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Sturm-Liouville problems with coupled boundary conditions and Lagrange interpolation series: II

W.N. EVERITT – G. NASRI-ROUDSARI

Dedicated to the achievements and memory of Professor Gaetano Fichera

RIASSUNTO: *Il lavoro tratta l'applicazione del teorema di Kramer sul campionamento al problema di Sturm-Liouville con condizioni al contorno accoppiate. Facemdo seguito alla parte I, questa parte II tratta il caso in cui lo spettro del problema autoaggiunto associato contenga qualche autovalore doppio. Si riconosce che può essere definito un nucleo analitico di Kramer e che ad ogni nucleo analitico può essere associata una funzione analitica di interpolazione per costruire la serie di interpolazione di Lagrange.*

ABSTRACT: *This paper is concerned with the application of the Kramer sampling theorem to Sturm-Liouville problems with coupled boundary conditions. In following the first paper with this title the analysis in this successor paper covers the case when the spectrum of the boundary value problem is allowed to be double, i.e. at least one eigenvalue is of multiplicity two. In all cases it is shown that Kramer analytic kernels can be defined, and that each kernel has an associated analytic interpolation function to give the Lagrange interpolation series.*

KEY WORDS AND PHRASES: *Sturm-Liouville – Coupled boundary conditions – Lagrange interpolation.*
A.M.S. CLASSIFICATION: 34B24 – 41A05 (primary) – 34L05 – 42A15 (secondary)

1 – Introduction

This paper is a continuation of the work of Everitt, Schöttler[1] and Butzer [7], and Everitt and Nasri-Roudsari [4]. The project as a whole is to study the form of the now classical Lagrange interpolation series generated by regular and singular Sturm-Liouville boundary value problems with symmetric (self-adjoint) boundary conditions. For the cases when the interval end-points are given as either regular or limit-circle the problems can be classified as follows (in all these cases it is known that the spectrum of the generated self-adjoint boundary value problem is discrete, i.e. consists only of eigenvalues):

1. The two boundary conditions are **separated**; in this case the spectrum of the generated self-adjoint operator is simple; such problems are considered in [7].
2. The two boundary conditions are **coupled**; in this case the spectral multiplicity of the self-adjoint operator may be one, i.e. all eigenvalues are simple, or may be two, i.e. at least one eigenvalue has geometric multiplicity two:
   (i) for the first of these two spectral cases the Lagrange interpolation analysis is given in [4],
   (ii) this paper is concerned with the Lagrange interpolation analysis for the second spectral case when there is at least one double eigenvalue.

Notation 1.1. Let $\mathbb{N} = \{1, 2, \cdots\}, \mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$, and let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex number fields.

For any open set $U \subseteq \mathbb{C}$ let $\mathbf{H}(U)$ denote the class of Cauchy analytic functions that are holomorphic (analytic, regular) on U; thus $\mathbf{H}(\mathbb{C})$ denotes the class of all entire functions on $\mathbb{C}$. A property is said to hold "locally" on U if it is satisfied on all compact subsets of U.

Notation 1.2. Let L denote Lebesgue integration. Let $I = (a, b)$ be an arbitrary open interval of $\mathbb{R}$; a property is said to hold "locally" on I if it holds on all compact subintervals of I, e.g. $L^1_{\text{loc}}(a, b)$.

The function w is said to be a weight function on I if $w : I \to \mathbb{R}$, w is Lebesgue measurable on I, and $w(x) > 0$ for almost all $x \in I$. If w

[1] Now named G. Nasri-Roudsari

is a weight function on I, then $L^2(I;w)$ denotes the class of all complex-valued, Lebesgue measurable functions $f : I \to \mathbb{C}$ such that

$$\int_I w|f|^2 \equiv \int_a^b w(x)|f(x)|^2\,dx < +\infty\,. \tag{1.1}$$

With the standard interpretation of vectors as equivalence classes, $L^2(I;w)$ is a Hilbert function space with norm and inner-product

$$\|f\|_w^2 := \int_I w|f|^2 \quad\text{and}\quad (f,g)_w := \int_a^b w(x)f(x)\overline{g}(x)\,dx\,. \tag{1.2}$$

NOTATION 1.3. The notation "$(x \in K)$" is to be read as "for all $x \in K$".

REMARK 1.1. The results in this paper follow closely on the earlier work of the authors in [4]. There are related results in the paper [1]; however this paper of Annaby and Hassan works only with regular boundary value problems and employs quite different analytical methods.

2 – Kramer analytic kernels

In [4, Section 1] there is an account of the earlier contributions to the connections between the original Kramer sampling theorem, Sturm-Liouville problems and Lagrange interpolation results.

Here we restrict ourselves to a statement of the analytic form of the Kramer theorem:

THEOREM 2.1. *Let $I = (a,b)$ be an arbitrary open interval of $\mathbb{R}$ and let w be a weight function on I. Let the mapping $K : I \times \mathbb{C} \to \mathbb{C}$ satisfy the following properties:*

1. $K(\cdot,\lambda) \in L^2(I;w) \quad (\lambda \in \mathbb{C})$
2. $K(x,\cdot) \in \mathbf{H}(\mathbb{C}) \quad (x \in (a,b))$
3. *There exists a sequence* $\{\lambda_n : n \in \mathbb{Z}\}$ *satisfying*
   (i) $\lambda_n \in \mathbb{R} \quad (n \in \mathbb{Z})$
   (ii) $\lambda_n < \lambda_{n+1} \quad (n \in \mathbb{Z})$
   (iii) $\lim_{n\to\pm\infty} \lambda_n = \pm\infty$

(iv) *the sequence of functions* $\{K(\cdot,\lambda_n) : n \in \mathbb{Z}\}$ *forms a locally linearly independent, and a complete orthogonal set in the Hilbert space* $L^2(I;w)$

(v) *the mapping*

$$\lambda \longmapsto \int_a^b w(x)|K(x,\lambda)|^2 dx \tag{2.1}$$

is locally bounded on $\mathbb{C}$.

Define the set of functions $\{K\}$ *as the collection of all functions* $F : L^2(I;w) \times \mathbb{C} \to \mathbb{C}$ *determined by, for* $f \in L^2(I;w)$,

$$F(f;\lambda) \equiv F(\lambda) := \int_a^b w(x)K(x,\lambda)f(x)\,dx \quad (\lambda \in \mathbb{C})\,. \tag{2.2}$$

Then for all $F \in \{K\}$ *the following results hold:*

(a) $F(f,\cdot) \in \mathbf{H}(\mathbb{C}) \quad (f \in L^2(I;w))$.

(b) *If* $S_n : \mathbb{C} \to \mathbb{C}$ *is defined by, for all* $n \in \mathbb{Z}$,

$$S_n(\lambda) := \|K(\cdot,\lambda_n)\|_w^{-2} \int_a^b w(x)K(x,\lambda)\overline{K}(x,\lambda_n)\,dx \quad (\lambda \in \mathbb{C}) \tag{2.3}$$

then $S_n \in \mathbf{H}(\mathbb{C})$ $(n \in \mathbb{Z})$.

(c) *For all* $F \in \{K\}$

$$F(f,\lambda) \equiv F(\lambda) = \sum_{n\,\in\,\mathbb{Z}} F(\lambda_n)S_n(\lambda) \tag{2.4}$$

where the infinite series is

$$\begin{cases} \text{(i)} & \textit{absolutely convergent for each } \lambda \in \mathbb{C} \\ \text{(ii)} & \textit{locally uniformly convergent on } \mathbb{C}. \end{cases} \tag{2.5}$$

(d) *If* $\{c_n : n \in \mathbb{Z}\}$ *is a sequence of complex numbers that satisfies the condition*

$$\sum_{n\,\in\,\mathbb{Z}} \frac{|c_n|^2}{\|K(\cdot,\lambda_n)\|_w^2} < +\infty \tag{2.6}$$

then there exists a unique $F \in \{K\}$, *equivalently a unique* $f \in L^2(I; w)$, *such that*

$$F(f, \lambda_n) = c_n \quad (n \in \mathbb{Z}) .$$

PROOF. See [6]; the ideas for these results stem from [7, Theorem 1.1]. See also the remarks made after [4, Theorem 1]. □

COROLLARY 2.1. *Let* $\mathbb{Z}_1$ *be a nonempty, strict subset of the integer set* $\mathbb{Z}$; *let* $L_1^2(I; w)$ *denote the Hilbert function space spanned by the set of orthogonal functions* $\{K(\cdot, \lambda_n) : n \in \mathbb{Z}_1\}$; *let* $\{K\}_1$ *be the function set defined by* $F(f_1; \cdot) \in \{K\}_1$ *if*

$$F_1(f_1; \lambda) \equiv F_1(\lambda) := \int_a^b w(x)K(x, \lambda)f_1(x)\, dx \quad (\lambda \in \mathbb{C}) \tag{2.7}$$

for all $f_1 \in L_1^2(I; w)$.

Then for all $F_1 \in \{K\}_1$

$$F_1(\lambda) = \sum_{n \in \mathbb{Z}_1} F_1(\lambda_n) S_n(\lambda) \quad (\lambda \in \mathbb{C}) \tag{2.8}$$

under the same conditions of convergence as given by (i) *and* (ii) *of* (2.5).

PROOF. The same proof as for Theorem 2.1 applies for this Corollary 2.1. □

REMARK 2.1. A kernel K that satisfies the conditions of Theorem 2.1 or Corollary 2.1 is said to be a Kramer analytic kernel.

DEFINITION 2.1. Let K be a Kramer analytic kernel under the conditions of, and with the properties given, in Theorem 2.1 above. Then K is said to have an analytic interpolation function if there exists $G : \mathbb{C} \to \mathbb{C}$ with the properties:

1. $G \in \mathbf{H}(\mathbb{C})$.
2. $G(\lambda) = 0$ if and only if $\lambda \in \{\lambda_n : n \in \mathbb{Z}\}$.
3. $G'(\lambda_n) \neq 0$ $(n \in \mathbb{Z})$.

4. With the function sequence $\{S_n(\cdot) : n \in \mathbb{Z}\}$ defined as in (2.3)

$$S_n(\lambda) = \frac{G(\lambda)}{G'(\lambda_n)(\lambda - \lambda_n)} \quad (\lambda \in \mathbb{C} \text{ and } n \in \mathbb{Z})\,. \tag{2.9}$$

Remark 2.2.

1. Theorem 2.1 is stated in terms of the integer set $\mathbb{Z}$, but the results are equally valid if $\mathbb{Z}$ is replaced by the integer sets $\mathbb{N}$ or $\mathbb{N}_0$, but now with only one limit point at $+\infty$.
2. Corollary 2.1 is required for the phenomenon of degeneracy introduced in [4, Section 8, Definition 3] and required also in this paper.
3. If the Kramer analytic kernel K has an analytic interpolation function G, then the sampling series (2.8) assumes the classic form of a Lagrange interpolation series

$$F(\lambda) = \sum_{n \in \mathbb{Z}} F(\lambda_n) \frac{G(\lambda)}{G'(\lambda_n)(\lambda - \lambda_n)} \tag{2.10}$$

with the same convergence properties (i) and (ii) of (2.5).
4. Note that both the function sequence $\{S_n : n \in \mathbb{Z}\}$ and the analytic interpolation function G depend on the choice of the Kramer analytic kernel K, but not on the choice of the element F in the infinite series (2.4) and (2.10).

3 – Sturm-Liouville theory

These boundary value problems concern the classic Sturm-Liouville differential equation

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \quad (x \in I = (a,b)) \tag{3.1}$$

where

$$\begin{cases} \text{(i)} & \lambda \in \mathbb{C} \text{ is the spectral parameter}\,, \\ \text{(ii)} & I = (a,b) \text{ is an open interval of } \mathbb{R} \text{ with } -\infty \le a < b \le +\infty\,, \\ \text{(iii)} & \text{the coefficients } p, q, w : (a,b) \to \mathbb{R}\,, \\ \text{(iv)} & p^{-1}, q, w \in L^1_{\text{loc}}(a,b)\,, \\ \text{(v)} & w \text{ is a weight function on } (a,b)\,. \end{cases} \tag{3.2}$$

For a discussion on the significance of these conditions see the remarks given in [4, Page 324].

A Sturm-Liouville boundary value problem involves the search for solutions of the differential equation (3.1) that satisfy certain boundary conditions involving the solution at both end-points a and b of the interval I. The number and form of these boundary conditions, in order to obtain a self-adjoint problem generating a self-adjoint differential operator in the Hilbert space $L^2(I;w)$, depends upon the classification of the end-points a and b of (3.1) within $L^2(I;w)$. To this end we impose two structural conditions involving this classification and the associated boundary conditions.

Condition 3.1. The end-point a of the differential equation (3.1) is to be regular or limit-circle in $L^2(I;w)$; independently the end-point b is to be regular or limit-circle in $L^2(I;w)$; see [7, Equation (1.5)] and [4, Page 324].

Remark 3.1. The end-point classification determined by Condition 3.1 leads to a minimal, closed, symmetric operator in $L^2(I;w)$, generated by the differential equation (3.1), with deficiency indices $d^{\pm}=2$; in turn this requires that all self-adjoint extensions A of this minimal, symmetric operator are determined by applying two linearly independent, symmetric boundary conditions; of these two conditions **either** both are separated with one applied at end-point a and with one applied at b, **or** both conditions are coupled.

Condition 3.2. The two linearly independent, symmetric boundary conditions are to be chosen to be coupled, i.e. both conditions connect the solution of (3.1) at both end-points a and b.

Remark 3.2.

1. Condition 3.1 is both necessary and sufficient for the existence and application of coupled, symmetric boundary conditions.
2. Condition 3.1 implies that the spectrum $\sigma(A)$ of the self-adjoint operator $A : D(A) \subset L^2(I;w) \to L^2(I;w)$, determined by the coupled boundary conditions of Condition 3.2, is always discrete, i.e. consists only of eigenvalues.

3. Without loss of generality we take the spectrum $\sigma(A)$ of the operator A to be of the form

$$\sigma(A)=\{\lambda_n : n \in \mathbb{Z}\} \text{ with } \lambda_n<\lambda_{n+1}(n\in\mathbb{Z}) \text{ and } \lim_{n\to\pm\infty} \lambda_n=\pm\infty\,, \tag{3.3}$$

i.e. the spectrum is unbounded both above and below. The results then presented here can be readily adapted to the case when the index set for the eigenvalues has only one limit point, either at $+\infty$ or $-\infty$.

4. The multiplicity of the spectrum $\sigma(A)$ is either 1 or 2; the first case is fully considered in the paper [7]; the second case is the subject of this present paper.
5. It is essential to distinguish between the **geometric** multiplicity and the **algebraic** multiplicity of an eigenvalue; the former is the dimension of the eigenspace of the eigenvalue; the latter, for Sturm-Liouville boundary value problems, is defined in Section 5 below.

Condition 3.3. For the self-adjoint operator A under consideration it is required that the spectral multiplicity is 2, i.e. there exists at least one eigenvalue in the spectrum $\{\lambda_n : n \in \mathbb{Z}\}$ with geometric multiplicity 2.

Remark 3.3. 1. Given, for $n \in \mathbb{Z}$, that the eigenvalue λ_n has geometric multiplicity 2, the differential equation (3.1) with $\lambda = \lambda_n$, has two linearly independent solutions both of which satisfy the coupled boundary conditions.

2. The existence of such boundary value problems with eigenvalues of geometrical multiplicity 2 is shown by the example in [4, Section 11].

4 – Boundary conditions

As in the earlier paper [4, Section 2] we make use of the canonical form of coupled boundary conditions for Sturm-Liouville problems given in [2].

Both the regular and singular cases of these boundary conditions, under Condition 3.1, can be considered together in one form. To this end let the maximal domain (linear manifold) $\Delta \subset L^2(I;w)$ of the differential equation (3.1) be defined by

$$\Delta := \Big\{ f : (a,b) \to \mathbb{C} : \text{(i)} \quad f, pf' \in AC_{\mathrm{loc}}(a,b) \qquad \text{(ii)} \quad f, w^{-1}(-(pf')' + qf) \in L^2((a,b);w) \Big\}. \tag{4.1}$$

Here AC denotes absolute continuity with respect to Lebesgue measure.

Note that under Condition 3.1 all solutions y of the differential equation (3.1) satisfy $y \in \Delta$, and this result holds for all values of the spectral parameter $\lambda \in \mathbb{C}$.

The Lagrange, skew-symmetric form $[\cdot,\cdot] : \Delta \times \Delta \times (a,b) \to \mathbb{C}$ is defined by,

$$[f,g](x) := f(x)(p\overline{g}')(x) - (pf'(x))\overline{g}(x) \quad (f, g \in \Delta \text{ and } x \in (a,b)). \tag{4.2}$$

From Green's formula it is known that

$$[f,g](a^+) := \lim_{x \to a^+} [f,g](x) \qquad [f,g](b^-) := \lim_{x \to b^-} [f,g](x) \tag{4.3}$$

both exist and are finite in $\mathbb{C}$, for all $f, g \in \Delta$.

Now define functions θ, φ with the following properties

$$\begin{aligned} &\text{(i)} \quad \theta, \varphi : (a,b) \to \mathbb{R} \\ &\text{(ii)} \quad \theta, \varphi \in \Delta \\ &\text{(iii)} \quad [\theta, \varphi](a^+) = 1 \text{ and } [\theta, \varphi](b^-) = 1 . \end{aligned} \tag{4.4}$$

It is shown in [2, Section 2] that θ, φ with these properties exist.

We quote now from [2, Section 2], recalling from above that all solutions y of (3.1) satisfy $y \in \Delta$,

Theorem 4.1. *All symmetric* (*formally self-adjoint*), *coupled boundary conditions for the Sturm-Liouville differential equation* (3.1), *under the end-point classification of Condition* 3.1 *above, can be written in the form*

$$\mathbf{Y}(b) = \exp(i\alpha)\mathbf{T}\mathbf{Y}(a) \tag{4.5}$$

where for all solutions y of the differential equation (*here the superscript* τ *denotes a matrix transpose*)

$$\mathbf{Y}(x) := [[y,\theta](x),[y,\varphi](x)]]^\tau \quad (x \in (a,b)) \tag{4.6}$$

$$\mathbf{Y}(a) := [[y,\theta](a^+),[y,\varphi](a^+)]]^\tau \quad \mathbf{Y}(b) := [[y,\theta](b^-),[y,\varphi](b^-)]]^\tau \tag{4.7}$$

and

$$\begin{cases} \text{(i)} & \textit{the parameter } \alpha \in [-\pi,\pi] \\ \text{(ii)} & \textit{the matrix } \mathbf{T} = [t_{rs}] \textit{ with } t_{rs} \in \mathbb{R} \textit{ for } r,s = 1,2 \\ \text{(iii)} & \det(\mathbf{T}) = 1. \end{cases} \tag{4.8}$$

Proof. See [2, Section 2]. ∎

Remark 4.1. 1. For additional details of these coupled boundary conditions see [4, Section 2]; in particular, for the more explicit form that the two conditions in (4.5) take when one or more of the end-points a and/or b is classified in the regular case.

2. The condition (iii) of (4.8), $\det(\mathbf{T}) = 1 \neq 0$, implies that the two boundary conditions in (4.5) are linearly independent, and that both of them are coupled across the interval (a,b).

5 – Boundary value problems

Consider now the boundary value problem determined by, under the conditions (3.2), the Conditions 3.1 and 3.2, and the conditions (4.8), the differential equation and the coupled boundary conditions

$$-(py')' + qy = \lambda wy \text{ on } (a,b) \tag{5.1}$$

$$\mathbf{Y}(b) = \exp(i\alpha)\mathbf{T}\mathbf{Y}(a)\,. \tag{5.2}$$

A solution of this problem is a pair $\{\lambda,\psi\}$ where $\lambda \in \mathbb{C}$, and ψ is a non-null solution of the equation (5.1) satisfying the boundary conditions (5.2); λ is the eigenvalue and ψ the associated eigenfunction.

To define the self-adjoint operator A in the Hilbert function space $L^2((a,b);w)$ generated by this boundary value problem, let (recall the definitions (4.1) of Δ, and (4.6) and (4.7) of the boundary conditions)

$$D(A) := \{f \in \Delta : \mathbf{F}(b) = \exp(i\alpha)\mathbf{TF}(a)\} \tag{5.3}$$

and

$$Af := w^{-1}(-(pf')' + qf) \quad (f \in D(A))\,. \tag{5.4}$$

The general theory of differential operators as given in [14, Chapter V] leads to the following for the properties of the operator A:

1. The operator A is self-adjoint and unbounded in the space $L^2((a,b);w)$.
2. The spectrum of A is discrete, say $\sigma(A) = \{\lambda_n\}$, is of multiplicity less than or equal to two, and is limiting at $+\infty$, or at $-\infty$ or at both $\pm\infty$.
3. The set of eigenfunctions (eigenvectors of A), say $\{\psi_n\}$, is, allowing for multiplicity, orthogonal and complete in $L^2((a,b);w)$.

To these general results we add one notation and one additional condition, both having been mentioned earlier in this paper; the notation requirement is given in (3) of Remark 3.2, i.e. $\sigma(A) = \{\lambda_n : n \in \mathbb{Z}\}$ with $\lim_{n\to\pm\infty} \lambda_n = \pm\infty$ (the analysis for the other two cases requires only notational changes); the condition requirement is given in Condition 3.3, i.e. the spectral multiplicity of the boundary value problem is two.

We now define a pair of linearly independent solutions $\{u,v\}$ of the differential equation (5.1) satisfying the generalized initial conditions for all $\lambda \in \mathbb{C}$, involving the pair $\{\theta,\varphi\}$ defined in (4.4),

$$[u,\theta](a^+,\lambda) = 0 \qquad [v,\theta](a^+,\lambda) = 1 \tag{5.5}$$

$$[u,\varphi](a^+,\lambda) = 1 \qquad [v,\varphi](a^+,\lambda) = 0\,. \tag{5.6}$$

The proof of the existence of this pair of solutions is given in [8, Section 5]. Note that we could have defined these initial conditions equally well at the end-point b, or indeed an interior point; the choice between a and b is arbitrary but using an interior point of the interval leads to complications in the formulae that follow below.

We can now state

LEMMA 5.1. *Let the solutions* u, v *be defined by* 5.5) *and* (5.6); *then the following properties hold:*

1. *The functions* $u(\cdot,\lambda)$, $(pu')(\cdot,\lambda)$ *and* $v(\cdot,\lambda)$, $(pv')(\cdot,\lambda)$ *all belong to* $AC_{\mathrm{loc}}(a,b)$ *for all* $\lambda \in \mathbb{C}$.
2. *The functions* $u(\cdot,\lambda)$ *and* $v(\cdot,\lambda)$ *both belong to* $\Delta \subset L^2((a,b);w)$ *for all* $\lambda \in \mathbb{C}$.
3. *The functions* $u(x,\cdot)$, $(pu')(x,\cdot)$ *and* $v(x,\cdot)$, $(pv')(x,\cdot)$ *all belong to* $\mathbf{H}(\mathbb{C})$ *for all* $x \in (a,b)$.
4. *If the Wronskian* $W : (a,b) \times \mathbb{C} \to \mathbb{C}$ *is defined by*

$$W(x,\lambda) := (upv' - pu'v)(x,\lambda) \quad (x \in (a,b) \text{ and } \lambda \in \mathbb{C}) \tag{5.7}$$

then

$$W(x,\lambda) = -1 \quad (x \in (a,b) \text{ and } \lambda \in \mathbb{C}); \tag{5.8}$$

thus the pair $\{u,v\}$ *form a basis of solutions of the differential equation* (5.1) *on* $(a,b) \times \mathbb{C}$.

5. *The functions* $[u,\theta](b^+,\cdot), [u,\varphi](b^+,\cdot)$ *and* $[v,\theta](b^+,\cdot), [v,\varphi](b^+,\cdot)$ *all belong to* $\mathbf{H}(\mathbb{C})$, *and are all locally bounded on* $\mathbb{C}$.

PROOF. For the proof of these results see [7] and [8]. The proof of (5.8) is given in [4, Section 3] and requires the use of the Plücker identity as discussed in [7, Section 4]. ∎

DEFINITION 5.1. Let the matrix $\mathbf{T}$ be given according to the boundary condition parameters (4.8); let the boundary functions θ, φ be given according to (4.4); let the basis solutions u, v be determined according to (4.5) and (4.6).

Define the function $D(\mathbf{T},\cdot) : \mathbb{C} \to \mathbb{C}$ by, for all $\lambda \in \mathbb{C}$,

$$\begin{aligned} D(\mathbf{T},\lambda) := {} & t_{11}[u(\cdot,\lambda),\varphi(\cdot)](b^-) + t_{22}[v(\cdot,\lambda),\theta(\cdot)](b^-) + \\ & - t_{12}[v(\cdot,\lambda),\varphi(\cdot)](b^-) - t_{21}[u(\cdot,\lambda),\theta(\cdot)](b^-)\,. \end{aligned} \tag{5.9}$$

LEMMA 5.2. *From Definition* 5.1 *we note*

$$\begin{cases} \text{(i)} & D(\mathbf{T},\cdot) \in \mathbf{H}(\mathbb{C}) \\ \text{(ii)} & D(\mathbf{T},\lambda) \in \mathbb{R} \quad (\lambda \in \mathbb{R})\,. \end{cases} \tag{5.10}$$

Proof. This proof follows from the Definition 5.1 and the earlier results of this section. ∎

We now have (see 5 of Remark 3.2 for the definition of **geometric** multiplicity of eigenvalues):

Theorem 5.1. *Let the coupled boundary value problem of Section* 4 *be determined, as in Theorem* 4.1, *by the matrix* $\mathbf{T}$ *and the parameter* α, *see in particular* (4.8).

Then

1. *A necessary and sufficient condition for* λ *to be an eigenvalue of the boundary value problem, i.e.* $\lambda \in \sigma(A) \equiv \{\lambda_n : n \in \mathbb{Z}\}$, *is that the entire function*

$$D(T, \cdot) - 2\cos(\alpha) : \mathbb{C} \to \mathbb{C} \tag{5.11}$$

has a zero at the point λ, *i.e.* $D(\mathbf{T}, \lambda) - 2\cos(\alpha) = 0$. *The eigenvalue* λ *is said to have* **algebraic** *multiplicity* $m \in \mathbb{N}$ *if the entire function* (5.11) *has a zero of order* m *at* λ.

The algebraic and geometric multiplicities of any eigenvalue are equal, that is with a prime $'$ *denoting differentiation with respect to* $\lambda \in \mathbb{C}$*:*

(a) λ *is a simple eigenvalue if and only if*

$$D(\mathbf{T}, \lambda) = 2\cos(\alpha) \text{ and } D'(\mathbf{T},\lambda) \neq 0 \tag{5.12}$$

(b) λ *is a double eigenvalue if and only if*

$$D(\mathbf{T}, \lambda) = 2\cos(\alpha) \text{ with } D'(\mathbf{T},\lambda) = 0 \text{ and } D''(\mathbf{T},\lambda) \neq 0\,. \tag{5.13}$$

Proof. The proof of 1 is to be found in [2, Section 3, Theorem 3.1]. The proof of 2, for the regular case, will be made available in the forthcoming papers [12] and [11]; see the Acknowledgement 3, at the end of the paper, for the singular case. ∎

REMARK 5.1. Since there is no numerical difference between algebraic and geometric multiplicity of eigenvalues of the Sturm-Liouville boundary value problem, reference is now made simply to the multiplicity of eigenvalues.

We can also characterize the existence and multiplicity of the eigenvalues using the basic solutions $\{u, v\}$ of the differential equation (5.1), as defined in (5.5) and (5.6). For any $\lambda \in \mathbb{C}$, the general solution of (5.1) can be written in the form

$$y(x,\lambda) = \beta(\lambda)u(x,\lambda) + \gamma(\lambda)v(x,\lambda) \quad (x \in (a,b))\,. \tag{5.14}$$

Substituting for this solution in the boundary conditions (5.2) leads to consideration of the 2×2 matrix, defined for all $\lambda \in \mathbb{C}$,

$$\mathbf{R}(\lambda) := \begin{bmatrix} [u(\cdot,\lambda),\theta(\cdot)](b^-) - \exp(i\alpha)t_{12} & [v(\cdot,\lambda),\theta(\cdot)](b^-) - \exp(i\alpha)t_{11} \\ [u(\cdot,\lambda),\varphi(\cdot)](b^-) - \exp(i\alpha)t_{22} & [v(\cdot,\lambda),\varphi(\cdot)](b^-) - \exp(i\alpha)t_{21} \end{bmatrix}. \tag{5.15}$$

If the solution (5.14) is to satisfy the boundary conditions, then the multipliers $\{\beta(\lambda), \gamma(\lambda)\}$ have to satisfy the linear, homogeneous matrix equation

$$\mathbf{R}(\lambda)[\beta(\lambda)\ \gamma(\lambda)]^{\tau} = 0\,. \tag{5.16}$$

Thus for any $\lambda \in \mathbb{C}$, the existence and multiplicity of eigenvalues of the boundary value problem is determined by the rank of the matrix $\mathbf{R}(\lambda)$; in this respect we have:

COROLLARY 5.1. *The matrix* $\mathbf{R}(\cdot)$ *has the properties:*

1. $\mathbf{R}(\cdot) \in \mathbf{H}(\mathbb{C})$.
2. $\det(\mathbf{R}(\lambda)) \neq 0$ $(\lambda \in \mathbb{C} \setminus \mathbb{R})$.
3. $\lambda \in \sigma(A) \equiv \{\lambda_n : n \in \mathbb{Z}\}$ *if and only if* $\det(\mathbf{R}(\lambda)) = 0$.
4. $\det(\mathbf{R}(\lambda)) = -1 - \exp(2i\alpha) + \exp(i\alpha)D(\mathbf{T},\lambda) \quad (\lambda \in \mathbb{C})$.
5. $D(\mathbf{T},\lambda) - 2\cos(\alpha) = \exp(i\alpha)\det(\mathbf{R}(\lambda)) \quad (\lambda \in \mathbb{C})$.
6. *For* $n \in \mathbb{Z}$ *let* $\lambda_n \in \sigma(A)$; *then the multiplicity of* λ_n *is given by* $2 - \operatorname{rank}(\mathbf{R}(\lambda_n))$.

PROOF. The proof of the results given in this Corollary follow from calculation and the earlier results of this section. ∎

6 – Spectral multiplicity two

We start this section with some remarks to set the parameters for this spectral case of the Sturm-Liouville boundary value problem

CONDITION 6.1. The boundary value problem is set by:

1. The Sturm-Liouville differential equation determined by (3.1) and (3.2).
2. The coupled boundary conditions determined by (4.5), (4.7) and (4.8).
3. The parameter α, see (i) of (4.8), is determined by restriction to one of the three cases $\alpha \in \{-\pi, 0, +\pi\}$ in order to give coupled boundary conditions that are **real** (if $\alpha \in (-\pi, 0) \cup (0, +\pi)$ then the coupled boundary conditions are **complex** and, necessarily, the spectrum $\sigma(A)$ of the boundary value problem is simple, see [4, Section 4, Theorem 5]); for the analysis given below we finally determine α by

$$\alpha = 0\,; \tag{6.1}$$

the results for the other two cases for α follow identical lines.

With the parameter α determined by (6.1) the eigenvalues of the boundary value problem, and the associated self-adjoint operator A, are determined by the zeros of the entire function $D(\mathbf{T}, \cdot) - 2 : \mathbb{C} \to \mathbb{C}$, or equivalently the roots of the equation

$$D(\mathbf{T}, \lambda) - 2 = 0\,; \tag{6.2}$$

this is a consequence of Theorem 5.1. As before these zeros or roots are denoted by $\{\lambda_n : n \in \mathbb{Z}\}$ with the properties, see (3.3),

$$\lambda_n \in \mathbb{R} \text{ and } \lambda_n < \lambda_{n+1}\ (n \in \mathbb{Z}),\ \text{with} \lim_{n \to \pm\infty} \lambda_n = \pm\infty\,. \tag{6.3}$$

Equivalently these eigenvalues are also the zeros of the entire function $\det(\mathbf{R}(\cdot)) : \mathbb{C} \to \mathbb{C}$ or the roots of the equation

$$\det(\mathbf{R}(\lambda)) = 0\,, \tag{6.4}$$

see 3 of Corollary 5.1. Here the matrix $\mathbf{R}(\cdot)$ is given by, see (5.15) with $\alpha = 0$,

$$\mathbf{R}(\lambda) := \begin{bmatrix} [u(\cdot,\lambda),\theta(\cdot)](b^-) - t_{12} & [v(\cdot,\lambda),\theta(\cdot)](b^-) - t_{11} \\ [u(\cdot,\lambda),\varphi(\cdot)](b^-) - t_{22} & [v(\cdot,\lambda),\varphi(\cdot)](b^-) - t_{21} \end{bmatrix}, \tag{6.5}$$

with $\mathbf{T} = [t_{rs}]$ and $\det(\mathbf{T}) = 1$.

Recall from Theorem 5.1 that, with the prime $'$ denoting differentiation in the complex plane $\mathbb{C}$:

1. The eigenvalue λ_n is **simple** if and only if

$$D(\mathbf{T},\lambda_n) - 2 = 0 \text{ and } D'(\mathbf{T},\lambda_n) \neq 0\,. \tag{6.6}$$

2. The eigenvalue λ_n is **double** if and only if

$$D(\mathbf{T},\lambda_n) - 2 = 0 \text{ and } D'(\mathbf{T},\lambda_n) = 0 \text{ and } D''(\mathbf{T},\lambda_n) \neq 0\,. \tag{6.7}$$

NOTATION 6.1. With the previous notation of $\sigma(A) = \{\lambda_n : n \in \mathbb{Z}\}$ as the spectrum of the boundary value problem, the set $\{\lambda_n\}$ of real numbers does not distinguish between simple and double eigenvalues; if for some $n \in \mathbb{Z}$ the eigenvalue λ_n is double then we write $\lambda_n \equiv \lambda_n^d$; in this case (6.7) takes the form

$$D(\mathbf{T},\lambda_n^d) - 2 = 0 \text{ and } D'(\mathbf{T},\lambda_n^d) = 0 \text{ and } D''(\mathbf{T},\lambda_n^d) \neq 0\,. \tag{6.8}$$

If λ_n is a simple eigenvalue then a (non-null) eigenfunction of the boundary value problem (equivalently a non-null eigenvector of the differential operator A) is denoted by ψ_n.

If λ_n^d is a double eigenvalue then a linearly independent pair of eigenfunctions of the boundary value problem (equivalently a linearly independent pair of eigenvectors of the differential operator A) is denoted by $\{\psi_{n,1},\psi_{n,2}\}$ or $\{\psi_{n,1}^d,\psi_{n,2}^d\}$.

CONDITION 6.2. We return now to Condition 3.3 and require that, for the boundary value problem determined as in Condition 6.1, there exists at least one integer $n \in \mathbb{Z}$ such that $\lambda_n \equiv \lambda_n^d$ is a double eigenvalue.

To link these notations with the matrix $\mathbf{R}(\cdot)$, defined by (5.15), we have

LEMMA 6.1. *If* λ_n *is a simple eigenvalue then* $\operatorname{rank}(\mathbf{R}(\lambda_n)) = 1$ *and the matrix* $\mathbf{R}(\lambda_n)$ *has at least one non-null row.*

If λ_n^d *is a double eigenvalue then* $\operatorname{rank}(\mathbf{R}(\lambda_n^d)) = 0$ *and* $\mathbf{R}(\lambda_n^d)$ *is the null matrix.*

PROOF. These results follow from Corollary 5.1. □

The example given in [4, Section 11], i.e. the differential equation

$$-y''(x) = \lambda y(x) \quad (x \in (-\pi + \pi) \tag{6.9}$$

with the coupled boundary conditions determined by

$$\alpha = 0 \quad \text{and} \quad \mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{6.10}$$

has eigenvalues given by (note in this case the spectrum is bounded below so that the eigenvalue index set is determined by $\mathbb{N}_0$ rather than $\mathbb{Z}$):

$$\lambda_0 = 0 \quad \text{and} \quad \lambda_n \equiv \lambda_n^d = n^2 \ (n \in \mathbb{N}) \tag{6.11}$$

where λ_0 is a simple eigenvalue.

This example shows that there are Sturm-Liouville boundary value problems, with coupled boundary conditions, with double eigenvalues; in particular, this example also shows that it is possible to have boundary value problems with both simple and double eigenvalues.

7 – Sturm-Liouville kernels

We now follow the definition of the basic Kramer analytic kernels $\{K_r : r = 1, 2\}$, for a Sturm-Liouville boundary value problem with coupled boundary conditions, given in [4, Section 4, Theorem 6] (recall from the condition (6.1) that the parameter $\alpha = 0$):

Definition 7.1. Define the two mappings $K_r : (a, b) \times \mathbb{C} \to \mathbb{C}$ $(r = 1, 2)$, using the notations of Lemma 5.1, for all $x \in (a, b)$ and all $\lambda \in \mathbb{C}$, by

$$\begin{aligned}(7.1)\quad & K_1(x, \lambda) := \\ & := ([u(\cdot, \lambda), \theta(\cdot)](b^-) - t_{12})v(x, \lambda) - ([v(\cdot, \lambda), \theta(\cdot)](b^-) - t_{11})u(x, \lambda)\end{aligned}$$

$$\begin{aligned}(7.2)\quad & K_2(x, \lambda) := \\ & := ([u(\cdot, \lambda), \varphi(\cdot)](b^-) - t_{22})v(x, \lambda) - ([v(\cdot, \lambda), \varphi(\cdot)](b^-) - t_{21})u(x, \lambda).\end{aligned}$$

This definition leads to (compare [4, Theorem 6]):

Theorem 7.1. *The two kernels defined in Definition* 7.1 *possess the following properties:*

1. *For all* $\lambda \in \mathbb{C} \setminus \{\lambda_n : n \in \mathbb{Z}\}$ *the pair* $\{K_1(\cdot, \lambda), K_2(\cdot, \lambda)\}$ *is a linearly independent system of solutions for the differential equation* (3.1).
2. *For all* $\lambda \in \sigma(A) = \{\lambda_n : n \in \mathbb{Z}\}$ *the pair* $\{K_1(\cdot, \lambda), K_2(\cdot, \lambda)\}$ *is a linearly dependent system of solutions.*
3. *For all* $\lambda \in \mathbb{C}$ *and for* $r = 1, 2$ *the kernels* $K_r(\cdot, \lambda) \in \Delta \subset L^2((a, b); w)$, *where* Δ *is defined in* (4.1).
4. *For* $r = 1, 2$, *the mapping* K_r *satisfies the coupled boundary conditions, see* (4.5) *and* 2 *of Condition* 6.1,

$$\mathbf{K}_r(b^- \lambda) = \mathbf{T}\mathbf{K}_r(a^+, \lambda)$$

   if and only if $\lambda \in \sigma(A) = \{\lambda_n : n \in \mathbb{Z}\}$.
5. *For* $r = 1, 2$ *the kernels* $K_r(x, \cdot) \in \mathbf{H}(\mathbb{C})$ *for all* $x \in (a, b)$.
6. *For* $r = 1, 2$ *the mappings*

$$\lambda \longmapsto \int_a^b w(x)|K_r(x, \lambda)|^2 \, dx$$

   are locally bounded on $\mathbb{C}$.

Proof. For the proof of these properties see the corresponding proof of [4, Theorem 6, as given in Section 6]. In particular, the proof of 4 requires use of the identity for the kernel K_1, see [4, Section 6, (52)] (with a similar result for the kernel K_2),

$$\mathbf{K}_1(b^-,\lambda) - \mathbf{T}\mathbf{K}_1(a^+,\lambda) = [0, \{D(\mathbf{T},\lambda) - 2\}]^\tau \quad (\lambda \in \mathbb{C})\,; \tag{7.3}$$

this identity is required later in the proof of Theorem 7.3. □

Remark 7.1. At this stage it should be noted that properties 7 and 8 of [4, Theorem 6] are no longer valid for this case of spectral multiplicity 2 and, consequently, are omitted in the statement of Theorem 7.1.

The property of **degeneracy** for the kernels $\{K_1, K_2\}$ is defined in [4, Definition 3, Section 8]:

Definition 7.2. For $r = 1, 2$ the eigenvalue λ_n is said to be degenerate for the kernel K_r if

$$K_r(x, \lambda_n) = 0 \ (x \in (a, b))\,. \tag{7.4}$$

It is shown in [4] that degeneracy, for coupled boundary value problems, cannot occur for the case of complex boundary conditions, when $\alpha \in (-\pi, 0) \cup (0, +\pi)$, but can be present for the case of real boundary conditions, when $\alpha \in \{-\pi, 0, +\pi\}$. Examples of degeneracy are given in [4, Section 10] for the case of real boundary conditions, when the spectrum is simple.

Degeneracy can also occur for real coupled boundary conditions, when the spectrum has multiplicity 2, as follows:

1. At a simple eigenvalue λ_n, i.e. when $\operatorname{rank}(\mathbf{R}(\lambda_n)) = 1$, one but not both of the kernels may be degenerate; this occurs when one of the rows of the matrix $\mathbf{R}(\lambda_n)$ is null.
2. At a double eigenvalue λ_n^d, i.e. when $\operatorname{rank}(\mathbf{R}(\lambda_n^d)) = 0$, both kernels K_1 and K_2 are degenerate; this occurs since then both rows of $\mathbf{R}(\lambda_n)$ are null.

To avoid the difficulties created in the case 2 above we have to introduce a revised form of the Kramer analytic kernels in the following definition (using calligraphic font $\mathcal{K}$ to denote the revised kernels):

DEFINITION 7.3. With the entire function $D(\mathbf{T},\cdot)$ given by (5.9) define the two mappings $\mathcal{K}_r : (a,b)\times\mathbb{C}\to\mathbb{C}$ for $r=1,2$, by, following Definition 7.1,

$$\mathcal{K}_r(x,\lambda) := \frac{K_r(x,\lambda)}{D'(\mathbf{T},\lambda)} \quad (x\in(a,b) \text{ and } \lambda\in\mathbb{C})\,. \tag{7.5}$$

It is necessary to show that the definitions made in the formulae (7.5) are in good order in view of the term in the denominator of (7.5). This term in the denominator is introduced to overcome the difficulties created by the property that $\mathbf{R}(\cdot)$ is the null matrix at a double eigenvalue. To make good the definition (7.5) we have

LEMMA 7.1. *Write the matrix* $\mathbf{R}(\cdot)$, *defined by* (6.5), *in the form*

$$\mathbf{R}(\lambda) = \begin{bmatrix} \rho_{11}(\lambda) & \rho_{12}(\lambda) \\ \rho_{21}(\lambda) & \rho_{22}(\lambda) \end{bmatrix} \quad (\lambda\in\mathbb{C})\,, \tag{7.6}$$

noting that $\rho_{rs}(\cdot)\in\mathbf{H}(\mathbb{C})$ $(r,s=1,2)$.

Then, then if $\lambda_n\equiv\lambda_n^d$ *is a double eigenvalue of the boundary value problem*

$$\rho_{rs}(\lambda_n^d) = 0 \quad (r,s=1,2)\,, \tag{7.7}$$

but

(i) *for* $r=1$ $\quad\rho'_{1s}(\lambda_n^d)\neq 0\quad$ *for* $s=1$ *or* 2 *or both*
(ii) *for* $r=2$ $\quad\rho'_{2s}(\lambda_n^d)\neq 0\quad$ *for* $s=1$ *or* 2 *or both.*

That is each row of $\mathbf{R}(\cdot)$ *has at least one element that has a simple zero at* λ_n^d.

PROOF. Consider the case $r=1$.

If both $\rho_{11}(\cdot)$ and $\rho_{12}(\cdot)$ have at least a zero of order 2 at λ_n^d then $\det(\mathbf{R}(\cdot))$ has at least a zero of order 3 at λ_n^d, i.e. $\det''(\mathbf{R}(\lambda_n^d))=0$. However, from 5 of Corollary 5.1, with $\alpha=0$, we have $\det''(\mathbf{R}(\lambda)) = D''(\mathbf{T},\lambda)$ for all $\lambda\in\mathbb{C}$ and so $D''(\mathbf{T},\lambda_n^d)=0$, in contradiction to (5.13) of Theorem 5.1. Thus (i) above must hold.

There is a similar proof for (ii) above. ∎

This last result enables us to prove that the kernels $\{\mathcal{K}_1, \mathcal{K}_2\}$ are well defined on $\mathbb{C}$, with the required analytic properties.

THEOREM 7.2. *Let the kernels* $\{\mathcal{K}_1, \mathcal{K}_2\}$ *be defined by* (7.5); *then, for* $r = 1, 2$

$$\mathcal{K}_r(x, \cdot) \in \mathbf{H}(\mathbb{C}) \quad (x \in (a, b)) \tag{7.8}$$

and

$$\mathcal{K}_r(\cdot, \lambda) \in \Delta \subset L^2((a, b); w) \quad (\lambda \in \mathbb{C})\,. \tag{7.9}$$

PROOF. The kernels $\{\mathcal{K}_1, \mathcal{K}_2\}$ inherit the property (7.8) from the same property for the original kernels $\{K_1, K_2\}$, as given in 5 of Theorem 7.1, except for those points of $\mathbb{C}$ where the entire function $D'(\mathbf{T}, \cdot)$ has zeros. These zeros, all simple from Theorem 5.1, occur only at the double eigenvalues $\{\lambda_n^d\}$ of the boundary value problem.

Note that in terms of the notation (7.6) for the matrix $\mathbf{R}(\cdot)$ and Definition 7.1 for the kernels $\{K_1, K_2\}$, we can write, using also (6.5), for $r = 1, 2$

$$\mathcal{K}_r(x, \lambda) = \frac{\rho_{r1}(\lambda)}{D'(\mathbf{T}, \lambda)} v(x, \lambda) - \frac{\rho_{r2}(\lambda)}{D'(\mathbf{T}, \lambda)} u(x, \lambda) \quad (x \in (a, b)\,. \tag{7.10}$$

Clearly, from 5 of Theorem 7.1, both of the kernels $\{K_1, K_2\}$ are regular analytic functions except, possibly, at the double eigenvalues $\{\lambda_n^d\}$. However from Lemma 7.1 both $\{\rho_{r1}(\cdot), \rho_{r2}(\cdot)\}$, for $r = 1, 2$, have zeros at all the points $\{\lambda_n^d\}$; but, for each $r = 1, 2$, at least on of these coefficients has a simple zero. Thus both limits, for $r = 1, 2$,

$$\lim_{\lambda \to \lambda_n^d} \frac{\rho_{r1}(\lambda)}{D'(\mathbf{T}, \lambda)} = a_{r1} \text{ (say)} \qquad \lim_{\lambda \to \lambda_n^d} \frac{\rho_{r2}(\lambda)}{D'(\mathbf{T}, \lambda)} = a_{r2} \text{ (say)} \tag{7.11}$$

exist and are finite; also for each $r = 1, 2$ at least one such limit is not zero.

Thus we have, for all points in the set $\{\lambda_n^d\}$, the definition

$$\mathcal{K}_r(x, \lambda_n^d) := \lim_{\lambda \to \lambda_n^d} \mathcal{K}_r(x, \lambda) = a_{r1} v(x, \lambda) - a_{r2} u(x, \lambda) \quad (x \in (a, b)) \tag{7.12}$$

implies that both the kernels $\{\mathcal{K}_1, \mathcal{K}_2\}$ are now well defined at all points of $\mathbb{C}$ and that the required result (7.8) is satisfied.

With $\mathcal{K}_r(\cdot, \lambda)$ now well defined on the interval (a, b) for all $\lambda \in \mathbb{C}$ and for $r = 1, 2$, the property (7.9) follows from 3 of Theorem 7.1. □

THEOREM 7.3. *Let all the conditions given in Condition* 6.1 *hold, and let the kernels* $\{\mathcal{K}_1, \mathcal{K}_2\}$ *be defined by* (7.5).

1. *At a simple eigenvalue* λ_n *of the boundary value problem, i.e.*

$$D(\mathbf{T}, \lambda_n) = 0, \quad D'(\mathbf{T}, \lambda_n) \neq 0 \quad \textit{and} \quad \operatorname{rank}(\mathbf{R}(\lambda_n)) = 1\,,$$

the kernels $\{\mathcal{K}_1(\cdot, \lambda_n), \mathcal{K}_2(\cdot, \lambda_n)\}$ *are linearly dependent on* (a, b) *and satisfy the boundary conditions, for* $r = 1, 2$,

$$\begin{aligned} &[[\mathcal{K}_r(\cdot, \lambda_n), \theta](b^-), [\mathcal{K}_r(\cdot, \lambda_n), \varphi](b^-)]^\tau = \\ &\qquad = \mathbf{T}[[\mathcal{K}_r(\cdot, \lambda_n), \theta](a^+), [\mathcal{K}_r(\cdot, \lambda_n), \varphi](a^+)]^\tau\,; \end{aligned} \tag{7.13}$$

at least one kernel is not degenerate (*see Definition* 7.2)*; let this non-degenerate kernel be denoted by* $\mathcal{K}_t(\cdot, \lambda_n)$, *where* $t = 1$ *or* 2; *then the function* $\psi_n : (a, b) \to \mathbb{R}$ *defined by*

$$\psi_n(x) := \mathcal{K}_t(x, \lambda_n) \quad (x \in (a, b)) \tag{7.14}$$

is an eigenfunction of the boundary value problem for λ_n; *this eigenfunction is unique up to linear dependence; the operator* A *in* $L^2((a, b);$ $w)$ *has* λ_n *as a simple eigenvalue and* $\{\psi_n\}$ *is a basis for the one-dimensional eigenspace of* A, *for the eigenvalue* λ_n.

2. *At a double eigenvalue* λ_n^d *of the boundary value problem, i.e.*

$$\begin{gathered} D(\mathbf{T}, \lambda_n^d) = 0, \quad D'(\mathbf{T}, \lambda_n^d) = 0, \quad D''(\mathbf{T}, \lambda_n^d) \neq 0 \\ \textit{and} \quad \operatorname{rank}(\mathbf{R}(\lambda_n^d)) = 0\,, \end{gathered}$$

the kernels $\{\mathcal{K}_1(\cdot, \lambda_n^d), \mathcal{K}_2(\cdot, \lambda_n^d)\}$ *are linearly independent on* (a, b) *and satisfy the boundary conditions, for* $r = 1, 2$,

$$\begin{aligned} &[[\mathcal{K}_r(\cdot, \lambda_n^d), \theta](b^-), [\mathcal{K}_r(\cdot, \lambda_n^d), \varphi](b^-)]^\tau = \\ &\qquad = \mathbf{T}[[\mathcal{K}_r(\cdot, \lambda_n^d), \theta](a^+), [\mathcal{K}_r(\cdot, \lambda_n^d), \varphi](a^+)]^\tau\,; \end{aligned} \tag{7.15}$$

both are non-degenerate; then the functions $\{\psi_{n,1}, \psi_{n,2}\}$ defined by

$$\psi_{n,r}(x) := \lim_{\lambda \to \lambda_n^d} \mathcal{K}_r(x, \lambda) \quad (x \in (a,b) \text{ and } r = 1,2) \tag{7.16}$$

are linearly independent on (a,b) and both are eigenfunctions of the boundary value problem for λ_n^d; the operator A in $L^2((a,b);w)$ has λ_n^d as a double eigenvalue and $\{\psi_{n,1}, \psi_{n,2}\}$ is a basis for the two-dimensional eigenspace of A, for the eigenvalue λ_n^d.

PROOF. We number the proof as in the statement of the theorem.

1. From Definition 7.1 of the original kernels $\{K_1, K_2\}$, Definition 7.3 of the revised kernels $\{\mathcal{K}_1, \mathcal{K}_2\}$, and the definition (6.5) of the matrix $\mathbf{R}(\cdot)$ we can write (again the superscript τ indicates a matrix transpose)

$$[\mathcal{K}_1(x, \lambda_n), \mathcal{K}_2(x, \lambda_n)]^\tau = D'(\mathbf{T}, \lambda_n)^{-1}\mathbf{R}(\lambda_n)[v(x, \lambda_n), -u(x, \lambda_n)]^\tau \qquad (x \in (a,b)),$$

where we recall that $\{v(\cdot, \lambda_n), u(\cdot, \lambda_n)\}$ is a basis for solutions of the differential equation (3.1) with $\lambda = \lambda_n$. From this identity, the given properties $D'(\mathbf{T}, \lambda_n) \neq 0$, $\det(\mathbf{R}(\lambda_n)) = 0$ and $\operatorname{rank}(\mathbf{R}(\lambda_n) = 1$ it follows that the pair $\{\mathcal{K}_1(\cdot, \lambda_n), \mathcal{K}_2(\cdot, \lambda_n)\}$ are linearly dependent solutions of the differential equation, but that at least one of the kernels $\mathcal{K}_t(\cdot, \lambda_n)$ is a non-null solution on (a,b).

From the identity (7.3) for $\mathcal{K}_1$ we obtain, for λ in a sufficiently small neighbourhood N of the point λ_n in which $D'(\mathbf{T}, \cdot)$ does not have any zeros,

$$\begin{aligned} &[[\mathcal{K}_1(\cdot, \lambda), \theta t](b^-), [\mathcal{K}_1(\cdot, \lambda), \varphi](b^-)]^\tau \\ &\quad - \mathbf{T}[[\mathcal{K}_1(\cdot, \lambda), \theta](a^+), [\mathcal{K}_1(\cdot, \lambda), \varphi](a^+)]^\tau \\ &= [0, D'(\mathbf{T}, \lambda)^{-1}\{D(\mathbf{T}, \lambda) - 2\}]^\tau \ (\lambda \in N); \end{aligned}$$

in this result let λ tend to λ_n. There is a similar identity for the kernel $\mathcal{K}_2$. Since the term $D(\mathbf{T}, \lambda) - 2$ tends to zero as λ tends to λ_n it now follows that the kernels $\{\mathcal{K}_r(\cdot, \lambda_n) : r = 1,2\}$ and the function $\psi_n(\cdot)$ all satisfy the coupled boundary conditions of the boundary value problem.

From the definition (7.14), the function ψ_n is now seen to satisfy all the required properties.

The properties of the operator A now follow from the definitions and results given in Section 5 above.

This completes the proof of part 1 of Theorem 7.2.

2. The case when the eigenvalue is double presents some additional difficulties.

We have seen in Theorem 7.2 that the kernels $\{\mathcal{K}_1(x,\cdot),\mathcal{K}_2(x,\cdot)\}$ are well defined on $\mathbb{C}$ for all $x \in (a,b)$. It is clear from the Definitions 7.1 and 7.3 that they are solutions of the differential equation (3.1; the same remarks hold for the functions $\{\psi_{n,1},\psi_{n,2}\}$ defined by (7.12) and (7.16).

To show that the kernels $\{\mathcal{K}_1(\cdot,\lambda_n^d),\mathcal{K}_2(\cdot,\lambda_n^d)\}$ and the functions $\{\psi_{n,1},\psi_{n,2}\}$ satisfy the boundary conditions (7.15) it is sufficient to argue with the kernel functions. Again, for the kernel $\mathcal{K}_1$, we have from the identity (7.3), for λ in a sufficiently small deleted neighbourhood $N\setminus\{\lambda_n^d\}$ of the point λ_n^d,

$$\begin{aligned}
&[[\mathcal{K}_1(\cdot,\lambda),\theta](b^-),[\mathcal{K}_1(\cdot,\lambda),\varphi](b^-)]^{\tau}\\
&\qquad -\mathbf{T}[[\mathcal{K}_1(\cdot,\lambda),\theta](a^+),[\mathcal{K}_1(\cdot,\lambda),\varphi](a^+)]^{\tau}\\
&= [0,D'(\mathbf{T},\lambda)^{-1}\{D(\mathbf{T},\lambda)-2\}]^{\tau} \qquad (\lambda\in N\setminus\{\lambda_n^d\})\,;
\end{aligned}$$

in this result let λ tend to λ_n^d. The term $D'(\mathbf{T},\lambda)^{-1}\{D(\mathbf{T},\lambda)-2\}$ tends to zero as λ tends to λ_n^d since the simple zero in the denominator is dominated by the double zero on the numerator; see (5.13) with $\alpha=0$. Thus the right-hand side above tends to the null vector as λ tends to λ_n^d and the boundary condition (7.15) is satisfied by $\mathcal{K}_1$. There is a similar argument for the kernel $\mathcal{K}_2$.

To prove that the kernels $\{\mathcal{K}_1(\cdot,\lambda_n^d),\mathcal{K}_2(\cdot,\lambda_n^d)\}$ and the functions $\{\psi_{n,1},\psi_{n,2}\}$ are linearly independent on the interval (a,b) we employ the Wronskian of $\{\psi_{n,1},\psi_{n,2}\}$ as solutions of the differential equation, as follows: let $x\in(a,b)$ then

$$\begin{aligned}
W(\psi_{n,1},\psi_{n,2})(x) &= \lim_{\lambda\to\lambda_n^d} W\Big(\frac{K_1(x,\lambda)}{D'(\mathbf{T},\lambda)},\frac{K_2(x,\lambda)}{D'(\mathbf{T},\lambda)}\Big) =\\
&= \lim_{\lambda\to\lambda_n^d}\frac{1}{D'(\mathbf{T},\lambda)^2}\det\begin{bmatrix} K_1(x,\lambda) & K_2(x,\lambda)\\ K_1'(x,\lambda) & K_2'(x,\lambda)\end{bmatrix}
\end{aligned}$$

where the prime $'$ denotes differentiation with respect to the variable $x \in (a,b)$. Now substitute from (7.1) and (7.2) for the terms in the matrix and a long calculation with all the terms involved leads to

$$\begin{aligned} W(\psi_{n,1}, \psi_{n,2})(x) = \lim_{\lambda \to \lambda_n^d} \frac{1}{D'(\mathbf{T},\lambda)^2} & W(v,u)(x)\{[v,\theta](b^-)[u,\varphi](b^-) \\ & - [u,\theta](b^-)[v,\varphi](b^-) + t_{11}t_{22} - t_{12}t_{21} - t_{11}[u,\varphi](b^-) \\ & - t_{22}[v,\theta](b^-) + t_{12}[v,\varphi](b^-) + t_{21}[u,\theta](b^-)\} \end{aligned}$$

where, for example, $[v,\theta](b^-)$ represents $[v(\cdot,\lambda),\varphi(\cdot)](b^-)$, and the t_{rs} are the elements of the matrix $\mathbf{T}$. We now have to use, respectively, 5.8, [4, Lemma 3, Section 6], (iii) of (4.8) and(5.9) to give

$$\begin{gathered} W(v,u)(x) = 1 \quad (x \in (a,b) \\ [v,\theta](b^-)[u,\varphi](b^-) - [u,\theta](b^-)[v,\varphi](b^-) = 1 \\ t_{11}t_{22} - t_{12}t_{21} = 1 \\ t_{12}[v,\varphi](b^-) + t_{21}[u,\theta](b^-) - t_{11}[u,\varphi](b^-) - t_{22}[v,\theta](b^-) = -D(\mathbf{T},\lambda)\,. \end{gathered}$$

Taken together these results yield

$$\begin{aligned} W(\psi_{n,1}, \psi_{n,2})(x) &= \lim_{\lambda \to \lambda_n^d} \frac{1}{D'(\mathbf{T},\lambda)^2}\{2 - D(\mathbf{T},\lambda)\} = \\ &= r \quad (\text{say}) \neq 0 \end{aligned}$$

using the holomorphic properties of $D(\mathbf{T},\cdot)$ and the results given in (5.13). Now the value of the Wronskian $W(\psi_{n,1}, \psi_{n,2})(x)$ is independent of $x \in (a,b)$, since $\psi_{n,1}$ and $\psi_{n,2}$ are solutions of the differential equation (3.1) with $\lambda = \lambda_n^d$, and hence

$$W(\psi_{n,1}, \psi_{n,2})(x) = r \neq 0 \quad (x \in (a,b))\,.$$

Thus the two solutions $\{\psi_{n,1}, \psi_{n,2}\}$ are linearly independent, and the same result holds for the kernels $\{\mathcal{K}_1(\cdot,\lambda_n^d), \mathcal{K}_1(\cdot,\lambda_n^d)\}$.

The properties of the operator A now follow from the definitions and results given in Section 5 above.

This completes the proof of part 2 of Theorem 7.3. □

8 – Degeneracy

We now develop the properties of the two revised kernels $\{\mathcal{K}_1, \mathcal{K}_2\}$ in respect of the property of degeneracy.

The definition of degeneracy for a Sturm-Liouville Kramer analytic kernel is given in Definition 7.2; see also [4, Section 8, Definition 3].

Remark 8.1.

1. From the remarks following Definition 7.2 it follows that both the original kernels K_r, for $r = 1, 2$, are degenerate for any double eigenvalue λ_n^d.
2. From part 2 of Theorem 7.3 it follows that both the revised kernels $\mathcal{K}_r$, for $r = 1, 2$, are not degenerate for any double eigenvalue λ_n^d.
3. Thus for the pair of revised kernels $\mathcal{K}_r$ $(r = 1, 2)$ degeneracy can only occur, if at all, at simple eigenvalues; this property is similar to the consideration of degeneracy for real, coupled boundary value problems, with simple spectrum, as given in [4, Sections 8 and 9].

Following the notation given in [4, Section 8] we define:

Definition 8.1.

1. For $n \in \mathbb{Z}$ the simple eigenvalue λ_n of the boundary value problem is said to be degenerate for the kernel $\mathcal{K}_r$ if

$$\mathcal{K}_r(x, \lambda_n) = 0 \quad (x \in (a, b)) . \tag{8.1}$$

2. For $r = 1, 2$ let $\mathbb{Z}_r \subseteq \mathbb{Z}$ denote the index set of all non-degenerate eigenvalues of the kernel $\mathcal{K}_r$.
3. Let $\mathbb{Z}_d \subseteq \mathbb{Z}$ denote the index set of all double eigenvalues of the boundary value problem.

We have

Theorem 8.1.

1. *If for $n \in \mathbb{Z}$ the simple eigenvalue is degenerate for the kernel $\mathcal{K}_1$, i.e. $n \in \mathbb{Z} \setminus \mathbb{Z}_1$, then λ_n is not degenerate for the kernel $\mathcal{K}_2$, i.e. $n \in \mathbb{Z}_2$; and* vice versa. *Thus*

$$(\mathbb{Z} \setminus \mathbb{Z}_1) \cap (\mathbb{Z} \setminus \mathbb{Z}_2) = \emptyset \quad \textit{and} \quad \mathbb{Z} = \mathbb{Z}_1 \cup \mathbb{Z}_2 . \tag{8.2}$$

2. *If for $n \in \mathbb{Z}$ the simple eigenvalue λ_n is degenerate for the kernel $\mathcal{K}_1$, and ψ_n is the corresponding non-null eigenfunction of the boundary value problem, then following the definition* (7.14) *and writing*

$$\mathcal{K}_r(x, \lambda_n) = k_{n,r}\psi_n(x) \quad (x \in (a, b)) \,,$$

it follows that

$$k_{n,1} = 0 \quad \textit{and} \quad k_{n,2} = 1 \,.$$

There is a similar result if the simple eigenvalue λ_n is degenerate for the kernel $\mathcal{K}_2$.

3. *For the index set $\mathbb{Z}_d$ the following results hold*

$$\emptyset \neq \mathbb{Z}_d \subseteq \mathbb{Z}_r \textit{ for } r = 1, 2 \,. \tag{8.3}$$

Proof.

1. If the simple eigenvalue λ_n is degenerate for the kernel $\mathcal{K}_1$ then, from the definitions (6.5) and (7.1), it follows that the first row of the matrix $\mathbf{R}(\lambda_n)$ is null. Since λ_n is simple we have $\operatorname{rank}(\mathbf{R}(\lambda_n)) = 1$ and thus the second row of this matrix is not null and so, from definition (7.2) the kernel $\mathcal{K}_2$ is not degenerate.
2. The non-null eigenfunction ψ_n is defined by (7.14) in terms of the non-degenerate kernel $\mathcal{K}_t$; in this case $\mathcal{K}_1$ is degenerate and so $t = 2$; this gives the required result.
3. The result $\mathbb{Z}_d \neq \emptyset$ follows from Conditions 3.3 and 6.2; for $r = 1, 2$ the inclusion $\mathbb{Z}_d \subseteq \mathbb{Z}_r$ holds since, from Theorem 7.3, both the kernels $\mathcal{K}_1$ and $\mathcal{K}_2$ are non-degenerate at all double eigenvalues $\{\lambda_n^d : n \in \mathbb{Z}_d\}$. □

Definition 8.2. For any set of vectors $\{f_r \in L^2((a, b); w) : r \in \mathbb{I}\}$, where $\mathbb{I}$ is an index set, $\operatorname{span}\{f_r : r \in \mathbb{I}\}$ denotes the linear hull of the set $\{f_r : r \in \mathbb{I}\}$, and $\overline{\operatorname{span}}\{f_r : r \in \mathbb{I}\}$ denotes the closure of the set $\operatorname{span}\{f_r : r \in \mathbb{I}\}$ in the normal topology of $L^2((a, b; w)$.

Definition 8.3. For $r = 1, 2$ define (see the notation for eigenfunctions given in Theorem 7.3):

1. The sets of eigenfunctions resulting from simple eigenvalues

$$S_r := \{\psi_n : n \in \mathbb{Z}_r \setminus \mathbb{Z}_d\} \quad (r = 1, 2)\} \, . \tag{8.4}$$

2. The sets of eigenfunctions resulting from double eigenvalues

$$D_r := \{\psi_{n,r} : n \in \mathbb{Z}_d\} \quad (r = 1, 2)\} \, . \tag{8.5}$$

Definition 8.4. For $r = 1, 2$ define the subspaces $L^2_r((a,b);w)$ of $L^2((a,b);w)$ by the direct sums:

$$L^2_r((a,b;w) := \overline{\text{span}}\{S_r\} \dotplus \overline{\text{span}}\{D_r\} \quad (r = 1, 2) \, . \tag{8.6}$$

Remark 8.2.

1. For $r = 1, 2$ the set S_r is the collection of eigenfunctions determined by all non-degenerate simple eigenvalues of the kernel $\mathcal{K}_r$; note that $S_1 \cap S_2 = \emptyset$ if and only if every simple eigenvalue is degenerate for either $\mathcal{K}_1$ or $\mathcal{K}_2$.
2. For $r = 1, 2$ the set D_r is the collection of eigenfunctions determined by the kernel $\mathcal{K}_r$ at all double eigenvalues; note that in all cases $D_1 \cap D_2 = \emptyset$.
3. The union

$$\Big(\bigcup_{r=1,2} S_r \Big) \cup \Big(\bigcup_{r=1,2} D_r \Big) \tag{8.7}$$

   is the collection of all eigenfunctions of the coupled boundary value problem.
4. We have

$$\overline{\text{span}}\Big[\Big(\bigcup_{r=1,2} S_r \Big) \cup \Big(\bigcup_{r=1,2} D_r \Big)\Big] = L^2((a,b);w) \, . \tag{8.8}$$

5. Since each eigenfunction, of the coupled boundary value problem, that belongs to the sub-space $L^2_1((a,b);w)$, i.e. that belongs to the set $S_1 \cup D_1$, arises from a different eigenvalue, all such eigenfunctions form an orthogonal set in $L^2_1((a,b(;w)$, and so form an orthogonal set in $L^2((a,b);w)$. There is a similar result for the set of eigenfunctions $S_2 \cup D_2$ in the sub-space $L^2_2((a,b);w)$.

9 – Kramer kernels for multiple eigenvalues

The problem of defining Kramer kernels for Sturm-Liouville boundary value problems with double eigenvalues, can be seen by reference to the conditions required for Theorem 2.1. If $K : I \times \mathbb{C} \to \mathbb{C}$ then for any $\lambda \in \mathbb{C}$ the symbol $K(\cdot, \lambda) : I \to \mathbb{C}$ is a single-valued function defined on the interval I. If now K is a Sturm-Liouville kernel and λ_n^d is a double eigenvalue the it is impossible for $K(\cdot, \lambda_n^d)$ to represent the two-dimensional eigenspace associated with λ_n^d.

For this reason the definition of the two kernels $\{\mathcal{K}_r : r = 1, 2\}$ is chosen so that the property of the ordered pair $\{\mathcal{K}_1(\cdot, \lambda_n^d), \mathcal{K}_2(\cdot, \lambda_n^d)\}$ is to represent a basis of the two-dimensional eigenspace associated with the double eigenvalue λ_n^d. However, it has then to be accepted that neither of these two kernels can support a complete set of eigenfunctions of the boundary value problem, as required by condition 3(iv) of Theorem 2.1.

The difficulties in this problem are compounded by the possible existence of degenerative simple eigenvalues for one or both of the kernels $\mathcal{K}_1$ and $\mathcal{K}_2$. The idea behind the definition, see (8.4) and (8.5), of the eigenfunction sets $\{S_r : r = 1, 2\}$ and $\{D_r : r = 1, 2\}$ is to associate the maximal set of eigenfunctions with each kernel, notwithstanding the problem of degeneracy.

This maximal technique is also used in the paper [4, Section 8] on coupled boundary value problems with simple spectra, but only to cover the problem of degeneracy. In general, neither of the kernels $\{K_r : r = 1, 2\}$ defined in [4, Section 4, (48) and (49)] support a complete set of eigenfunctions of the coupled boundary value problem. This deficiency is overcome in [4], in this case for simple spectra, by the device of forming linear combinations of the two kernels to define a kernel K that has global eigenfunction representation, in the form, see [4, Section 8, Theorem 7],

$$K := \gamma_1 K_1 + \gamma_2 K_2 \,. \tag{9.1}$$

However this device does not work for the coupled case with double eigenvalues since, again, no kernel of the form (9.1) can represent the two-dimensional eigenspace of any eigenvalue of multiplicity two.

To overcome this difficulty we show that the kernels $\{\mathcal{K}_r : r = 1, 2\}$, see (7.6) of Definition 7.3, are Kramer analytic kernels, respectively, in the sub-spaces $L_r^2((a, b); w)$, see (8.6) of Definition 8.2. Thereafter we consider the orthogonal sum of these two subspaces to define the Hilbert space

$$\mathbf{L}^2((a,b);w) := L_1^2((a,b);w) \oplus L_2^2((a,b);w) \tag{9.2}$$

and construct Kramer analytic kernels in $\mathbf{L}^2((a,b);w)$.

Note that even if degeneracy does not occur in either $\mathcal{K}_1$ nor in $\mathcal{K}_2$ both of the spaces $L_r^2((a,b);w)$, for $r = 1, 2$, remain strict sub-spaces of the space $L^2((a,b);w)$ due to the presence of at least one double eigenvalue.

10 – Kramer analytic kernels for the spaces $L_r^2((a,b);w)$

We have

Theorem 10.1. *For $r = 1, 2$ let:*

(i) *The kernels $\mathcal{K}_r : (a,b) \times \mathbb{C} \to \mathbb{C}$ be defined as in Definition* 7.3

(ii) *The subspaces $L_r^2((a,b);w)$ be defined as in Definition* 8.2.

Then for $r = 1, 2$ the kernels $\mathcal{K}_r$ are Kramer analytic kernels in the subspaces $L_r^2((a,b);w)$ in the sense of conditions $1, 2$ *and* 3 *of Theorem* 2.1.

Proof. Consider the conditions $1, 2$ and 3 of Theorem 2.1:

1. This condition is satisfied from (7.9) of Theorem 7.2.
2. This condition is satisfied from (7.8) of Theorem 7.2.
3. For the sequence $\{\lambda_n : n \in \mathbb{Z}\}$ of the boundary value problem the properties (i), (ii) and (iii) are satisfied; see the results and notations of Section 3 above;
   (iv) for $r = 1, 2$, the sequence of functions $\{\mathcal{K}_r(\cdot, \lambda_n) : n \in \mathbb{Z}_r\}$, i.e. the collection of eigenfunctions $S_r \cup D_r$, is locally linearly independent (individually they are solutions of the differential equation (3.1) for different values of the parameter λ), and are a complete orthogonal set in the Hilbert space $L_r^2((a,b);w)$ from their properties as eigenfunctions, and the definition (8.6) of Definition 8.4 for the space $L_r^2((a,b);w)$

(v) for $r = 1, 2$, the local boundedness of the mappings

$$\lambda \longmapsto \int_a^b w(x)|\mathcal{K}_r(x,\lambda)|^2\,dx$$

follows from the properties now established for these kernels, the definition (7.5) of Definition 7.3, and the corresponding property of the original kernels K_r established in [4, Theorem 6]. □

Remark 10.1.

1. Both $\mathcal{K}_1$ and $\mathcal{K}_2$ are now seen to be Kramer analytic kernels, see Remark 2.1, in the sense Theorem 2.1 or, possibly, of Corollary 2.1.
2. All the properties of Kramer analytic kernels given in the results of Theorem 2.1 now apply to the kernels $\mathcal{K}_1$ and $\mathcal{K}_2$.

11 – Analytic interpolation functions

The properties of analytic interpolation functions for Kramer analytic kernels are given in Definition 2.1.

For the two Kramer analytic kernels $\mathcal{K}_1$ and $\mathcal{K}_2$ we define an analytical interpolation function G by

$$G(\lambda) := \frac{D(\mathbf{T},\lambda) - 2}{D'(\mathbf{T},\lambda)} \quad (\lambda \in \mathbb{C})\,. \tag{11.1}$$

We have

Lemma 11.1. *With the definition* (11.1) *of the function G:*

1. $G \in \mathbf{H}(\mathbb{C})$.
2. $G(\lambda) = 0$ *if and only if* $\lambda \in \{\lambda_n : n \in \mathbb{Z}\}$, *i.e.* λ *is an eigenvalue of the coupled boundary value problem.*
3. $G'(\lambda_n) \neq 0$ $(n \in \mathbb{Z})$.

Proof. We have:

1. This result follows from essentially from Theorem 5.1; $D(\mathbf{T},\cdot) \in \mathbf{H}(\mathbb{C})$; the zeros of $D'(\mathbf{T},\cdot)$ are all simple and occur only at the double eigenvalues $\{\lambda_n^d\}$ of the boundary value problem; at these double

eigenvalues $\{\lambda_n^d\}$ the function $D(\mathbf{T},\cdot)-2$ has double zeros; thus $G(\cdot)$ has no finite singularities on $\mathbb{C}$ and so $G \in \mathbf{H}(\mathbb{C})$.

2. From the definition (11.1) it follows that G can only have zeros at the zeros of the numerator $D(\mathbf{T},\cdot)-2$, i.e. at the eigenvalues $\{\lambda_n\}$; at a simple eigenvalue λ_n the denominator $D'(\mathbf{T},\lambda_n) \neq 0$, the numerator $D(\mathbf{T},\lambda_n)-2=0$ and so $G(\lambda_n)=0$; at a double eigenvalue λ_n^d the denominator $D'(\mathbf{T},\cdot)$ has a simple zero, the numerator $D(\mathbf{T},\cdot)-2$ has a double zero and so $G(\lambda_n^d)=0$.
3. A calculation shows that

$$G'(\lambda) = 1 - \frac{D(\mathbf{T},\lambda)-2}{D'(\mathbf{T},\lambda)^2} D''(\mathbf{T},\lambda);$$

at a simple eigenvalue λ_n this formula gives $G'(\lambda_n) = 1 \neq 0$; at a double eigenvalue λ_n^d take the limit

$$\lim_{\lambda \to \lambda_n^d} \frac{D(\mathbf{T},\lambda)-2}{D'(\mathbf{T},\lambda)^2} = \lim_{\lambda \to \lambda_n^d} \frac{D'(\mathbf{T},\lambda)}{2D'(\mathbf{T},\lambda)D''(\mathbf{T},\lambda)} = \frac{1}{2D''(\mathbf{T},\lambda_n^d)}$$

and so $G'(\lambda_n^d) = \frac{1}{2} \neq 0$.

This last result leads to

THEOREM 11.1. *Let $G : \mathbb{C} \to \mathbb{C}$ be defined by* (11.1); *then G is an analytic interpolation function for both the Kramer analytic kernels $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively in the spaces $L_1^2((a,b);w)$ and $L_2^2((a,b);w)$.*

PROOF. We give the proof only for the kernel $\mathcal{K}_1$; there is a similar proof for the kernel $\mathcal{K}_2$.

The four conditions for G to be an analytic interpolation function are given in Definition 2.1; the first three of these conditions are seen to hold from Lemma 11.1. It remains then to prove that the fourth condition is satisfied by G, i.e.

$$\frac{G(\lambda)}{G'(\lambda_n)(\lambda-\lambda_n)} = S_{n,1}(\lambda) \quad (\lambda \in \mathbb{C} \text{ and } n \in \mathbb{Z}_1) \tag{11.2}$$

where, from (2.3), for the kernel $\mathcal{K}_1$

$$S_{n,1}(\lambda) = \|\mathcal{K}_1(\cdot,\lambda_n)\|_w^{-2} \int_a^b w(x)\mathcal{K}_1(x,\lambda)\overline{\mathcal{K}}_1(x,\lambda_n)\,dx =$$

$$= \frac{\displaystyle\int_a^b w(x)\mathcal{K}_1(x,\lambda)\overline{\mathcal{K}}_1(x,\lambda_n)\,dx}{\displaystyle\int_a^b w(x)|\mathcal{K}_1(x,\lambda_n)|^2\,dx} \quad (\lambda \in \mathbb{C} \text{ and } n \in \mathbb{Z}_1)\,. \tag{11.3}$$

Define the sequence $\{G_n : n \in \mathbb{Z}\}$ of entire functions by

$$G_{n,1}(\lambda) := (\lambda - \lambda_n) \int_a^b w(x)\mathcal{K}_1(x,\lambda)\overline{\mathcal{K}}_1(x,\lambda_n)\,dx \qquad (\lambda \in \mathbb{C} \text{ and } n \in \mathbb{Z})\,; \tag{11.4}$$

we note that, but disregard it for the proof since we work only with $\mathbb{Z}_1$,

$$G_{n,1}(\lambda) = 0 \quad (\lambda \in \mathbb{C} \text{ and } n \in \mathbb{Z} \setminus \mathbb{Z}_1)\,. \tag{11.5}$$

From [4, Section 6, (56) and (57)], using Green's formula and (4.2) and (4.3) above, we obtain

$$G_{n,1}(\lambda) = [\mathcal{K}_1(\cdot,\lambda), \mathcal{K}_1(\cdot,\lambda_n)]_{a+}^{b-} \quad (\lambda \in \mathbb{C} \text{ and } n \in \mathbb{Z}_1)\,. \tag{11.6}$$

For any $\lambda \in \mathbb{C}$ substitute in the Plücker identity, see [7, Section 4], to obtain where ξ stands for either a^+ or b^-

$$\begin{aligned}[\mathcal{K}_1(\cdot,\lambda), \mathcal{K}_1(\cdot,\lambda_n)](\xi) = &-[\mathcal{K}_1(\cdot,\lambda), \theta(\cdot)](\xi)[\varphi(\cdot), \mathcal{K}_1(\cdot,\lambda_n)](\xi) + \\ &+ [\mathcal{K}_1(\cdot,\lambda), \varphi(\cdot)](\xi)[\theta(\cdot), \mathcal{K}_1(\cdot,\lambda_n)](\xi)\,.\end{aligned} \tag{11.7}$$

We now follow the sequence of steps given in [4, Section 7] to obtain the following results, recalling that we have now taken $\alpha = 0$ (these results require the use of the Plücker identity), valid firstly for all $\lambda \in \mathbb{C}$:

$$[\mathcal{K}_1(\cdot,\lambda), \theta(\cdot)](b^-) = t_{11}[\mathcal{K}_1(\cdot,\lambda), \theta(\cdot)](a^+) + t_{12}[\mathcal{K}_1(\cdot,\lambda), \varphi(\cdot)](a^+)$$

$$\begin{aligned}[\mathcal{K}_1(\cdot,\lambda), \varphi(\cdot)](b^-) = &\; t_{21}[\mathcal{K}_1(\cdot,\lambda), \theta(\cdot)](a^+) + \\ &+ t_{22}[\mathcal{K}_1(\cdot,\lambda), \varphi(\cdot)](a^+) + \frac{D(\mathbf{T},\lambda) - 2}{D'(\mathbf{T},\lambda)},\end{aligned}$$

and secondly for all $n \in \mathbb{Z}_1$

$$[\theta(\cdot), \mathcal{K}_1(\cdot, \lambda_n)](b^-) = t_{11}t[\theta(\cdot), \mathcal{K}_1(\cdot, \lambda_n)](a^+) + t_{12}[\theta(\cdot), \mathcal{K}_1(\cdot, \lambda_n)](a^+)$$

$$[\varphi(\cdot), \mathcal{K}_1(\cdot, \lambda_n)](b^-) = t_{21}[\varphi(\cdot), \mathcal{K}_1(\cdot, \lambda_n)](a^+) + t_{22}[\varphi(\cdot), \mathcal{K}_1(\cdot, \lambda_n)](a^+)\,.$$

We now substitute these last four results into (11.7), using both $\xi = a^+$ and $\xi = b^-$, and noting (11.6) to give (details of the calculation are omitted)

$$\begin{aligned} G_{n,1}(\lambda) &= \frac{D(\mathbf{T}, \lambda) - 2}{D'(\mathbf{T}, \lambda)}[\theta(\cdot), \mathcal{K}_1(\cdot, \lambda_n)](b^-) \\ &= G(\lambda)[\theta(\cdot), \mathcal{K}_1(\cdot, \lambda_n)](b^-) \quad (\lambda \in \mathbb{C} \text{ and } n \in \mathbb{Z}_1)\,. \end{aligned} \tag{11.8}$$

We now prove that

$$[\theta(\cdot), \mathcal{K}_1(\cdot, \lambda_n)](b^-) \neq 0 \quad (n \in \mathbb{Z}_1)\,. \tag{11.9}$$

Assume, to the contrary, that for some $m \in \mathbb{Z}_1$ we have $[\theta(\cdot), \mathcal{K}_1(\cdot, \lambda_m)]$ $(b^-) = 0$; then from (11.8) it follows that $G_{m,1}(\lambda) = 0$ for all $\lambda \in \mathbb{C}$, and then from (11.4) we obtain

$$(\lambda - \lambda_m) \int_a^b w(x)\mathcal{K}_1(x, \lambda)\overline{\mathcal{K}}_1(x, \lambda_m)\, dx = 0 \quad (\lambda \in \mathbb{C})\,.$$

This last result implies that

$$\lim_{\lambda \to \lambda_m} \int_a^b w(x)\mathcal{K}_1(x, \lambda)\overline{\mathcal{K}}_1(x, \lambda_m)\, dx = 0$$

and then, see [7, Section 3, (3.6)] for justification of the inverse of the limit process,

$$\int_a^b w(x)|\mathcal{K}_1(x, \lambda_m)|^2\, dx = 0\,.$$

This last result is in contradiction to

$$\int_a^b w(x)|\mathcal{K}_1(x, \lambda_n)|^2\, dx > 0 \quad (n \in \mathbb{Z}_1)$$

and (11.9) is established.

From (11.8) it now follows that, since the terms $[\theta(\cdot), \mathcal{K}_1(\cdot, \lambda_n)](b^-)$ are independent of λ for all $n \in \mathbb{Z}_1$,

$$G_{n,1}(\cdot) \in \mathbf{H}(\mathbb{C}) \text{ for all } n \in \mathbb{Z}_1\,, \tag{11.10}$$

and letting $\lambda \to \lambda_{n,1}$ in (11.4) gives

$$G'_{n,1}(\lambda_n) = \int_a^b w(x)|\mathcal{K}_1(x, \lambda_n)|^2\, dx > 0 \text{ for all } n \in \mathbb{Z}_1\,. \tag{11.11}$$

Finally to prove (11.2) for any $n \in \mathbb{Z}_1$ multiply (11.3) top and bottom by $(\lambda - \lambda_n) \neq 0$, use (11.4), (11.11) and (11.8) to give

$$\begin{aligned} S_{n,1}(\lambda) &= \frac{G_{n,1}(\lambda)}{(\lambda - \lambda_n)G'_{n,1}(\lambda_n)} \\ &= \frac{G(\lambda)}{(\lambda - \lambda_n)G'(\lambda)} \quad (\lambda \in \mathbb{C} \ \text{ and } \ n \in \mathbb{Z}_1)\,. \end{aligned} \tag{11.12}$$

This completes the proof of the theorem. ∎

Remark 11.1. Restricting the working environment to the original Hilbert function space $L^2((a, b); w)$, the Theorems 10.1 and 11.1 represent one form of the best results to obtain Kramer analytic kernels, with analytic interpolation functions, from Sturm-Liouville boundary value problems with coupled boundary conditions and spectral multiplicity two.

We show in the next section that if the space $L^2((a, b); w)$ is replaced by a vector Hilbert space then an extended form of these results can be obtained.

12 – Kramer kernels in vector Hilbert spaces

Given the two Hilbert function spaces $L^2_1((a, b); w)$ and $L^2_2((a, b); w)$, see Definitions 8.1, 8.3 and 8.4, we can define the vector space $\mathbf{L}^2((a.b); w)$ by

$$\mathbf{L}^2((a.b); w) := L^2_1((a, b); w) \oplus L^2_2((a, b); w)\,. \tag{12.1}$$

Elements $\mathbf{f}$ of $\mathbf{L}^2((a.b);w)$ are determined by

$$\mathbf{f} = \{f_1, f_2\} \quad (f_1 \in L_1^2((a,b);w) \text{ and } f_2 \in L_2^2((a,b);w))$$

with the usual vector addition and scalar multiplication, for all vectors $\mathbf{f}, \mathbf{g} \in \mathbf{L}^2((a.b);w)$ and scalars $\alpha \in \mathbb{C}$,

$$\begin{gathered}\mathbf{f} + \mathbf{g} \equiv \{f_1, f_2\} + \{g_1, g_2\} := \{f_1 + g_1, f_2 + g_2\} \\ \alpha\mathbf{f} \equiv \alpha\{f_1, f_2\} := \{\alpha f_1, \alpha f_2\}\,.\end{gathered}$$

The inner-product on $\mathbf{L}^2((a.b);w)$ is defined by

$$(\mathbf{f}, \mathbf{g})_{\mathbf{L}^2} := (f_1, g_1)_1 + (f_2, g_2)_2 = \int_a^b w f_1 \overline{g}_1 + \int_a^b w f_2 \overline{g}_2\,.$$

With the norm $\|\cdot\|_{\mathbf{L}^2}$ derived in the usual way from this inner-product the space $\mathbf{L}^2((a.b);w)$ is complete and thus a vector Hilbert space over $\mathbb{C}$.

Given the two Kramer analytic kernels $\mathcal{K}_1$ and $\mathcal{K}_2$, see Definition 7.3, and recalling that

$$\mathcal{K}_r(\cdot, \lambda) \in L_r^2((a,b);w) \quad (\lambda \in \mathbb{C} \text{ and } r = 1,2)$$

we define the vector kernel $\mathbf{K}(\cdot) : \mathbb{C} \to \mathbf{L}^2((a,b);w)$ by

$$\mathbf{K}(\lambda) := \{\mathcal{K}_1(\cdot, \lambda), \mathcal{K}_2(\cdot, \lambda)\} \quad (\lambda \in \mathbb{C})\,; \tag{12.2}$$

i.e. $\mathbf{K}(\lambda) \in \mathbf{L}^2((a,b);w)$ $(\lambda \in \mathbb{C})$.

Following the notation in Theorem 2.1 we define the set $\{\mathbf{K}\}$ as the collection of all functions $F : \mathbf{L}^2((a.b);w) \times \mathbb{C} \to \mathbb{C}$ determined by

$$F(\mathbf{f};\lambda) \equiv F(\lambda) := (\mathbf{K}(\lambda), \mathbf{f})_{\mathbf{L}^2} \quad (\mathbf{f} \in \mathbf{L}^2((a,b);w) \text{ and } \lambda \in \mathbb{C})\,. \tag{12.3}$$

We note that, from previous results for the two Kramer analytic kernels $\mathcal{K}_1$ and $\mathcal{K}_2$,

$$F(\mathbf{f};\cdot) \in \mathbf{H}(\mathbb{C}) \text{ for all } \mathbf{f} \in \mathbf{L}^2((a,b);w)\,. \tag{12.4}$$

THEOREM 12.1. *Let the vector Hilbert space* $\mathbf{L}^2((a.b);w)$ *be defined by* (12.1), *the vector kernel* $\mathbf{K}$ *defined by* (12.2), *and the set* $\{\mathbf{K}\}$ *of functions defined by* (12.3).

Let the analytic interpolation function $G : \mathbb{C} \to \mathbb{C}$ *be defined by* (11.1).

The for all $F \in \{\mathbf{K}\}$ *there is the Lagrange interpolation representation*

$$F(\lambda) = \sum_{n \in \mathbb{Z}} F(\lambda_n) \frac{G(\lambda)}{G'(\lambda_n)(\lambda - \lambda_n)} \tag{12.5}$$

where the infinite series is

$$\begin{cases} \text{(i)} & \textit{absolutely convergent for each } \lambda \in \mathbb{C} \\ \text{(ii)} & \textit{locally uniformly convergent on } \mathbb{C}. \end{cases} \tag{12.6}$$

PROOF. We use the earlier results giving the properties of the Kramer analytic kernels $\mathcal{K}_1$ and $\mathcal{K}_2$, the interpolation sequences $\{S_{n,1}(\cdot) : n \in \mathbb{Z}_1\}$ and $\{S_{n,2}(\cdot) : n \in \mathbb{Z}_2\}$, and the analytic interpolation function G.

We have for all $\lambda \in \mathbb{C}$

$$\begin{aligned} F(\lambda) &= (\mathbf{K}(\lambda), \mathbf{f})_{\mathbf{L}^2} = (\mathcal{K}_1(\cdot,\lambda), f_1)_1 + (\mathcal{K}_2(\cdot,\lambda), f_2)_2 = \\ &= F_1(\lambda) + F_2(\lambda) \\ &= \sum_{n \in \mathbb{Z}_1} F_1(\lambda_n) S_{n,1}(\lambda) + \sum_{n \in \mathbb{Z}_2} F_2(\lambda_n) S_{n,2}(\lambda) \\ &= \sum_{n \in \mathbb{Z}_1} F_1(\lambda_n) \frac{G(\lambda)}{G'(\lambda_n)(\lambda - \lambda_n)} + \sum_{n \in \mathbb{Z}_2} F_2(\lambda_n) \frac{G(\lambda)}{G'(\lambda_n)(\lambda - \lambda_n)} \qquad (12.7) \\ &= \sum_{n \in \mathbb{Z}} F_1(\lambda_n) \frac{G(\lambda)}{G'(\lambda_n)(\lambda - \lambda_n)} + \sum_{n \in \mathbb{Z}} F_2(\lambda_n) \frac{G(\lambda)}{G'(\lambda_n)(\lambda - \lambda_n)} \qquad (12.8) \\ &= \sum_{n \in \mathbb{Z}} \{F_1(\lambda_n) + F_2(\lambda_n)\} \frac{G(\lambda)}{G'(\lambda_n)(\lambda - \lambda_n)} \\ &= \sum_{n \in \mathbb{Z}} F(\lambda_n) \frac{G(\lambda)}{G'(\lambda_n)(\lambda - \lambda_n)}\,. \qquad (12.9) \end{aligned}$$

The passage from (12.6) to (12.7) can be made since, from the degeneracy properties of $\mathcal{K}_1$ and $\mathcal{K}_2$,

$$F_1(\lambda_n) = 0 \ (n \in \mathbb{Z} \setminus \mathbb{Z}_1) \quad \text{and} \quad F_2(\lambda_n) = 0 \ (n \in \mathbb{Z} \setminus \mathbb{Z}_2)\,.$$

The convergence properties (12.5) for the infinite series (12.4), equivalently the series (12.8), follows from the corresponding properties for the infinite series (12.6); see (2.4) and (2.5) of Theorem 2.1. ∎

Remark 12.1. Notwithstanding the analytic result (12.4) and the expansion result (12.8) the kernel $\mathbf{K}(\cdot)$, defined by (12.3), is not a Kramer analytic kernel in the sense of the properties given in Section 2 above.

Remark 12.2. For a different use of the method of involving vector Hilbert spaces, but only in connection with regular Sturm-Liouville boundary value problems with coupled boundary conditions, see the results in [1].

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Indirizzo degli Autori:

W. N. Everitt – School of Mathematics and Statistics – University of Birmingham – Edgbaston – Birmingham B15 2TT – England, UK
E-mail: w.n.everitt@bham.ac.uk

G. Nasri-Roudsari – c/o Lehrstuhl A für Mathematik – R-W T H Aachen – D-52056 Aachen – Germany
E-mail: gabriele.nasri-roudsari@dresdner-bank.com