(R^N, R^n)} \leq C(r, \tau) p(t) (1+t)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}} \,. \tag{4.10}$$

With (4.10) in hand, similar to the proof of Lemma 3.1, we have from the assumption (1.8) that for each $j = 1, 2, \cdots, N$, $t \geq \tau > 0$

$$\Big\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} f_j(u(s, x)) \Big\|_{L^p(R^N, R^n)} \leq C(r, \tau) p(s)^2 (1+s)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{N+|\alpha|+2}{2}} \,. \tag{4.11}$$

Having obtained (4.11), Theorem 1.4 can be proved by repeating the techniques used in the proof of Theorem 1.3. The details are omitted. This completes the proof of Theorem 1.4.

At last, we prove Theorem 1.5. For this purpose, we first cite the following result which is essentially due to M. Schonbek [5]

Lemma 4.1. *Let $\phi_0(T_1, x) \in L^2(R^N, R^n), T_1 > 1$ be a sufficiently large fixed constant, and $\phi(t, x)$ be the unique smooth solution to the Cauchy problem* (1.10) *with initial data $u_0(x) \equiv \phi_0(T_1, x)$. Suppose that there exist a vector-valued function $h(T_1, \xi) := (h_1(T_1, \xi), \cdots, h_n(T_1, \xi))^t$ and a matrix-valued function $l(T_1, \xi) := (l_{ij}(T_1, \xi))_{n \times N}$ such that the Fourier transform of $\phi_0(T_1, x)$ for $|\xi| \leq \delta, \delta > 0$ admits the representation*

$$\hat{\phi}_0(T_1, \xi) = l(T_1, \xi) \cdot \xi + h(T_1, \xi) \,, \tag{4.12}$$

*where $l(T_1, \xi)$ and $h(T_1, \xi)$ satisfy the following conditions:*

(i) $|h(T_1,\xi)| \leq L_3|\xi|^2\sqrt{1+T_1}$ *for some* $T_1$*-independent constant* $L_3 > 0$;
(ii) $l(T_1,\xi)$ *is homogeneous of degree zero; and*
(iii) $L_4 := \int_{|\omega|=1} |l(T_1,\omega)\cdot\omega|^2 d\omega$ *is a positive constant independent of* $T_1$.

*Then if* $L_5 := \sup_{|y|=1} |l(T_1,y)|$ *is independent of* $T_1$*, we have that*

$$\Big\|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\phi(t,x)\Big\|_{L^2(R^N,R^n)} \geq L_6(1+t)^{-\frac{N+2(|\alpha|+1)}{4}}, \quad t \geq T_1 > 1\,. \tag{4.13}$$

*Here*

$$L_6 := \sqrt{\frac{\omega_N L_4}{2(N+2(|\alpha|+1))}}\Big(\frac{\sqrt{2}L_4}{8L_3L_5}\Big)^{\frac{N+2(|\alpha|+1)}{2}}\exp\Big(-\frac{L_4^2}{16L_3^2L_5^2}\Big)\,, \tag{4.14}$$

*and* $\omega_N$ *denotes the volume of the* $N$*-dimensional unit sphere.*

Remark 4.1. It is easy to see that $L_6$ is independent of $T_1$. Such a property will play an important role in our subsequent analyses.

To exploit Lemma 4.1 to prove our Theorem 1.5, we need the following results on some weighted $L^1(R^N,R^n)$-norm estimates of the globally smooth solution $u(t,x)$ to the Cauchy problem (1.1), (1.2)

Lemma 4.2 (Some weighted $L^1(R^N,R^n)$-norm estimates). *If the assumptions stated in Theorem* 1.5 *are satisfied, we can deduce that*

$$\int_{R^N} |x||u(t,x)|dx \leq C(r,\tau)\ln(1+t)\,, \tag{4.15}$$

*and*

$$\int_{R^N} |x|^2|u(t,x)|dx \leq C(r,\tau)(1+t)^{\frac{1}{2}}\,. \tag{4.16}$$

Proof. We only prove (4.15). (4.16) can be treated similarly.

To prove (4.15), for the standard mollifier $\rho(x)$, if we set

$$(4.17) \qquad (\mathrm{sgn} u(t,x))^\epsilon := \frac{1}{\epsilon}\begin{pmatrix} \int_R \rho\Big(\frac{u_1-v}{\epsilon}\Big)\mathrm{sgn}\ v dv \\ \int_R \rho\Big(\frac{u_2-v}{\epsilon}\Big)\mathrm{sgn}\ v dv \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \int_R \rho\Big(\frac{u_n-v}{\epsilon}\Big)\mathrm{sgn}\ v dv \end{pmatrix},$$

then multiplying (1.1) by $|x|((\mathrm{sgn}\ u(t,x))^\epsilon)^t$ and integrating the results with respect to $x$ over $R^N$, one can get

$$(4.18) \qquad \begin{aligned} &\frac{d}{dt}\Big\{\sum_{l=1}^{n}\int_{R^N}|x|\int_0^{u_l}(\mathrm{sgn}\ v)^\epsilon dv\, dx\Big\}+\\ &\quad+\sum_{l=1}^{n}\sum_{j=1}^{N}\int_{R^N} d_{ll}|x|\frac{d}{du_l}(\mathrm{sgn}\ u_l)^\epsilon[(u_l)_{x_j}]^2dx=\\ &=-\sum_{j=1}^{N}\int_{R^N}|x|(\mathrm{sgn}\ u)^\epsilon\cdot f_j(u)_{x_j}dx+\\ &\quad-\sum_{j=1}^{N}\sum_{l=1}^{n}\int_{R^N} d_{ll}\frac{x_j}{|x|}(\mathrm{sgn}\ u_l)^\epsilon(u_l)_{x_j}dx\,. \end{aligned}$$

Since

$$(4.19) \qquad \frac{d}{du_l}\Big(\mathrm{sgn}\ u_l\Big)^\epsilon=\frac{2}{\epsilon}\rho\Big(\frac{u_l}{\epsilon}\Big)\geq 0\,,$$

we have from (4.18) that

$$(4.20) \quad \frac{d}{dt}\Big\{\sum_{l=1}^{n}\int_{R^N}|x|\int_0^{u_l}(\mathrm{sgn}\ v)^\epsilon dv\, dx\Big\}\leq C(r,\tau)\int_{R^N}(1+|x||u|)|u_{x_j}|dx\,.$$

Letting $\epsilon\to 0^+$ in (4.20), we have from Theorem 1.4 that

$$(4.21) \qquad \begin{aligned} \frac{d}{dt}\int_{R^N}|x||u(t,x)|dx &\leq C(r,\tau)(1+t)^{-1}\times\\ &\quad\times\Big(1+(1+t)^{-\frac{N}{2}}\int_{R^N}|x||u(t,x)|dx\Big)\,, \end{aligned}$$

and by employing the Grownwall's inequality, we can deduce from (4.21) that (4.15) is true. This completes the proof of Lemma 4.2.

The following lemma gives some integral identities for the globally smooth solution $u(t,x)$ obtained in [3], i.e.

LEMMA 4.3. *Under the assumptions of Theorem* 1.5, *we have*

$$\int_{R^N} u(t,x)dx = \int_{R^N} u_0(x) = 0\,, \tag{4.22}$$

*and for each* $j = 1, 2, \cdots, N, k = 1, 2, \cdots, n$

$$\int_{R^N} x_j u_k(t,x)dx = \int_{R^N} x_j u_{0k}(x)dx + \int_0^t \int_{R^N} f_{jk}(u(s,x))dx\,ds\,. \tag{4.23}$$

From Lemma 4.2 and Lemma 4.3, for each $k = 1, 2, \cdots, n$, we can easily deduce that $\hat{u}_k(t,\xi)$ are differentiable up to the second order. Consequently, we have from the Taylor expansion formula that

$$\begin{aligned} \hat{u}_k(t,\xi) &= \hat{u}_k(t,0) + \sum_{j=1}^N \xi_j \frac{\partial \hat{u}_k(t,0)}{\partial \xi_j} + \frac{1}{2}\sum_{i,j=1}^N \xi_i\xi_j \frac{\partial^2 \hat{u}_k(t,\bar{\xi})}{\partial\xi_i\partial\xi_j} = \\ &= -i\sum_{j=1}^N \xi_j \Big[\int_{R^N} x_j u_{0k}(x)dx + \int_0^t\int_{R^N} f_{jk}(u(s,x))dx\,ds\Big] + \\ &\quad + \frac{1}{2}\sum_{i,j=1}^N \xi_i\xi_j \frac{\partial^2 \hat{u}_k(t,\bar{\xi})}{\partial\xi_i\partial\xi_j}\,. \end{aligned} \tag{4.24}$$

So

$$\hat{u}(t,\xi) = l(t,\xi)\cdot\xi + h(t,\xi) \tag{4.25}$$

with

$$l(t,\xi) := \begin{pmatrix} l_{11}(t,\xi) & l_{12}(t,\xi) & \cdots & l_{1N}(t,\xi) \\ l_{21}(t,\xi) & l_{22}(t,\xi) & \cdots & l_{2N}(t,\xi) \\ \cdots & \cdots & \cdots & \cdots \\ l_{n1}(t,\xi) & l_{n2}(t,\xi) & \cdots & l_{nN}(t,\xi) \end{pmatrix}, \tag{4.26}$$

and

$$h(t,\xi)=(h_1(t,\xi),h_2(t,\xi),\cdots,h_n(t,\xi))^t\,. \tag{4.27}$$

Here

$$\begin{cases} l_{lk}(t,\xi):=\displaystyle\int_{R^N}x_k u_{0l}(x)dx+\int_0^t\int_{R^N}f_{kl}(u(s,x))dx\,ds\,,\\ h_l(t,\xi):=\displaystyle\frac{1}{2}\sum_{i,j=1}^N \xi_i\xi_j\frac{\partial^2\hat{u}_l(t,\bar{\xi})}{\partial\xi_i\partial\xi_j}\,,\\ k=1,2,\cdots,N,\quad l=1,2,\cdots,n\,. \end{cases} \tag{4.28}$$

It is easy to see from Lemma 4.2 that

$$|h(t,\xi)|\le C(r,\tau)|\xi|^2\int_{R^N}|x|^2|u(t,x)|dx\le d_1(r,\tau)(1+t)^{\frac{1}{2}}|\xi|^2\,. \tag{4.29}$$

Now we show that, under the assumption of (1.21) and for each fixed $t>1$, there is a time-indepent constant $L_7>0$ such that

$$\int_{|\omega|=1}|l(t,\omega)\cdot\omega|^2d\omega\ge L_7>0\,. \tag{4.30}$$

In fact, under the assumption of (1.21), if we let $\xi_0=e_{\bar{k}}$, we can conclude that

$$|l(t,\xi_0)\cdot\xi_0|=\Big(\sum_{l=1}^N|l_{\bar{k}l}(t,\xi_0)|^2\Big)^{\frac{1}{2}}\ge|l_{\bar{k}\bar{j}}(t,\xi_0)|\ge L_1\,. \tag{4.31}$$

Next we show that we can choose a time-independent neighborhood $B_\gamma(\xi_0)$ of $\xi_0$ such that for each $\xi\in B_\gamma(\xi_0)$

$$|l(t,\xi)\cdot\xi|\ge\frac{L_1}{2}>0\,. \tag{4.32}$$

Indeed, since for each $j=1,2,\cdots,N,k=1,2,\cdots,n$,

$$\begin{aligned}\Big|\int_0^t\int_{R^N}f_{jk}(u(s,x))dx\,ds\Big|&\le C(r,\tau)\int_0^t\|u(s,x)\|^2_{L^2(R^N,R^n)}ds\le\\ &\le C(r,\tau)\int_0^t(1+s)^{-\frac{N+2}{2}}ds\le C(r,\tau)\,,\end{aligned} \tag{4.33}$$

and noticing that $l(t,\xi)$ is independent of $\xi$, we have for each $\xi \in B_\gamma(\xi_0)$ that

$$\begin{aligned}|l(t,\xi)\cdot\xi| &\geq |l(t,\xi_0)\cdot\xi_0| - |l(t,\xi)\cdot(\xi-\xi_0)| \geq \\ &\geq L_1 - C(r,\tau)\gamma \geq \frac{L_1}{2}\end{aligned} \tag{4.34}$$

provided that $\gamma > 0$ is sufficiently small such that

$$\gamma \leq \frac{L_1}{2C(r,\tau)}\,.$$

Having obtained (4.34), we can easily deduce that

$$\begin{aligned}&\int_{|\omega|=1} |l(t,\omega)\cdot\omega|^2 d\omega \geq \int_{\{\omega:|\omega|=1,\omega\in B_\gamma(\xi_0)\}} |l(t,\omega)\cdot\omega|^2 d\omega \geq \\ &\geq \frac{L_1^2}{4}\text{meas}\{\omega : |\omega| = 1, \omega \in B_\gamma(\xi_0)\} := \\ &:= L_7.\end{aligned} \tag{4.35}$$

This proves (4.30).

On the other hand, we can get that

$$\begin{aligned}\sup_{y\in R^N} |l(t,y)| &\leq C(N)\Big(\int_{R^N} |x||u_0(x)|dx + \\ &\quad + \sum_{j=1}^N \int_0^t \int_{R^N} |f_j(u(s,x))t|dx\,ds\Big) \leq \\ &\leq C(N,r,\tau)\Big(\int_{R^N} |x||u_0(x)|dx + \int_0^t (1+s)^{-\frac{N+2}{2}} ds\Big) \leq \\ &\leq C(N,r,\tau)\Big(\int_{R^N} |x||u_0(x)|dx + 1\Big) := \\ &:= L_8\,.\end{aligned} \tag{4.36}$$

Thus for arbitrarily fixed $T_1 > 1$, if we let $v(t,x)$ be the solution of the Cauchy problem (1.10) with initial data $v(0,x) = u(T_1,x)$, where $u(t,x)$ is the globally smooth solution to the Cauchy problem (1.1), (1.2) obtained in [3], then we have from Lemma 4.1 and (4.25)-(4.30), (4.36) that

$$\Big\|\frac{\partial^{|\alpha|}}{\partial x^\alpha} v(t,x)\Big\|_{L^2(R^N,R^n)} \geq L_9(1+t)^{-\frac{N+2(|\alpha|+1)}{4}}, \quad t \geq T_1\,. \tag{4.37}$$

Here, again from Lemma 4.1, $L_9$ is independent of $T_1$.

Now let

$$w(t,x) := u(t+T_1,x) - v(t,x)\,,$$

we can easily deduce that $w(t,x)$ satisfies the following Cauchy problem

$$\begin{cases} w_t = D\Delta w - \sum_{j=1}^{N} f_j(u(t+T_1,x))_{x_j}\,, \\ w(0,x) = u(T_1,x) - v(0,x) = 0\,. \end{cases} \tag{4.38}$$

and consequently

$$w(t,x) = -\sum_{j=1}^{N}\int_0^t K_{x_j}(t-s,x) * f_j(u(s+T_1,x))ds\,. \tag{4.39}$$

From (4.39), we can deduce that

$$\begin{aligned} &\Big\|\frac{\partial^{|\alpha|}}{\partial x^\alpha} w(t,x)\Big\|_{L^2(R^N,R^n)} \le \\ &\le \sum_{j=1}^{N}\int_0^{\frac{t}{2}} \Big\|\frac{\partial^{|\alpha|}}{\partial x^\alpha} K_{x_j}(t-s,x)\Big\|_{L^2(R^N,R^n)} \times \\ &\quad\times \|f_j(u(T_1+s,x))\|_{L^1(R^N,R^n)} ds + \\ &\quad+ \sum_{j=1}^{N}\int_{\frac{t}{2}}^{t} \Big\|\frac{\partial^{|\alpha|}}{\partial x^\alpha} f_j(u(T_1+s,x))\Big\|_{L^2(R^N,R^n)} \times \\ &\quad\times \|K_{x_j}(t-s,x)\|_{L^1(R^N,R^n)} ds := \\ &:= J_1 + J_2\,. \end{aligned} \tag{4.40}$$

Similar to the proof of Lemma 3.1, we have from Theorem 1.4 that

$$\begin{cases} \|f_j(u(T_1+s,x))\|_{L^1(R^N,R^n)} \le L_{10}(T_1+s+1)^{-\frac{N+2}{2}}\,, \\ \Big\|\frac{\partial^{|\alpha|}}{\partial x^\alpha} f_j(u(T_1+s,x))\Big\|_{L^2(R^N,R^n)} \le L_{11}(T_1+s+1)^{-\frac{3N+2(|\alpha|+2)}{4}}\,. \end{cases} \tag{4.41}$$

Consequently

$$\begin{aligned} J_1 &\le C(r,\tau)\int_0^{\frac{t}{2}} (t-s)^{-\frac{N+2(|\alpha|+1)}{4}}(1+T_1+s)^{-\frac{N+2}{2}} ds \le \\ &\le \frac{L_{12}}{2\sqrt{1+T_1}}(1+t)^{-\frac{N+2(|\alpha|+1)}{4}}\,, \end{aligned} \tag{4.42}$$

and

$$
\begin{aligned}
J_2 &\leq C(r,\tau)\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}(1+T_1+s)^{-\frac{3N+2(|\alpha|+2)}{4}}ds \leq \\
&\leq \frac{L_{12}}{2\sqrt{1+T_1}}(1+t)^{-\frac{N+2(|\alpha|+1)}{4}}\,.
\end{aligned}
\tag{4.43}
$$

Substituting (4.42) and (4.43) into (4.40), we arrive at

$$
\left\|\frac{\partial^{|\alpha|}}{\partial x^{\alpha}}w(t,x)\right\|_{L^2(R^N,R^n)} \leq \frac{L_{12}}{2\sqrt{1+T_1}}(1+t)^{-\frac{N+2(|\alpha|+1)}{4}}, \quad t\geq 0\,. \tag{4.44}
$$

Here $L_{12}$ is independent of $t$ and $T_1$.

Taking $T_1$ sufficiently large such that

$$
\frac{L_{12}}{\sqrt{1+T_1}} \leq \frac{L_9}{2}\,, \tag{4.45}
$$

we have from (4.37), (4.44), and (4.45) that for $t \geq T_1$

$$
\begin{aligned}
&\left\|\frac{\partial^{|\alpha|}}{\partial x^{\alpha}}u(t+T_1,x)\right\|_{L^2(R^N,R^n)} \geq \\
&\quad\geq \left\|\frac{\partial^{|\alpha|}}{\partial x^{\alpha}}v(t,x)\right\|_{L^2(R^N,R^n)} - \left\|\frac{\partial^{|\alpha|}}{\partial x^{\alpha}}w(t,x)\right\|_{L^2(R^N,R^n)} \geq \\
&\quad\geq \frac{L_9}{2}(1+t)^{-\frac{N+2(|\alpha|+1)}{4}} \geq \\
&\quad\geq \frac{L_9}{2}(1+t+T_1)^{-\frac{N+2(|\alpha|+1)}{4}}\,.
\end{aligned}
\tag{4.46}
$$

Since $u(t,x) \not\equiv 0$ and $T_1$ is a fixed constant, we have for $\tau \leq t \leq 2T_1$ that

$$
\begin{aligned}
\left\|\frac{\partial^{|\alpha|}}{\partial x^{\alpha}}u(t,x)\right\|_{L^2(R^N,R^n)} &\geq \min_{[\tau,2T_1]}\left\|\frac{\partial^{|\alpha|}}{\partial x^{\alpha}}u(t,x)\right\|_{L^2(R^N,R^n)} := \\
&:= L_{13}(T_1,\tau) \geq L_{13}(T_1,\tau)(1+t)^{-\frac{N+2(|\alpha|+1)}{4}}\,.
\end{aligned}
\tag{4.47}
$$

Thus if we let

$$
L_{14}(T_1,\tau) := \min\left\{L_{13}(T_1,\tau), \frac{1}{2}L_9\right\},
$$

we can get from (4.46) and (4.47) that for $t \geq \tau > 0$

$$\Big\|\frac{\partial^{|\alpha|}}{\partial x^\alpha}u(t,x)\Big\|_{L^2(R^N,R^n)} \geq L_{14}(T_1,\tau)(1+t)^{-\frac{N+2(|\alpha|+1)}{4}}\,. \tag{4.48}$$

This is (1.22) and hence completes the proof of Theorem 1.5.

## 5 – Concluding remarks

For general multidimensional parabolic conservation laws, the results on the global existence and optimal temporal decay estimates to solutions of the corresponding Cauchy problem obtained in this paper and [2], [3], in some sense, are quite complete. But in all these results, some smallness conditions must be imposed on the initial data. However, the techniques used in the above papers, especially those used in this paper, can also be used to discuss the optimal temporal decay estimates of solutions to certain multidimensional parabolic conservation laws with large initial data. For example, for the following Cauchy problem (for simplicity of the presentation of the result, we let $D = \varepsilon I$ in (1.1))

$$u_t + \sum_{j=1}^{N} f_j(u)_{x_j} = \varepsilon\Delta u, \quad x \in R^N, \quad t > 0\,. \tag{5.1}$$

with initial data

$$u(t,x)|_{t=0} = u_0(x), \quad x \in R^N\,, \tag{5.2}$$

if we assume that

($H_1$) (5.1) admits a strictly convex quadratic entropy $\eta(u)$ such that

$$\begin{cases} C^{-1}|u|^2 \leq \eta(u) \leq C|u|^2, & u \in \overline{B}_\gamma(0)\,, \\ \eta''(u) \geq dI, \quad d > 0, & u \in \overline{B}_\gamma(0)\,, \end{cases} \tag{5.3}$$

($H_2$) The solution $u(t,x)$ to the Cauchy problem (5.1), (5.2) satisfies the following time independent $L^\infty(R^+ \times R^N, R^n)$ *a priori* estimate

$$\|u(t,x)\|_{L^\infty(R^+\times R^N,R^n)} \leq \gamma\,, \tag{5.4}$$

and

($H_3$) The flux functions $f_j(u)(j=1,\cdots,N)$ satisfies

$$\begin{cases} f(u)=O(|u|^3) & \text{as } |u|\to 0 \text{ when } N=1,\\ f_j(u)=O(|u|^2) & \text{as } |u|\to 0 \text{ when } N\geq 2\,, \end{cases}$$

then we can get the following result

Theorem 5.1. *Suppose* ($H_1$)-($H_3$) *hold, then for* $u_0(x)\in L^1\cap L^\infty(R^N,R^n)$*, the Cauchy problem* (5.1), (5.2) *admits a unique globally smooth solution* $u(t,x)$ *and* $u(t,x)$ *satisfies the optimal temporal decay estimate* (1.9)*. Moreover,* $u(t,x)$ *satisfies the estimates* (1.15) *and in this case,*

$$p(t)=\begin{cases} \ln\,(1+t), & N=1,2\,,\\ 1, & N\geq 3\,. \end{cases}$$

*Furthermore, similar results parallel to Theorem* 1.4 *and Theorem* 1.5 *also hold.*

We just give an outline of the proof of Theorem 5.1.

The first step. Global existence result.

Such a result follows easily from the standard local existence result and hypothesis ($H_2$).

The second step. Estimates of (1.6) type.

To get such estimates, we first get from the integral representation for $u(t,x)$ and hypothesis ($H_2$) that for each $k\in Z^+$, there exists a $\tau\in(0,1)$ such that

$$\|\Delta^{\frac{k}{2}}u(\tau,x)\|_{L^2\cap L^1(R^N,R^n)}\leq C_k(\gamma,\tau)\,. \tag{5.5}$$

Having obtained (5.5), we can deduce from hypothesis ($H_1$) and the standard energy estimate that the estimate of type $(1.6)_2$ hold and consequently from Nirenberg's inequality, $(1.6)_1$ follows.

The third step. Optimal temporal decay estimates (1.9).

With the results obtained in the above two steps in hand and by employing the techniques used in this paper, to get (1.9), we only need to get the optimal temporal $L^2(R^N,R^n)$-norm decay estimate. Under the hypothesis ($H_3$), the cases $N=1,2$ can be done similar to that of [6], [7] and the case $N \geq 3$ can be treated similar to that of Theorem 2.1. The only difference in this case is that we first choose $h(s) = (1+t-s)^{-1-\varepsilon}$ to obtain

$$\|u(t,x)\|_{L^2(R^N,R^n)} \leq C(\gamma,\tau)(1+t)^{-\frac{N-2}{4}+\varepsilon}, \quad t \geq \tau > 0\,, \tag{5.6}$$

and then the iteration technique gives the desired optimal temporal $L^2(R^N, R^n)$-norm decay estimate.

The forth step. Asymptotic profile of $u(t,x)$ and the results parallel to Theorem 1.4 and Theorem 1.5.

Having obtained (1.9), the desired results in this step can be obtained by mimicing the arguments used in the proof of Theorem 1.3, Theorem 1.4, and Theorem 1.5.

Remark 5.1. Although hypothesis $(H_1)$ is quite restrictive, some physical systems are indeed equipped with such an entropy.

Remark 5.2. For some concrete parabolic conservation laws, the time independent $L^\infty(R^+ \times R^N, R^n)$ *a priori* estimate, i.e., hypothesis $(H_2)$, can be obtained through K. Chueh, C. Conley, and J. Smoller's theory of positively invariant regions [1] or the method of energy estimates as in [4].

Remark 5.3. For the viscous solution to the system of nonlinear elasticity

$$\begin{cases} v_t - u_x = \varepsilon v_{xx}, \\ u_t - \sigma(v)_x = \varepsilon u_{xx}, \\ v\sigma''(v) > 0 \ \text{ for } \ v \neq 0, \quad \sigma'(v) > 0\,, \end{cases} \tag{5.7}$$

with initial data

$$(v(t,x), u(t,x))|_{t=0} = (v_0(x), u_0(x))\,, \tag{5.8}$$

we can easily verify that the hypotheses $(H_1)$-$(H_3)$ are satisfied and hence Theorem 5.1 can be applied to this case.

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