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Existence and multiplicity results for some nonlinear elliptic equations: a survey

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Dedicated to the memory of Professor Gaetano Fichera

RIASSUNTO: *Discutiamo alcuni risultati di esistenza e molteplicità per equazioni ellittiche nonlineari del tipo* (1).

ABSTRACT: *We survey some existence and multiplicity results dealing with nonlinear elliptic equations like* (1).

1 – Introduction

In this paper we will outline some recent advances concerning the existence of solutions of some classes of Nonlinear Elliptic Equations. We will mainly deal with the existence of positive solutions of problems like

$$-\Delta u + au = \lambda h(x)u^{q-1} + k(x)u^{\alpha-1}, \quad x \in D, \tag{1}$$

where

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- Δ denotes the Laplace operator;
- $\lambda \geq 0$ is a real parameter and $h(x)$ and $k(x)$ are given functions that will be precised in each case;
- D is either a bounded smooth domain $\Omega \subset \mathbb{R}^n$ or it is all of $\mathbb{R}^n$. In the former case we will associate (1) with homogeneous Dirichlet boundary conditions $u(x) = 0,\ x \in \partial\Omega$; in the latter we will look for *bound states*, namely for solutions u such that $u(x) \to 0$ as $|x| \to \infty$;
- $1 < q < \alpha \leq 2^*$. In particular, a main part of the discussion will be focussed on the case that $q < 2 < \alpha$, when the nonlinearity is the sum of a concave and a convex term;
- a is a non-negative constant. Precisely, when $D = \Omega$ we can take w.l.o.g. $a = 0$. When $D = \mathbb{R}^n$, we shall take $a > 0$, say $a = 1$, if $\alpha < 2^*$ and $a = 0$ if $\alpha = 2^*$.

Here and in the sequel we set

$$2^* = \begin{cases} \dfrac{2n}{n-2} & \text{if } n \geq 3 \\ d + \infty & \text{if } n = 2. \end{cases}$$

We will also review some results concerning the existence of multiple solutions, possibly infinitely many, of problems with odd nonlinearities like

$$-\Delta u + au = \lambda h(x)|u|^{q-2}u + k(x)|u|^{\alpha-2}u, \quad x \in D, \tag{2}$$

Let us point out that (1) and (2) are model problems and the results we will discuss hold for broader classes of equations. We refer to the original papers for more general results. See also the Monographs [47] and [51].

The paper is divided into 6 Sections, organized as follows.

Section 2 contains some results dealing with (1) when $D = \Omega$. We mainly report on some results taken from [8], [9].

Section 3 is devoted to show how some of the preceding results can be extended to quasilinear problems, where the second order operator $-\Delta$ is substituted by the p-laplacian $-\Delta_p := -\operatorname{div}\,(|\nabla u|^{p-2}\nabla u)$. Here we mainly follow the papers [10], [23], [36]-[39].

Section 4 deals with second order elliptic equations on $\mathbb{R}^n$ in the case that the nonlinearity has a subcritical growth at infinity, namely when $\alpha < 2^*$. We outline the results of [35], [42].

Sections 5 and 6 are concerned with the case that $D = \mathbb{R}^n$, $a = 0$ and $\alpha = 2^*$ (critical growth). In the former we use the *Concentration Compactness* method, while in the latter we report some recent results of [11], [12] which are obtained by means of a perturbation method variational in nature.

There is a very large bibliography on the topics we will deal with and is not possible to make an exhaustive list of papers. We have reported only the works that are more closely related to the material discussed here. Moreover, to confine the survey within the limits of a reasonable length, we cannot discuss many other interesting topics on elliptic equations. Among them, we cite the problems dealing with the bifurcation from the essential spectrum, the existence of solutions of nonlinear Schrödinger equations with a potential (including the existence of multi-bump solutions when the potential has an oscillating behaviour), the existence of concentrated solutions for singularly perturbed problems, and the list could continue.

Notation

- Ω denotes a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.
- For $D = \Omega$ or $D = \mathbb{R}^n$, $L^r(D)$ denote Lebesgue spaces. and $H^{k,r}(D)$ denote the Sobolev spaces; we also set $H^k(D) = H^{k,2}(D)$.
- $\mathcal{D}^{1,2}(\mathbb{R}^n)$ denotes the closure of $C_0^\infty(\mathbb{R}^n)$ with respect the norm $\|\nabla u\|_{L^2(\mathbb{R}^n)}$.
- $\lambda_1 = \lambda_1(\Omega)$ denotes the first eigenvalue of the Laplace opeartor $-\Delta$ on $H_0^1(\Omega)$.
- u_+ denotes the positive part of the function u.

2 – Second order problems on bounded domains

In this section we will mainly deal with the existence of positive solutions to Dirichlet boundary value problems like

$$
\begin{cases}
-\Delta u = \lambda u^{q-1} + k u^{\alpha-1} & \text{in } \Omega,\\
u(x) > 0 & \text{in } \Omega,\\
u|_{\partial\Omega} = 0,
\end{cases}
\tag{3}
$$

where, for the sake of simplicity, we have taken $h \equiv 1$ and k to be a constant.

First, it is convenient to recall some known results in the limit cases in which either $q = 2$ or $k = 0$.

(a) When $k = 0$ and $1 < q < 2$, namely when the nonlinearity is purely concave, problem (3) has a unique solution for every $\lambda > 0$, which can be found e.g. by the method of lower and upper solutions. The uniqueness is proved in [27].

(b) When $q = 2$, $k > 0$ and $2 < \alpha < 2^*$ the problem can be faced by variational tools. Let us consider the Euler functional

$$f_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \frac{k}{\alpha}\int_\Omega u_+^\alpha dx.$$

It is immediate to see that f_λ has the Mountain-Pass geometry whenever $0 \leq \lambda < \lambda_1$. Furthermore, the condition $\alpha < 2^*$ ensures that the Palais-Smale, (PS) in short, compactness condition is satisfied. Then the Mountain-Pass Theorem [14] provides the existence of a critical point $u \neq 0$ of f_λ and hence a solution of (3), provided $0 \leq \lambda < \lambda_1$.

(c) In the limit case of the critical Sobolev exponent $\alpha = 2^*$, problem (3) (with $k = 1$) becomes

$$\begin{cases} -\Delta u = \lambda u + u^{(n+2)/(n-2)} & \text{in } \Omega \subset \mathbb{R}^n,\ n \geq 3, \\ u(x) > 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{4}$$

and can have no solution at all. This is the case if e.g. $\lambda = 0$ and the domain Ω is star-shaped. (See [45]). However, a celebrated result by Brezis and Nirenberg shows that

THEOREM 2.1 [25]. *If $n \geq 4$ problem (4) has a positive solution iff $\lambda \in (0, \lambda_1)$. If $n = 3$, there exists $\underline{\lambda} > 0$ such that problem (4) has a positive solution for all $\lambda \in (\underline{\lambda}, \lambda_1)$.*

Roughly, the functional f_λ with $\alpha = 2^*$ satisfies the (PS) condition along any sequence u_j such that

$$f_\lambda(u_j) \to c_\lambda, \quad \nabla f_\lambda(u_j) \to 0,$$

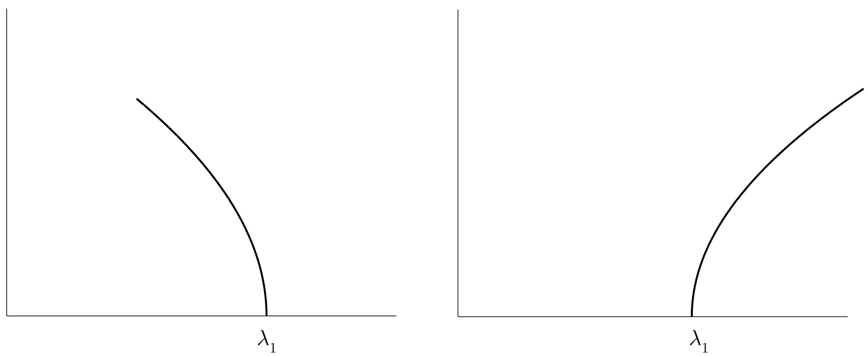
provided $c_\lambda < (1/n)S^{n/2}$, where S denotes the best Sobolev constant for the embedding of $H_0^1(\Omega)$ in $L^{2^*}(\Omega)$. One proves that the mountain-pass critical level c_λ is actually below $(1/n)S^{n/2}$ provided $0 < \lambda < \lambda_1$ an $n \geq 4$. When $n = 3$, one shows that this happens for $\underline{\lambda} < \lambda < \lambda_1$, for a suitable $\underline{\lambda} > 0$ depending on Ω.

Moreover if $n = 3$ there exist domains (for instance a ball) and $\underline{\lambda} < \lambda_1$ such that if $\lambda \in (0, \underline{\lambda})$ problem (4) has no positive solution. See [25].

For future references, let us recall that problem (3) with $q = 2$, can be also faced by the Bifurcation Theory. Actually, it is well known that λ_1 is a bifurcation point for (the positive) solutions of (3) and there is a continuum

$$\Sigma = \{(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega) : u \text{ is a solution of } (3)\}$$

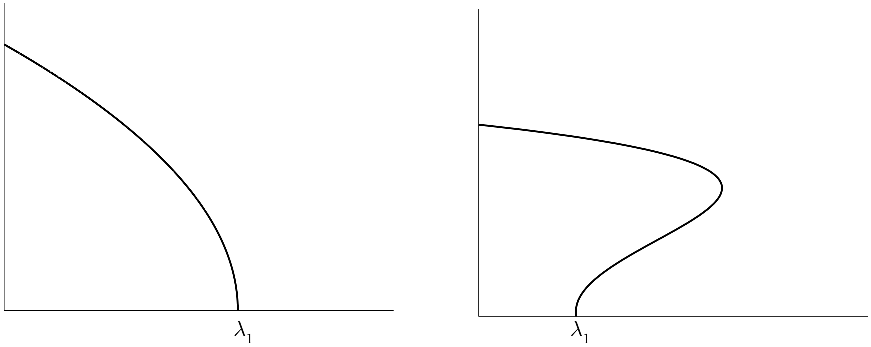
such that $(\lambda_1, 0) \in \overline{\Sigma}$. It is worth recalling that this result holds true without any restriction on α. The local behaviour of Σ near $(\lambda_1, 0)$ depends on the sign of k. If $k > 0$, resp. $k < 0$, the bifurcation is sub-critical, resp. super-critical. See figure 1 below. For example, the existence of solutions of (4) for λ in a left neighbourhood of λ_1 can be deduced by these arguments.



Subcritical bifurcation Supercritical bifurcation

Fig. 1

To control the global behaviour of Σ one has to take $\alpha < 2^*$. If this is the case one can show, see [40], there exists an *a priori* bound for the positive solutions of (3) and this permits to show that it possesses a solution for every $0 \le \lambda < \lambda_1$, see the left part of figure 2 below.



Bifurcation diagram when $k > 0$

Bifurcation diagram when $k_+ \not\equiv 0$, $\int k\varphi_1^\alpha < 0$

Fig. 2

Remark 2.2. For completeness, let us recall some further results.

1) If $k < 0$ and $\alpha > 2$ (with no further restrictions), problem (3) with $q = 2$ has a solution for all $\lambda > \lambda_1$: it suffices to consider, for $u \ge 0$, the positive part of $\lambda u + ku^{\alpha-1}$ and use the maximum principle.

2) When $k = k(x)$ and k_+, the positive part of k, is not identically $= 0$, problem (3) has still a solution for $0 \le \lambda < \lambda_1$. In addition, if φ_1 denotes the (normalized) positive eigenfuction of $-\Delta$ on $H_0^1(\Omega)$ and $\int k\varphi_1^\alpha < 0$, the bifurcation from λ_1 is *supercritical.* As a consequence, one can show that there exists $\varepsilon > 0$ such that (3) has at least two solutions for all $\lambda_1 < \lambda < \lambda_1 + \varepsilon$. See the right part of Figure 2. We refer to [1], see also [4], for more details. $\square$

2.1 – Concave-convex nonlinearities

The simultaneous effect of the concave and the convex terms has been investigated in [8]. Here we report on some of those results. Since here the positive factor k is not relevant, we will take in the sequel $k = 1$.

THEOREM 2.3 [8].

(*i*) *Let* $1 < q < 2 < \alpha$. *Then there exist* $\Lambda > 0$ *such that*

1. *for all* $\lambda \in (0, \Lambda)$ *problem* (3) *has a minimal solution* u_λ;
2. *for* $\lambda = \Lambda$ *problem* (3) *has at least one weak solution* $u_\Lambda \in H_0^1(\Omega) \cap L^{\alpha+1}(\Omega)$;
3. *for* $\lambda > \Lambda$ *problem* (3) *has no solution.*

(*ii*) *Let* $1 < q < 2 < \alpha \leq 2^*$. *Then for all* $\lambda \in (0, \Lambda)$ *problem* (3) *has a second solution* $v_\lambda > u_\lambda$.

The proof is based on the following steps:

STEP 1. Statement (*i*) is proved by using lower and upper solutions. Clearly, here one takes advantage of the presence of the concave term λu^{q-1}.

STEP 2. To prove (*ii*) one uses critical point theory. First, one uses [26] to show that the minimal solution u_λ is a local minimum of f_λ in $H_0^1(\Omega)$. Next, one verifies that f_λ has the Mountain-Pass geometry. Letting, as before, c_λ denote the Mountain-Pass level, one proves that f_λ satisfies the (PS) condition at level c_λ provided $\alpha \leq 2^*$. In particular, when $\alpha = 2^*$ this is true because, again, the presence of the term $-\frac{\lambda}{q}\int_\Omega u_+^q dx$ implies that $c_\lambda < (1/n)S^{n/2}$.

REMARK 2.4. The results stated in Theorem 2.3 can be made more precise. One can show that there is $A > 0$ such that for all $\lambda \in (0, \Lambda)$ problem (3) has at most one solution such that $\|u\|_\infty \leq A$. Furthermore, when $\alpha = 2^*$ and Ω is star-shaped, any solution w_λ different from the minimal solution u_λ is such that $\|w_\lambda\|_\infty \to \infty$ as $\lambda \downarrow 0$.

We can also prove a multiplicity result concerning the problem

$$(5) \qquad \begin{cases} -\Delta u = \lambda\, |u|^{q-2}u + |u|^{\alpha-2}u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

When $2 = q < \alpha < 2^*$ it is well known that the Lusternik-Schnierelman theory provides the existence of infinitely many solutions of (5) for all λ. See, e.g. [3], [14]. A similar result holds when the nonlinearity is merely $\lambda|u|^{q-2}u$ with $\lambda > 0$ and $1 < q < 2$, see [5].

Using again the Lusternik-Schnierelman theory applied to

$$\widetilde{f}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{q}\int_\Omega |u|^q dx - \frac{1}{\alpha}\int_\Omega |u|^\alpha dx,$$

one can extend the cited results to handle equations with concave-convex nonlinearities.

THEOREM 2.5 [8], [38]. *Let* $1 < q < 2 < \alpha \leq 2^*$*. Then there is* $\lambda^* > 0$ *such that for all* $\lambda \in (0, \lambda^*)$ (5) *has infinitely many solutions with* $\widetilde{f}_\lambda(u) < 0$*. Furthermore, if* $1 < q < 2 < \alpha < 2^*$*,* (5) *has also infinitely many solutions with* $\widetilde{f}_\lambda(u) > 0$*.*

REMARK 2.6. When $1 < q < 2 < \alpha < 2^*$, Bartsch and Willem [49] have shown that the preceding results hold true for all $\lambda > 0$. Moreover, Wang [49] has proved the existence of infinitely many solutions with negative energy for equations like (5) with $|u|^{\alpha-2}u$ substituted by any continuous $g(x,u)$ which is odd in u for $|u|$ small and such that $g = o(|u|^{q-1})$ as $|u| \to 0$, uniformly in $x \in \Omega$. □

Next, following [9, Sect. 2], we obtain some multiplicity results without symmetry assumptions. Precisely consider

$$(6) \qquad -\Delta u = \lambda |u|^{q-2}u + G'(u), \quad x \in \Omega, \; u = 0 \, x \in \partial\Omega,$$

where $G \in C^2(\mathbb{R}, \mathbb{R})$ satisfies:

(G_1) $G'(s)s \geq \alpha G(s) > 0$, $\forall\, s \in \mathbb{R}$, with $2 < \alpha < 2^*$;
(G_2) $G''(s)s^2 \geq \alpha G'(s)s$, $\forall\, s \in \mathbb{R}$;
(G_3) $G''(s)s^2 \leq c_1|s|^\alpha$, $\quad \forall\, s \in \mathbb{R} \quad (c_1 > 0)$.

From the above assumptions it follows that G is convex and

$$(7) \qquad G(s) = \frac{1}{\alpha}|s|^\alpha + o(|s|^\alpha), \text{ at } s = 0,\, s = \infty.$$

Of course, the difference with the nonlinearity handled in equation (5) is that now G' is not assumed to be odd. For $u \in H_0^1(\Omega)$ we set (below

$u_- := u - u_+$):

$$f_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{q}\int_\Omega |u|^q dx - \int_\Omega G(u)dx\,,$$
$$f_\lambda^+(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{q}\int_\Omega u_+^q dx - \int_\Omega G(u_+)dx\,,$$
$$f_\lambda^-(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{q}\int_\Omega u_-^q dx - \int_\Omega G(u_-)dx\,.$$

From the preceding arguments, see Theorem 2.3, it follows there exists $\delta > 0$ such that for all $\lambda \in (0, \delta)$ one has:

(i) f_λ^+ has a local minimum $u_1 > 0$ and a Mountain-Pass critical point $v_1 > 0$, with $f_\lambda^+(u_1) < 0 < f_\lambda^+(v_1)$;
(ii) f_λ^- has a local minimum $u_2 > 0$ and a Mountain-Pass critical point $v_2 > 0$, with $f_\lambda^-(u_2) < 0 < f_\lambda^-(v_2)$.

Taking advantage of this geometry we can prove the following result.

THEOREM 2.7 [9]. *There exists $\lambda^* > 0$ such that, for all $0 < \lambda < \lambda^*$, one has:*

(i) *if* (7) *holds, then* (6) *has a solution $u_3 \neq u_{1,2}$, with $f_\lambda(u_3) < 0$;*
(ii) *if, in addition,* ($G_1 - G_2 - G_3$) *hold, then* (6) *has another solution $v_3 \neq v_{1,2}$, with $f_\lambda(v_3) > 0$.*

The idea of the proof can be summarized as follows.

STEP 1. According to the results of [26], $u_{1,2}$ are also local minima of f_λ. Consider the *Mountain Pass Level* with base u_1 and u_2,

$$b_\lambda = \inf_{h\in\mathcal{H}} \max\{f_\lambda(h(t)) \,:\, t \in [0,1]\}$$

where $\mathcal{H} = \{h \in C^1([0,1], H_0^1(\Omega)) \,:\, h(0) = u_1,\ h(1) = u_2\}$. Since f_λ verifies the (PS) condition, then b_λ is a critical level that carries a critical point u_3 different from $u_{1,2}$. Moreover, the fact that $f_\lambda(tu) < 0$ for all $t \neq 0$ and small, implies that $b_\lambda < 0$ and hence $f_\lambda(u_3) < 0$. This proves (i).

STEP 2. For $u \in H_0^1(\Omega)$ we set

$$\phi_\lambda^\pm(u) := (\nabla f_\lambda^\pm(u)|u) = \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega |u_\pm|^q - \int_\Omega G'(u_\pm)u_\pm dx.$$

and

$$\phi_\lambda(u) := (\nabla f_\lambda(u)|u) = \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega |u|^q - \int_\Omega G'(u)u dx.$$

Using the properties of G it is possible to show that there exist $\lambda_0 > 0$, $A > 0$ and $\rho > 0$ such that, setting

$$M_\lambda^\pm = \{u \in H_0^1(\Omega) \, : \, \phi_\lambda^\pm(u) = 0, \ \|u\| \geq \rho\},$$

respectively

$$M_\lambda = \{u \in H_0^1(\Omega) \, : \, \phi_\lambda(u) = 0, \ \|u\| \geq \rho\},$$

there results, for all $0 < \lambda < \lambda_0$:

(a) $(\nabla\phi_\lambda^\pm(u)|u) < 0$, resp. $(\nabla\phi_\lambda(u)|u) < 0$, for all $u \in M_\lambda^\pm$, resp. M_λ; in particular $M_\lambda^\pm$, resp. M_λ, are smooth manifolds in $H_0^1(\Omega)$. Moreover they are radially diffeomorphic to the unit sphere in $H_0^1(\Omega)$.

(b) $f_\lambda^\pm(u) \geq A$, resp. $f_\lambda(u) \geq A$, for all $u \in M_\lambda^\pm$, resp. $u \in M_\lambda$.

(c) Critical points of $f_\lambda^\pm$, resp. f_λ, constrained on $M_\lambda^\pm$, resp. M_λ, are stationary points of $f_\lambda^\pm$, resp. f_λ. We will refer this property by saying that $M_\lambda^\pm$, resp. M_λ, is a *natural constraint* for $f_\lambda^\pm$, resp. f_λ.

REMARK 2.8. One could actually show that if λ is small enough the set $\{u \in H_0^1(\Omega) \setminus \{0\} \, : \, \phi_\lambda(u) = 0\}$ consists of two disjoint manifolds, radially diffeomorfic to the unit sphere in $H_0^1(\Omega)$. One contains the critical points of f_λ at negative levels; the other one is M_λ and carries the critical points of f_λ at positive levels. □

STEP 3. $f_\lambda^\pm$, resp. f_λ, satisfy the Palais-Smale condition on $M_\lambda^\pm$, resp. M_λ.

STEP 4. By the previous steps it follows that there exist $w_1 \in M_\lambda^+$ and $w_2 \in M_\lambda^-$ such that

$$f_\lambda^+(w_1) = \min_{u \in M_\lambda^+} f_\lambda^+ > 0 \tag{8}$$

$$f_\lambda^-(w_2) = \min_{u \in M_\lambda^-} f_\lambda^- > 0. \tag{9}$$

We can assume that $w_1 = v_1$ and $w_2 = v_2$, otherwise $w_{1,2}$ already provide two additional solutions of (6). The new feature is that, now, $v_{1,2}$ are characterized as minima of $f_\lambda^\pm$ on $M_\lambda^\pm$. Furthermore, by a suitable modification of the arguments of [26], one can show that $v_{1,2}$ are in fact local minima of f_λ on M_λ.

STEP 5. We use the preceding information to find another critical point of f_λ on M_λ. This critical point will be obtained by using the Mountain-Pass Theorem with base points v_1 and v_2. This is possible because of (8) and (9). More precisely, let $\mathcal{H}^* = \{h \in C([0,1], M_\lambda) : h(0) = v_1,\ h(1) = v_2\}$ and set

$$b_\lambda^* = \inf_{h \in \mathcal{H}^*} \max\{f_\lambda(h(t)) : t \in [0,1]\}.$$

There results $b_\lambda^* > \max\{f_\lambda(v_1), f_\lambda(v_2)\} > 0$ and since (PS) holds, the Mountain-Pass Theorem yields the existence of $v_3 \neq v_1, v_2$, which is a critical point of f_λ on M_λ, with $f_\lambda(v_3) = b_\lambda^* > 0$. Finally, since M_λ is a natural constraint, see Step 2, then v_3 is a free critical point of f_λ and hence a solution of (6).

REMARK 2.9. The results in Theorem 2.7 extends some results in [48]. Other results with lack of symmetry can be found in [20] and the references therein. □

3 – Quasilinear problems

In this section we consider the quasilinear (model) problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2} u + |u|^{\alpha-2} u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{10}$$

where $1 < q < p < \alpha \leq p^*$. Here

$$p^* = \begin{cases} \dfrac{np}{n-p} & \text{if } n > p \\ +\infty & \text{if } n \leq p. \end{cases}$$

As in Section 2, we begin with a short review of some results in the case $q = p$. Letting λ_1 denote now the first eigenvalue of the p-laplacian,

$$-\Delta_p u = \lambda |u|^{p-2} u, \quad \text{in}\, \Omega, \quad u|_{\partial\Omega} = 0,$$

one shows that there exists a positive solution of (10) for all $0 \leq \lambda < \lambda_1$, when $q = p$ and $\alpha < p^*$. Such a solution is found as Mountain-Pass critical point of the Euler functional

$$F_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega |u|^p dx - \frac{1}{\alpha}\int_\Omega |u|^\alpha dx.$$

In the case in which $\alpha = p^*$, the critical exponent case, one can follow the strategy used to prove Theorem 2.1 and estimate the mountain-pass level C_λ of F_λ (with $\alpha = p^*$). One shows that, if $n \geq p^2$, there results $C_\lambda < (1/n)S^{n/p}$, where the (PS) holds. This permits to obtain the following result which is the counterpart of Theorem 2.1.

THEOREM 3.1 [36]. *Assume $n \geq p^2$. Then for $0 < \lambda < \lambda_1$ the problem* (10) *with $p = q$ and $\alpha = p^*$ has a positive solution.*

REMARK 3.2. For $p < n < p^2$ (in the case $p = 2$ this corresponds to $n = 3$) there is a result similar to the one stated in Theorem 2.1 for $n = 3$. See [15]. Moreover there is a non-existence result for λ close to zero, when the domain is a ball and we look for radial solutions. This result is contained in a widely quoted, but unpublished, work by Atkinson, Peletier and Serrin. See [32] where a generalization of this result is proved. □

3.1 – The case $1 < q < p < \alpha \leq p^*$

Let us now consider the case $1 < q < p < \alpha \leq p^*$ which corresponds to the concave-convex case when $p = 2$. The following result extends Theorem 2.3 to p-laplacian equations, in the case that $\alpha < p^*$.

THEOREM 3.3 [32]. *Let $1 < q < p < \alpha < p^*$. Then there exist $\Lambda > 0$ such that*

1. *for all $\lambda \in (0, \Lambda)$ problem* (10) *has at least two positive solutions $u \in H_0^{1,p}(\Omega)$;*
2. *for $\lambda = \Lambda$ problem* (10) *has at least one positive solution $u \in H_0^{1,p}(\Omega)$;*
3. *for $\lambda > \Lambda$ problem* (10) *has no solution.*

Theorem 3.3 has been proved by an argument similar to that used in the semilinear case. The main difficulty arises dealing with the existence of a second solution. The new feature is an extension to p-Laplace equations of the result by Brezis-Nirenberg [26] that allows to say that the first solution sweeped between a lower and an upper solution is in fact a local minimum in the Sobolev space for the Euler functional.

When $\alpha = p^*$, to find a result global in λ as in Theorem 3.3, a restriction on p is in order.

THEOREM 3.4 [34]. *Let* $1 < q < p < \alpha = p^*$. *Then the statement* 1) *of Theorem* 3.3 *holds true provided that either* $2n/(n+2) < p < 3$ *and* $1 < q < p$, *or* $p \geq 3$ *and* $p^* - 2/(p-1) < q < p$.

One can also extend to the p-laplacian the first statement of Theorem 2.5. Actually, for $\lambda > 0$ small one can take advantage of the fact that the dominant term is $\lambda |u|^{q-2}u$ with $q < p$, to find critical points of F_λ in the region where $F_\lambda(u) < 0$. This permits to prove:

THEOREM 3.5 [38]. *Let* $1 < q < p < \alpha \leq p^*$. *Then there exists* $\lambda_0 > 0$ *such that for all* $\lambda \in (0, \lambda_0)$ *problem* (10) *has infinitely many solutions with* $F_\lambda(u) < 0$.

3.2 – The radial case

In this subsection we take $\Omega = B_1 := \{x \in \mathbb{R}^n : |x| < 1\}$ and look for *radial* solutions of (10). Precisely, letting $r = |x|$ and

$$A_p u = -\frac{1}{r^{N-1}} \left(r^{N-1}|u'(r)|^{p-2}u'(r)\right)',$$

we shall look for solutions $u = u(r)$ of

$$\text{(11)} \qquad \begin{cases} A_p u = \lambda |u|^{q-2}u + |u|^{\alpha-2}u, & 0 \leq r < 1, \\ u'(0) = 0,\ u(1) = 0, \end{cases}$$

where $1 < q < p < \alpha < p^*$. Plainly, if $u(r)$ solves (11), then $u(|x|)$ is a solution of (10).

To begin with, we deal with positive (radial) solutions of (11) and outline a procedure, based on Topological Degree arguments, that permits to obtain an alternative proof of Theorem 3.3. We follow [9].

Let us set

$$\widetilde{\Lambda} = \sup\{\lambda > 0 : \text{ (11) has a positive (radial) solution}\}.$$

Using lower and upper solutions, one readily shows that (11) has a positive radial solution for $\lambda > 0$ small enough. This implies that $\widetilde{\Lambda} > 0$. Furthermore, one can also show that $\widetilde{\Lambda} < \infty$.

The following Lemma provides an a-priori bound like the one by Gidas and Spruck [40].

LEMMA 3.6. *There exists $C > 0$ such that $\|u\|_\infty \le C$, for all positive* (*radial*) *solutions of* (11) *and all $\lambda \in [0, \widetilde{\Lambda}]$.*

The proof of Lemma 3.6 makes use of the following two facts.

1) One considers the Cauchy problem

$$\begin{cases} A_p u = |u(r)|^{\alpha-2} u(r), \\ u(0) = a,\ u'(0) = 0. \end{cases} \tag{12}$$

It is well known, see e.g. [33], that (12) has a unique solution u_a; moreover there exists $R_a > 0$ such that $u_a(R_a) = 0$ and $u'_a(R_a) < 0$.

2) As a consequence of the preceding result one shows

LEMMA 3.7. *Let $1 < p < \alpha < p^*$ and let u be any solution of*

$$\begin{cases} A_p u = |u|^{\alpha-2} u, 0 \le r < \infty, \\ u'(0) = 0,\ u(r) \ge 0. \end{cases}$$

Then $u \equiv 0$.

Consider the compact map K_λ defined on $X := \{ u \in C^1(B_1) : u(x) = u(|x|),\, u(1) = 0\}$ by setting

$$K_\lambda(u) = (A_p)^{-1} \left(\lambda |u|^{q-2}u + |u|^{\alpha-2}u\right) \quad \lambda \in (0, \Lambda).$$

Lemma 3.6 and the homotopy invariance of the degree imply that

$$\deg(I - K_\lambda, T_r, 0) = \text{const} \quad T_r := \{u \in X : \|u\|_X < r\},$$

for $r > 0$ large enough. Since for $\lambda > \widetilde{\Lambda}$ (11) has no solution, we infer that

$$\deg(I - K_\lambda, T_r, 0) = 0.$$

On the other side, using sub and super solutions and the arguments of [2], it is easy to show that for all $0 < \lambda < \widetilde{\Lambda}$, K_λ has a fixed point u_λ. Moreover, if this u_λ is isolated (otherwise we have done) its fixed point index is $= 1$. Then, by the excision property of the degree we infer:

Theorem 3.8 [9]. *For all $\lambda \in (0, \widetilde{\Lambda})$ the problem* (11) *admits at least two positive solutions.*

Remark 3.9. (a) Lemma 3.6 is the only point where one uses the fact to deal with radial solutions. Then the preceding arguments provide a different proof of Theorem 2.3, according with the uniform L^∞ estimates by Gidas and Spruck. See [40].

(b) Lemma 3.6 can also be used to find a a global continuum S of solutions of (11) emanating from $\lambda = 0$, $u = 0$ which has the behaviour indicated Fig.3. See [9, Theorem 3.12]. ∎

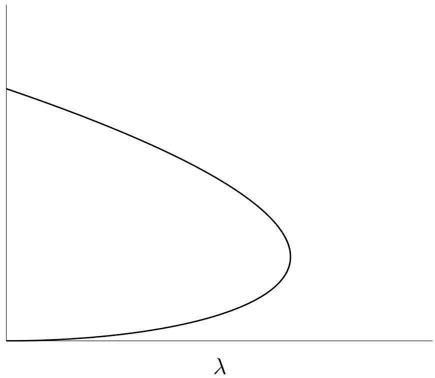


Fig. 3

The result indicated in Remark 3.9-(b) can be greatly improved. In order to highlight that the results do not depend on the oddness of the nonlinearity, we will consider a slightly modification of problem (11), namely

$$\begin{cases} A_p u = \lambda |u|^{q-2}u + g(r,u), & 0 < r < 1, \\ u'(0) = 0, \ u(1) = 0, \end{cases} \tag{13}$$

where $1 < q < p$ and $g \in C(\mathbb{R}^+ \times \mathbb{R})$ satisfy, uniformly with respect to $r \in [0,1]$,

(g_1) $\lim_{s\to 0} \dfrac{g(r,s)}{|s|^{p-2}s} = 0,$

(g_2) $g(r,s)s > 0, \quad \forall\, s \neq 0,$

(g_3) $g(r,s) \sim |s|^{\alpha-2}s$ as $|s| \to \infty$, with $p < \alpha < p^*$.

THEOREM 3.10 [10]. *Let $1 < q < p < \alpha < p^*$ and suppose that g satisfies (g_1). Then there exist infinitely many unbounded continua $\Gamma_k \subset \mathbb{R} \times X$, $k \in \mathbb{N}$, of solutions of* (13) *with the following properties:*

1) *If $\lambda \downarrow 0$ there exists $u^k_\lambda \in X$ such that $(\lambda, u^k_\lambda) \in \Gamma_k$ and $u^k_\lambda \to 0$.*
2) *If $(\lambda, u) \in \Gamma_k$, and $\lambda > 0$ then $u \neq 0$ and u has exactly $k-1$ simple zeros in the interval $(0,1)$.*
3) *There exists a constant $\rho > 0$ such that if $r \in (0,\rho]$, and $(\lambda, u) \in \Gamma_k$ with $\|u\|_\infty = \rho$, then $\lambda > \lambda(\rho) > 0$.*

If, in addition, $(g_2 - g_3)$ hold, then:

4) *$\exists\, C_k > 0$ such that $\|u\| \leq C_k$ for every $(\lambda, u) \in \Gamma_k$, $\lambda \geq 0$.*

COROLLARY 3.11. *Let $1 < q < p < \alpha < p^*$.*

1) *If (g_1) holds, $\exists\, \lambda^* > 0$ such that for all $\lambda \in [0, \lambda^*)$ problem* (13) *has infinitely many radial solutions.*
2) *If, in addition, $(g_2 - g_3)$ hold, then for all $\lambda \in [0, \lambda^*)$ problem* (13) *has infinitely many pairs of radial solutions.*

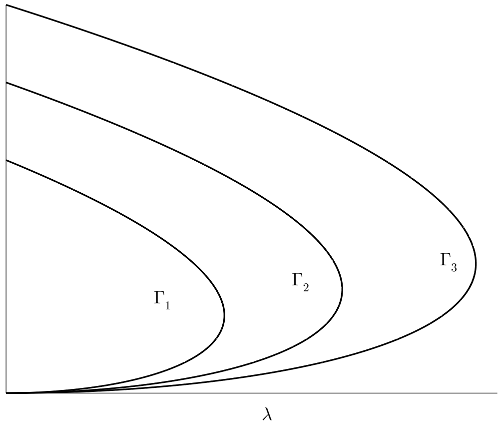


Fig. 4

The behaviour of the solutions set of (13) is indicated in Fig. 4 below. The proof is carried out in 3 steps.

STEP 1. For $j \in \mathbb{N}$ let $a_j = j^{p-q}$ and

$$h_j(u) = \begin{cases} a_j \cdot |u|^{p-2}u, & \text{if } |u| \leq 1/j, \\ |u|^{q-2}u & \text{if } |u| \geq 1/j. \end{cases}$$

Problem (13) is approximated by

$$\begin{cases} A_p u = \lambda h_j(u) + g(r,u), 0 < r < 1, \\ u'(0) = 0, \ u(1) = 0, \end{cases} \tag{14}$$

which can be handled by bifurcation theory (for the case $p \neq 2$, see [9]). It follows that for all $k \in \mathbb{N}$ there exists an unbounded continuum $S_{k,j}$ of solutions (λ, u) of (14) bifurcating from $\mu_{k,j} = \mu_k / a_j$, where μ_k is the k-th eigenvalue of

$$A_p u = \lambda |u|^{p-2}u, \quad u'(0) = 0, \, u(1) = 0.$$

Moreover, if $(\lambda, u) \in S_k$ and $u \neq 0$ then u has exactly $k-1$ zeros in $(0,1)$.

STEP 2. Using a topological lemma from [50], one performs a limiting procedure to find that S_k "converges" to a continuum Γ_k.

STEP 3. The branches S_k satisfy properties $1-4$ stated in Theorem 3.10. Moreover, it is possible to show these properties are preserved after the limiting procedure.

4 – Problems on $\mathbb{R}^n$: the sub-critical case

In this section we will be concerned with elliptic equations on all of $\mathbb{R}^n$ like

$$\begin{cases} -\Delta u + u = \lambda h(x) u^{q-1} + k(x) u^{\alpha-1}, \\ u > 0, \quad u \in H^1(\mathbb{R}^n). \end{cases} \tag{15}$$

Here we deal with the case in which the exponent α is sub-critical, namely $2 < \alpha < 2^*$. This will be always assumed in this section.

Let us begin with some results dealing with

$$\begin{cases} -\Delta u + u = k(x) u^{\alpha-1} \quad \text{in } \mathbb{R}^n, \\ u > 0, \quad u \in H^1(\mathbb{R}^n). \end{cases} \tag{16}$$

The main tool used to face (16) is critical point theory.
Let us suppose that $k \in L^\infty(\mathbb{R}^n)$ and that

$$k(x) > 0, \ \forall\, x \in \mathbb{R}^n, \qquad \lim_{|x|\to\infty} k(x) = k_\infty > 0. \tag{17}$$

Solutions $u \in H^1(\mathbb{R}^n)$ of (16) can be found as critical points of

$$f(u) := \frac{1}{2}\int_{\mathbb{R}^n} (|\nabla u|^2 + u^2) - \frac{1}{\alpha}\int_{\mathbb{R}^n} k(x)|u|^\alpha .$$

The functional f has the geometry of the Mountain-Pass Theorem: $u = 0$ is a strict local minimum and there exists $e \in H^1(\mathbb{R}^n)$ such that $f(e) < 0$. But, since we are dealing with a problem on all of $\mathbb{R}^n$ which is invariant under translations, there is a lack of (PS), even if $\alpha < 2^*$.

If $k(x) \equiv k_\infty > 0$ the difficulty can be overcome by looking for radially symmetric solutions. Actually, the Sobolev space $H^1_r(\mathbb{R}^n)$, the radial functions of $H^1(\mathbb{R}^n)$, is compactly embedded in $L^\alpha(\mathbb{R}^n)$ for any $\alpha \in]2, 2^*[$, and this permits to prove the existence of a solution of

$$-\Delta u + u = k_\infty u^{\alpha-1}, \quad u > 0, \quad u \in H^1_r(\mathbb{R}^n). \tag{18}$$

See [46], [22].

Remark 4.1. If $n = 4$ or $n \geq 6$ it has been shown that (18) possesses a non-radial solution. See [18].

When $k(x) \not\equiv k_\infty$ or, more in general, is not radial, the lack of (PS) can be bypassed by using the *Concentration-Compactness Principle* introduced by P.L. Lions [43], [44]. Roughly, let us set

$$S_\alpha = \inf\{\frac{1}{2}\int_{\mathbb{R}^n} (|\nabla u|^2 + u^2) : u \in H^1(\mathbb{R}^n),\ \int_{\mathbb{R}^n} k_\infty |u|^\alpha = 1\}.$$

Remark 4.2. The value S_α is achieved by a positive radially symmetric function $w \in H^1(\mathbb{R}^n)$ satisfying (18).

There results

Lemma 4.4. *$(PS)_c$ holds provided $c < \left(\frac{1}{2} - \frac{1}{\alpha}\right) S_\alpha^{\alpha/(\alpha-2)}$.*

Moreover, letting $c(f)$ denote the Mountain-Pass level of f one proves:

Lemma 4.4. *Let (17) hold and suppose that $k(x) > k_\infty$ for all $x \in \mathbb{R}^n$. Then $c(f) < \left(\frac{1}{2} - \frac{1}{\alpha}\right) S_\alpha^{\frac{\alpha}{\alpha-2}}$.*

From Lemmas 4.3 and 4.4 one immediately infers that $c(f)$ is a critical level for f and hence (16) has a positive solution.

A more general result has been obtained in [17] proving that (16) has a positive solution provided k satisfies (17) and $\exists\, C, \delta > 0$ such that

$$k(x) \geq k_\infty - C\mathrm{e}^{-\delta|x|}, \quad \text{for } |x| \gg 1. \tag{19}$$

The new feature is that if (19) holds, the critical point of f is found at a min-max level which is possibly greater that $\left(\frac{1}{2} - \frac{1}{\alpha}\right) S_\alpha^{\alpha/(\alpha-2)}$. For this reason the results of [17] require delicate arguments.

As for the case of problem on bounded domains, one can find multiplicity results provided the nonlinearity is odd. For example, if $2 < \alpha < 2^*$ the problem

$$-\Delta u + u = |u|^{\alpha-2}u, \quad u \in H^1(\mathbb{R}^n) \tag{20}$$

has infinitely many radial solutions, see [46]. Furthermore, if $n = 4$ or $n \geq 6$, (20) has infinitely many non-radial solutions, see [18].

Remark 4.5. If $n \geq 3$ and $\alpha \geq 2^*$, the only solution of (20) which belongs to $H^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$ is $u \equiv 0$. □

Different are the results if we deal with a concave nonlineaity. For example, following [24], consider the problem

$$(21) \qquad -\Delta u = h(x)u^{q-1}, \quad x \in \mathbb{R}^n, \quad u > 0, \quad (1 < q < 2)$$

where h satisfies:

$$(22) \qquad h \in L^\infty_{loc}(\mathbb{R}^n), \quad h \geq 0.$$

Using sub- and supersolutions, one can prove:

Theorem 4.6 [24]. *Suppose that* (22) *holds. Then problem* (21) *has a solution iff there exists* $U \in L^\infty(\mathbb{R}^n)$ *such that* $-\Delta U = h$*. Moreover,* (21) *has exactly one solution such that* $\limsup_{|x|\to\infty} u(x) = 0$.

For some other results dealing with problems on $\mathbb{R}^n$ with a concave (or, more in general, sublinear at infinity) nonlinearity, see [16].

Next, we will briefly discuss the existence of solutions of (15) in the *concave-convex* case, namely when $1 < q < 2 < \alpha < 2^*$. The main tool used to face problem (15) is critical point theory jointly with bifurcation results. Following [35], we will consider two cases:

(I) $h \geq 0$, $h \in L^s(\mathbb{R}^N)$, for all $s \in [1,\infty]$, and $k > 0$, $k \in L^t(\mathbb{R}^N)$, $t = \frac{2N}{2N-\alpha(N-2)}$.

(II) $h \geq 0$, $h \in L^s(\mathbb{R}^N)$, for all $s \in (1,\infty]$, $k \equiv 1$.

In the case (I), the behaviour is very similar to that found dealing with the Dirichlet problem in a bounded domain. In fact, the hypothesis on k permits to prove a global Palais-Smale condition and the classical variational methods can be applied.

One can also find a bifurcation result. Problem (15) is approximated with problems on balls with radious $R \to \infty$. Then a limiting procedure like the one emploied in Theorem 3.10 allows us to show:

Theorem 4.7 [35]. *Assume* h, k *satisfy* (I) *and let* $2 < \alpha < 2^*$. *Then there exists a nonempty closed connected set* $\Sigma \subset H^1(\mathbb{R}^n)$ *of positive solutions of* (15) *branching-off from* $(0,0)$*. Moreover*

1. *There exists a constant* $c^* > 0$ *such that if* $\overline{u} \in H^1(\mathbb{R}^n)$ *is a positive solution of* (15) *with* $\lambda = 0$*, then* $||\overline{u}||_{H^1} \geq c^*$*.* ($\Sigma$ *does not collapse into* $H^1 \times \{\lambda = 0\}$.)
2. *If* $(u_0, \lambda_0) \in \Sigma$*, and* $\lambda_0 > 0$*, then* $u_0 > 0$*.* (Σ *does not collapse into* $\{0\} \times (0, \infty)$.)
3. *There exists* $\Lambda > 0$ *such that* $\Sigma \subset H^1(\mathbb{R}^n) \times [0, \Lambda]$*.*

The abstract topological argument that allows us to pass to the limit on the set of approximate branches, requires some compactness. The idea is that if we take a bounded sequence $\{(u_j, \lambda_j)\}$ of solutions of the approximate problems we have in fact a Palais-Smale sequence for the energy functional corresponding to problem (15).

Dealing with case (II), the main difference is that the Palais-Smale condition holds only locally and for small values of λ, depending on a convenient norm of h. As a consequence, we can obtain a limit of branches in $H^1(\mathbb{R}^n)$, as in case (I), only for small λ. More precisely we obtain a continuum of solutions $\Sigma_e \subset H^1(\mathbb{R}^n) \times [0, \lambda_0)$. Here λ_0 has to be small to have the Palais-Smale condition.

Furthermore, by using uniform L^∞ estimates derived from [40] we have the following result that gives rise to a *global* branch in $C^{1,\alpha}$.

THEOREM 4.8 [35]. *Assume* h, k *satisfy* (II) *and let* $2 < \alpha < 2^*$*. Then, there exists* $\Lambda > 0$ *and a branch* $\Sigma \subset C^{1,\alpha} \times [0, \Lambda]$ *of positive solutions of* (15) *bifurcating from* $(0, 0)$*.*

The branch Σ is bounded in $C^{1,\alpha} \times [0, \Lambda]$, is nondegenerate in the sense of Theorem 4.7-1) and can be seen as an extension of Σ_e in the space $C^{1,\alpha}$. Unfortunately, we have no estimates to conclude that Σ is globally contained in $H^1(\mathbb{R}^n)$. As anticipated before, the difficulty to get these estimates arises because the Palais-Smale condition is obtained only under a critical level of the energy, and for λ small. *A priori*, we do not know if, branching off from $(0, 0)$, the solutions on Σ blow up in the energy norm at some value of $\lambda \in [0, \Lambda]$. However, the behaviour of Σ in $C^{1,\alpha} \times [0, \Lambda]$ is similar to the one indicated in Figure 3. Let us point out that the crossing of Σ with $\lambda = 0$ could be a function with finite energy, namely in $H^1(\mathbb{R}^n)$. This would be consistent with the perturbation method we will discuss in Section 6 later on. But, again, we cannot exclude that the crossing point is a solution of (15) which has not finite energy, such as $v \equiv 1$.

What is the precise behaviour of Σ is an open problem.

5 – Problems on $\mathbb{R}^n$: the case of critical exponent

In this section we deal with equations on $\mathbb{R}^n$ involving a nonlinearity like $u^{\alpha-1}$ with $\alpha = 2^*$. According to Remark 4.5, here the natural function space is $\mathcal{D}^{1,2}(\mathbb{R}^n)$. Our model problem will be

$$-\Delta u = \lambda h(x) u^{q-1} + u^{\frac{n+2}{n-2}}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \tag{23}$$

with $1 < q < 2^*$. Non-negative solutions of (23) can be found as critical points of the functional $f_\lambda : \mathcal{D}^{1,2}(\mathbb{R}^n) \longrightarrow \mathbb{R}$,

$$f_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 - \frac{\lambda}{q} \int_{\mathbb{R}^n} h(x) u_+^q - \frac{1}{2^*} \int_{\mathbb{R}^n} u_+^{2^*}.$$

Above, it is understood that h is a function such that the preceding functional makes sense. Specific assumptions will be made later on.

Likewise for the problems discussed in the preceding section, also dealing with (23) one of the main difficulty is the concerned with the (PS) condition. Let us recall that now, dealing with these *critical* nonlinearities, the analogous of S_α defined in the preceding section is the Sobolev constant

$$S = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^2 dx : u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \int_{\mathbb{R}^n} u^{2^*} dx = 1 \right\}.$$

Roughly, the Concentration Compactness Principle implies that f_λ satisfies (PS) at any level $c < \frac{1}{n} S^{n/2}$.

The first case we shall consider is the problem

$$\begin{cases} -\Delta u + a(x) u = u^{\frac{n+2}{n-2}}, \\ u > 0, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n). \end{cases} \tag{24}$$

Assume that

(a_0) $a(x) \geq 0$, and $a(x) \geq \nu > 0$ in some ball centered in a point $\bar{x} \in \mathbb{R}^N$.

(a_1) $a \in L^s$ for all $s \in (n/2-\varepsilon, n/2+\varepsilon)$, $\varepsilon > 0$ if $n > 3$, $s \in (n/2-\varepsilon, 3)$ if $n = 3$.

THEOREM 5.1 [21]. *If* (a_0) *and* (a_1) *hold then* (24) *has at least a solution provided* $\|a\|_{L^{n/2}}$ *is sufficiently small.*

The method of proof is still based on an application of the Mountain-Pass theorem jointly with the *concentration compactness principle.* Theorem 5.1 is completed by the following

REMARK 5.2. Let (a_0) hold and suppose that a does not satisfy (a_1) but merely $a \in L^{n/2}$. Then the miniminization problem

$$\inf\{\int_{\mathbb{R}^n}(|\nabla u|^2 + a(x)u^2)dx : u \in \mathcal{D}^{1,2}(\mathbb{R}^n)| \int_{\mathbb{R}^n} |u|^{\frac{n+2}{n-2}}dx = 1\},$$

has no solution. ∎

For another result dealing with (24), see Theorem 6.5 below.

We now consider (23) whith $1 < q < 2$ and h satisfying

(h_0) $h \in L^t(\mathbb{R}^N)$, with $t = 2n/(n+2-(q-1)(n-2))$.

(h_1) $h_+ \not\equiv 0$. (h_+ denotes the positive part of h.)

First of all, one shows the following geometric result.

LEMMA 5.3. *If* $1 < q < 2$, *there exist* $\rho > 0$, *and* $\varepsilon_0 > 0$ *such that*

$$-\infty < \inf\{f_\lambda(u) : u \in \mathcal{D}^{1,2}(\mathbb{R}^n),\ \|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)} \leq \rho\} < 0.$$

Concerning the (PS) condition one proves:

LEMMA 5.4. *Let* $\{u_m\} \subset \mathcal{D}^{1,2}(\mathbb{R}^n)$ *and* $c \in \mathbb{R}$ *be such that:*

1. $f_\lambda(u_m) \to c < 0$,
2. $\nabla f_\lambda(u_m) \to 0$.

Then $\{u_m\}$ *has a strongly convergent subsequence.*

An application of Lemmas 5.3 and 5.4 implies that f_λ achieves its infimum on the ball $\|u\|_{\mathcal{D}^{1,2}} \leq \rho$ yielding

THEOREM 5.5 [12]. *Let $1 < q < 2$ and suppose that (h_0, h_1) hold. Then there exists $\varepsilon_0 > 0$ such that for all $\lambda \in (0, \varepsilon_0)$ problem (23) has a non-negative solution $u_{0,\lambda}$ such that $u_{0,\lambda} \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ as $\lambda \to 0^+$. In addition, $u_{0,\lambda}$ is a local minimizer of f_λ.*

Theorem 5.5 can be completed by showing that (23) possesses a second non-negative solution $\widetilde{u}_\lambda$. Following the arguments of [8], one considers the translated functional

$$\widehat{f}_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 dx - \int_{\mathbb{R}^n} H(\lambda, x, v)dx, \tag{25}$$

where

$$H(\lambda, x, v) = \int_0^v g(\lambda, x, s)ds,$$

$$g(\lambda, x, s) = \begin{cases} \lambda h(x)\left((u_{0,\lambda} + s)^q - u_{0,\lambda}^q\right) + \left((u_{0,\lambda} + s)^\alpha - u_{0,\lambda}^\alpha\right), & s \geq 0 \\ 0, \quad s < 0 \end{cases}$$

Obviously, $\widehat{f}_\lambda$ has a local minimum at $v = 0$. Without loss of generality we can assume it is a strict local minimum. To find a Mountain-Pass critical point we have to investigate the (PS) condition. As anticipated before, one shows that

$$\widehat{f}_\lambda \text{ satisfies the (PS) condition at any level } c < \frac{1}{n}S^{n/2}.$$

Finally, one checks that, for $\lambda > 0$ small, the Mountain-Pass level $c(\widehat{f}_\lambda)$ of $\widehat{f}_\lambda$ is indeed smaller than $\frac{1}{n}S^{\frac{n}{2}}$. Therefore $c(\widehat{f}_\lambda)$ carries a second critical point of $\widehat{f}_\lambda$ and thus (23) has a second solution $\widetilde{u}_\lambda$.

For completeness, it is worth stating the following result.

THEOREM 5.6. *There exists $\bar{\lambda} > 0$ such that if u_λ is a nonnegative solution to problem (23) for $\lambda > \bar{\lambda}$, then $u_\lambda \equiv 0$.*

The proof is similar to the one in the bounded domain case. See [9] and [35].

We end this section with a result dealing with (23) when $2 < q < 2^*$. In such a case, the geometry of the functional f_λ is different from the previous one because now $0 \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ is a local minimum. It is possible to show that, if $\lambda > 0$ is sufficiently small, the Mountain-Pass critical level $c_\lambda(f_\lambda)$ is smaller than $\frac{1}{n} S^{\frac{n}{2}}$, where (PS) holds. This yields

THEOREM 5.7 [12]. *Assume that* $2 < q < 2^*$, (h_0), (h_1) *hold and let* $h > 0$. *Then there exists* $\varepsilon_0 > 0$ *such that if* $\lambda < \varepsilon_0$ *problem* (23) *has at least a positive solution.*

6 – Existence via perturbation

In this final section we will seek solutions of (23) by using a perturbation method in critical point theory discussed in [6], [7]. This approach has been followed in [11], [12] and gives rise to somewhat different results than those outlined in the previous section. To keep our notations as close as possible to that of [11], [12], we shall consider the problem (23) with $\lambda = \varepsilon$. Let us write the corresponding Euler functional f_ε in the form

$$f_\varepsilon(u) = f_0(u) + \varepsilon G(u)$$

where

$$f_0(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^n} u^{2^*} dx, \quad G(u) = \frac{1}{q} \int_{\mathbb{R}^n} h(x) u_+^q dx.$$

The unperturbed functional f_0 has an $n+1$ dimensional manifold of critical points, corresponding to the solutions of

$$\begin{cases} -\Delta u = u^{\frac{n+2}{n-2}}, \\ u > 0, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \end{cases} \tag{26}$$

Letting

$$z_0(x) = \left(\frac{1}{1+|x|^2} \right)^{\frac{(n-2)}{2}},$$

it is known that all the solutions of (26) have the form

$$z_{\mu,\xi}(x) = \mu^{-(n-2)/2} z_0 \left(\frac{x-\xi}{\mu} \right) \qquad \mu > 0,\ \xi \in \mathbb{R}^n.$$

Hence every element of $Z = \{z_{\mu,\xi} : \mu > 0,\ \xi \in \mathbb{R}^n\} \simeq \mathbb{R}^{n+1}$ is a critical point of f_0. Such a Z will be called *critical manifold* of f_0. In a certain sense, it is just this critical manifold that causes the lack of (PS).

We will now outline a procedure that allows us to find the points of Z from which critical points of f_ε, namely solutions of (23) branch off. First of all, it is worth pointing out that Z is *non-degenerate* in the following sense:

$$KerD^2 f_0(z) = T_z Z, \quad \forall\, z \in Z. \tag{27}$$

Above $T_z Z$ denotes the tangent manifold to Z at z.

Using this non degeneracy, one proves

LEMMA 6.1. *There exist $\varepsilon_0 > 0$ and a smooth function*

$$w = w(\mu, \xi, \varepsilon) : Z \times (-\varepsilon_0, \varepsilon_0) \to \mathcal{D}^{1,2}(\mathbb{R}^n)$$

such that

(a) $w(\mu, \xi, 0) = 0$
(a) $\nabla f_\varepsilon(z_{\mu,\xi} + w) \in T_{z_{\mu,\xi}} Z$.

Consider the finite dimensional functional $\Phi_\varepsilon : (0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$,

$$\Phi_\varepsilon(\mu, \xi) = f_\varepsilon(z_{\mu,\xi} + w(\mu, \xi, \varepsilon))$$

LEMMA 6.2. *If $(\mu^*, \xi^*) \in (0, +\infty) \times \mathbb{R}^n$ is a critical point of Φ_ε, then $z_{\mu^*,\xi^*} + w(\mu^*, \xi^*, \varepsilon)$ is a critical point of f_ε.*

Lemmas 6.1 and 6.2 are nothing but a kind of finite dimensional reduction like the Lyapunov-Schmidt one.

In order to find critical points of Φ_ε it is convenient to expand Φ_ε with respect to ε. One finds:

$$\Phi_\varepsilon(z) = b + \varepsilon G(z) + o(|\varepsilon|), \quad b := f_0(z).$$

In the specific situation of (23), there results

$$\Gamma(\mu, \xi) := G(z_{\mu,\xi}) = \frac{1}{q} \int_{\mathbb{R}^n} h(x) z_{\mu,\xi}^q dx.$$

Setting $\theta = q \cdot (n-2)/2$ (remark that $\theta < n$), one gets

$$\Gamma(\mu, \xi) = \frac{\mu^{n-\theta}}{q} \int_{\mathbb{R}^n} h(\mu y + \xi) z_0^q(y) dy. \tag{28}$$

The reader will recognize that Γ is the analogous of the functional that in Dynamical System theory is called Poincaré functional, i.e. the primitive of the Melnikov function.

As a consequence of the preceding arguments we infer

Lemma 6.3. *If $(\mu^*, \xi^*) \in (0, +\infty) \times \mathbb{R}^n$ is a local strict minimum or maximum of Γ, then for $|\varepsilon|$ small, $u_\varepsilon = z_{\mu^*,\xi^*} + w(\mu^*, \xi^*, \varepsilon)$ is a solution of (23).*

To find existence results, we shall distinguish whether $q > 2$ or not.

Theorem 6.4 [11]. *Let $2 < q < 2^*$ and suppose that $h \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $h \not\equiv 0$. Then for $|\varepsilon|$ small, equation (23) has a positive solution.*

Theorem 6.5 [11]. *Let $q = 2$ and suppose h is continuous and such that*

$$\omega := \text{ supp } (h) \text{ is compact.} \tag{h_2}$$

Moreover, we assume that either $\int_{\mathbb{R}^n} h \neq 0$ or $h \not\equiv 0$ and $n > 4$. Then for $|\varepsilon|$ small, equation (23) has a positive solution. Moreover, if $n > 4$ and h changes sign, then (23) has at least two positive solutions.

THEOREM 6.6 [12]. *Let* $1 < q < 2$ *and suppose* $h \not\equiv 0$ *is a continuous function such that* (h_2) *holds.*

Then there exists $\varepsilon_1 > 0$, $\mu_1 > 0$ *and* $\xi_1 \in \mathbb{R}^N$ *such that for all* $|\varepsilon| < \varepsilon_1$ *problem* (23) *has a positive solution* $u_{1,\varepsilon}$ *with* $u_{1,\varepsilon} \to z_{\mu_1,\xi_1}$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^n)$ *as* $\varepsilon \to 0$.

Furthermore, if h *changes sign then* (23) *has at least two positive solutions. Precisely, there exist* $\mu_2 > 0$, $\xi_2 \in \mathbb{R}^N$ *with* $(\mu_1, \xi_1) \neq (\mu_2, \xi_2)$ *and a second positive solution* $u_{2,\varepsilon}$ *of* (P_ε) *such that* $u_{2,\varepsilon} \to z_{\mu_2,\xi_2}$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^n)$ *as* $\varepsilon \to 0$.

Since $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$ are close to Z while the solution found in Theorem 5.5 bifurcates from $u = 0$, we can use Theorem 5.5 jointly with Theorem 6.6 to infer

COROLLARY 6.7. *If* h *changes sign and satisfies both the assumptions of Theorems* 5.5 *and* 6.5, *then* (23) *has at least three different non-negative solutions.*

The proof of Theorems 6.4 and 6.5 relies mainly on the following facts:

(Γ_1) $\lim_{\mu \to 0^+} \Gamma(\mu, \xi) = 0$;
(Γ_2) $\lim_{\mu + |\xi| \to \infty} \Gamma(\mu, \xi) = 0$;
(Γ_3) $\Gamma \not\equiv 0$.

For example, if $n > 4$ and $n/(n-2) < q < 2$, property (Γ_3) can be shown in the following way. Actually $z_0 \in L^q(\mathbb{R}^n)$ and one uses (28) to yield

$$\lim_{\mu \to 0^+} \frac{\Gamma(\mu, \xi)}{\mu^{n-\theta}} = \frac{h(\xi)}{q} \int_{\mathbb{R}^n} z_0^q .$$

Let us point out that when $1 < q < 2$ the functional f_ε is no more of class C^2 and the abstract setting needs to be modified. We refer to [12] for details.

REMARK 6.8. In general, critical points of f_ε are non-negative solutions of (23). To show that they are indeed *positive* solutions, one uses the strong maximum principle provided $q > 2$. To prove the positivity when $1 < q \leq 2$ we assume (h_2) and use the fact that the solutions are close to some $z \in Z$. The assumption (h_2) in the case $q = 2$ has been eliminated in the recent paper [29]. Of course, if $h > 0$, as in Theorem 5.5, we can directly conclude that the solution is positive. $\square$

As a concluding remark we recall that in some recent papers [11], [13], [42] the abstract approach has been used to face problems like

$$-\Delta u = (1 + \varepsilon K(x))u^{\frac{n+2}{n-2}}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n),$$

arising in Differential Geometry (e.g. the Scalar Curvature problem and the Yamabe problem). We will not discuss this kind of results here.

REFERENCES

[1] S. Alama – G. Tarantello: *On semilinear elliptic equations with indefinite nonlinearities*, Calc. Var. **1** (1993), 439-475.

[2] H. Amann: *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review, Vol. 18 No 4 (1976), 620-709.

[3] A. Ambrosetti: *Critical points and nonlinear variational problems*, Memoire No 49, Soc. Math. de France Tome 120, Fascicule 2, 1992.

[4] A. Ambrosetti: *Global branches of positive solutions of nonlinear eigenvalue problems*, in: *Progress in elliptic and parabolic PDE*, A. Alvino *et al.* (ed.), Pitman Res. Notes in Math. n. 350, Longman, 1996, 1-8.

[5] A. Ambrosetti – M. Badiale: *The dual variational principle and elliptic problems with discontinuous nonlinearities*, Jour. Math. Anal. Appl., 140-2 (1989), 363-373.

[6] A. Ambrosetti – M. Badiale: *Homoclinics: Poincaré-Melnikov type result via a variational approach*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **15** (1998) 233-252.

[7] A. Ambrosetti – M. Badiale: *Variational perturbative methods and bifurcation of bound states from the essential spectrum*, Proc. Royal Soc. Edinburgh, **128** A(1998), 1131-1161.

[8] A. Ambrosetti – H. Brezis – G. Cerami: *Combined Effects of Concave and Convex Nonlinearities in some Elliptic Problems*, Jour. Funct. Anal., **122** (1994), 519-543.

[9] A. Ambrosetti – J. García Azorero – I. Peral: *Multiplicity of solutions for semilinear and quasilinear elliptic problems*, Jour. Funct. Anal., **137** (1996), 219-242.

[10] A. Ambrosetti – J. García Azorero – I. Peral: *Quasilinear Equations with a Multiple Bifurcation*, Diff. and Integral Equations, **10-1** (1997), 37-50.

[11] A. Ambrosetti – J. García Azorero – I. Peral: *Perturbation of* $\Delta u + u^{(N+2)/(N-2)} = 0$, *the scalar curvature problem in* $\mathbb{R}^N$ *and related topics*, Jour. Funct. Anal., **165** (1999), 117-149.

[12] A. Ambrosetti – J. García Azorero – I. Peral: *Elliptic variational problems in* $\mathbb{R}^N$ *with critical growth*, to appear in Jour. Diff. Equations.

[13] A. Ambrosetti – A. Malchiodi: *A multiplicity result for the Yamabe problem on* S^n, Jour. Funct. Anal., **168** (1999), 529-561.

[14] A. Ambrosetti – P.H. Rabinowitz: *Dual variational methods in critical point theory and applications*, Jour. Funct. Anal., **14** (1973), 349-381.

[15] G. Arioli – F. Gazzola: *Some results on p-Laplace equations with a critical growth term*, Research Report in Mathematics No 14, 1996. Dept. Math. Stockholm University.

[16] M. Badiale – F. Dobarro: *Some existence results for sublinear elliptic problems in* $\mathbb{R}^N$, Funkc. Ekvacioj, **39** (1996), 183-202.

[17] A. Bahri – Y.Y. Li: *On a Min-Max Procedure for the Existence of a Positive Solution for Certain Scalar Fields Equations in* $\mathbb{R}^N$, Revista Matemática Iberoamericana, 6-1, 2 (1990) 1-16.

[18] T. Bartsch – M. Willem: *Infinitely many nonradial solutions of Euclidean scalar field equation*, Jour. Funct. Anal., **117** (1993), 447-460.

[19] T. Bartsch – M. Willem: *On an elliptic equation with concave and convex nonlinearities*, Proc. Amer. Math. Soc., **123** (1995), 3555-3561.

[20] T. Bartsch – Z. Q. Wang: *On the existence of sign changing solutions for semilinear Dirichlet problems*, Topol. Methods Nonlinear Anal., 7, 1 (1996), 115-131.

[21] V. Benci – G. Cerami: *Existence of positive solutions of the equation* $-\Delta u + a(x)u = u^{(N+2)/(N-2)}$ *in* $\mathbb{R}^N$, Jour. Funct. Anal., **88** (1990), 91-117.

[22] H. Berestycki – P. L. Lions: *Nonlinear scalar field equations*, Arch. Rat. Mech. Anal., **82** (1983), 313-376.

[23] L. Boccardo – M. Escobedo – I. Peral: *A Dirichlet problem involving critical exponent*, Nonlinear Analysis T.M.A., **24** (11) (1995), 1639-1648.

[24] H. Brezis – S. Kamin: *Sublinear elliptic equations in* $\mathbb{R}^N$, Manus. Math., **74** (1992), 87-106.

[25] H. Brezis – L. Nirenberg: *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., **36** (1983), 437-477.

[26] H. Brezis – L. Nirenberg: H^1 *versus* C^1 *local minimizers*, C.R.A.S. Paris, t. 317 (1993), 465-472.

[27] H. Brezis – L. Oswald: *Remarks on sublinear elliptic equations*, Nonlinear Analysis T.M.A., 10-1 (1986), 55-64.

[28] X. Cabre – P. Majer: *Truncation of nonlinearities in some supercritical elliptic problems*, C.R.A.S. Paris Serie I, t. 322 No 12 (1996), 1157-1162.

[29] S. Cingolani: *Positive solution to perturbed elliptic problems in* $\mathbb{R}^N$ *involving critical Sobolev exponent*, Preprint University of Bari 1999.

[30] M. Del Pino – R. Manasevich: *Global Bifurcation from the Eigenvalues of the p-Laplacian*, Jour. Diff. Equations, **92** (1991), 226-251.

[31] E. Di Benedetto: $C^{1,\alpha}$ *local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Analysis. T.M.A., 7-8 (1983), 827-850.

[32] H. Egnell: *Elliptic boundary problems with singular coefficients and critical nonlinearities*, Indiana Univ. Math. Jour., 38-2 (1989), 235-251.

[33] B. Franchi – E. Lanconelli – J. Serrin: *Existence and uniqueness of nonnegative solutions of quasilinear equations in* $\mathbb{R}^N$, Advances in Math., **118** (1996) 177-243.

[34] J. Garcia Azorero – J. Manfredi – I. Peral: *Sobolev versus Hölder Local Minimizers and Global Multiplicity for some Quasilinear Elliptic Equations*, to appear in Communications in Contemporary Mathematics.

[35] J. Garcia Azorero – E. Montefusco – I. Peral: *Bifurcation for the p-Laplacian in* $\mathbb{R}^N$, to appear in Advances in Diff. Equat. (1999)

[36] J. Garcia Azorero – I. Peral Alonso: *Existence and nonuniqueness for the p-Laplacian: Nonlinear Eigenvalues*, Commun. in P.D.E., **12** (1987), 1389-1430.

[37] J. Garcia Azorero – I. Peral Alonso: *Comportement asymptotique des valeurs propres du p-laplacien*, C.R.A.S. Paris Paris Série I, t. 307 (1988), 75-78.

[38] J. Garcia Azorero – I. Peral Alonso: *Multiplicity of solutions for elliptic problems with critical exponent or with a non-symmetric term*, Trans. Amer. Math. Soc., **323** (1991), 877-895.

[39] J. Garcia Azorero – I. Peral Alonso: *Some results about the Existence of a Second Positive Solution in a Quasilinear Critical Problem*, Indiana Univ. Math. Jour., 43-3 (1994), 941-957.

[40] B. Gidas – J. Spruck: *Global and Local Behavior of Positive Solutions of Nonlinear Elliptic Equations*, Commun. Pure Appl. Math., **34** (1981), 525-598.

[41] M. Guedda – L. Veron: *Bifurcation phenomena associated to the p-Laplace operator*, Trans. Amer. Math. Soc., **310** (1988), 419-431.

[42] A. Malchiodi: *The scalar curvature problem on* S^n*: an approach via Morse theory*, preprint S.I.S.S.A. - Trieste (July 1999).

[43] P. L. Lions: *The concentration-compactness principle in the calculus of variations. The limit case, part 1*, Rev. Matemática Iberoamericana, Vol. 1, No. 1 (1985), 145-201.

[44] P. L. Lions: *The concentration-compactness principle in the calculus of variations. The limit case, part 2*, Rev. Matemática Iberoamericana, Vol. 1, No. 2 (1985), 45-121.

[45] S. Pohozaev: *On the Eigenfunctions of Quasilinear Elliptic Problems*, Mat. Sbornik, Vol 82 (124) No. 2 (1970), 171-188.

[46] W. A. Strauss: *Existence of solitary waves in higher dimensions*, Comm. Math. Phys., **55** (1977), 149-162.

[47] M. Struwe: *Variational Methods*, Ergeb. der Math. u. Grenzgeb., Vol. 34, Springer V., 1996.

[48] Z. Q. Wang: *On a superlinear elliptic equation*, Ann. Inst. H. Poincaré - Anal. Non Linéaire, **8** (1991), 43-57.

[49] Z. Q. Wang: *Nonlinear boundary value problems with concave nonlinearities near the origin*, Nonlin. Diff. Equat. Appl., to appear.

[50] G. T. Whyburn: *Topological Analysis*, Princeton University Press, Princeton, New Jersey, 1964.

[51] M. Willem: *Minimax Theorems*, Progr. Nonlin. Diff. Equat. Appl., Vol. 24, Birkhäuser, 1996.

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