Rendiconti di Matematica, Serie VII
Volume 20, Roma (2000), 239-256

Statistical theory of dislocations in two-dimensional elastic bodies

H. ZORSKI

Dedicated to the memory of Gaetano Fichera

RIASSUNTO: *Si determinano le equazioni di evoluzione di una distribuzione continua di dislocazioni nella teoria lineare dell'elasticità. Le dislocazioni sono rappresentate da un gas di particelle immateriali dotate di una struttura: esse possiedono un'inerzia di campo. Si assume, unico assioma, una legge costitutiva per il tensore cinetico dello sforzo. Si ottiene un modello con due continui interagenti governati da un sistema di 6 equazioni differenziali per lo spostamento, la densità e la velocità del gas di dislocazioni. Si studiano le proprietà di questo sistema di equazioni e si ricava una legge di bilancio energetico. Si esamina nei dettagli il caso particolarmente interessante delle dislocazioni a vite.*

ABSTRACT: *We derive in this paper equations of continuously distributed dislocations in linear elastic media, starting from a finite number of dislocation lines perpendicular to the plane of the solid. Thus, dislocations are points with a structure and the non-material (but possessing field mass) dislocation "gas" is constructed by statistical means, following known procedures of the kinetic theory. A constitutive law for the kinetic stress tensor is postulated - the only one required in this theory. The result is a mixture of two interacting continua, governed by a system of* 6 *partial differential equations, for* 3 *displacements,* 2 *dislocation gas velocities and the dislocation density.*

Energy balance law is derived from the system of equations and some general properties of the latter are examined.

One particular case is examined in more detail, namely screw dislocations.

KEY WORDS AND PHRASES: *Elastic body – Dislocation field – Waves in mixture*
A.M.S. CLASSIFICATION: 73F

1 – The governing equations

We start with a single dislocation A (fig. 1). In the present two-dimensional theory, the surface of every dislocation is the half-plane $y \geq 0$, extending in the z direction from $-\infty$ to $+\infty$. The Burgers vector b (the same for all dislocations) is assumed to be constant over the surface and in time. The surface $\overset{A}{S}$ of the dislocation A is assumed to move rigidly without rotation, its motion therefore is determined by the two coordinates $\overset{A}{\zeta}_1(t), \overset{A}{\zeta}_2(t)$ of the boundary of $\overset{A}{S}$, i.e. by the dislocation line.

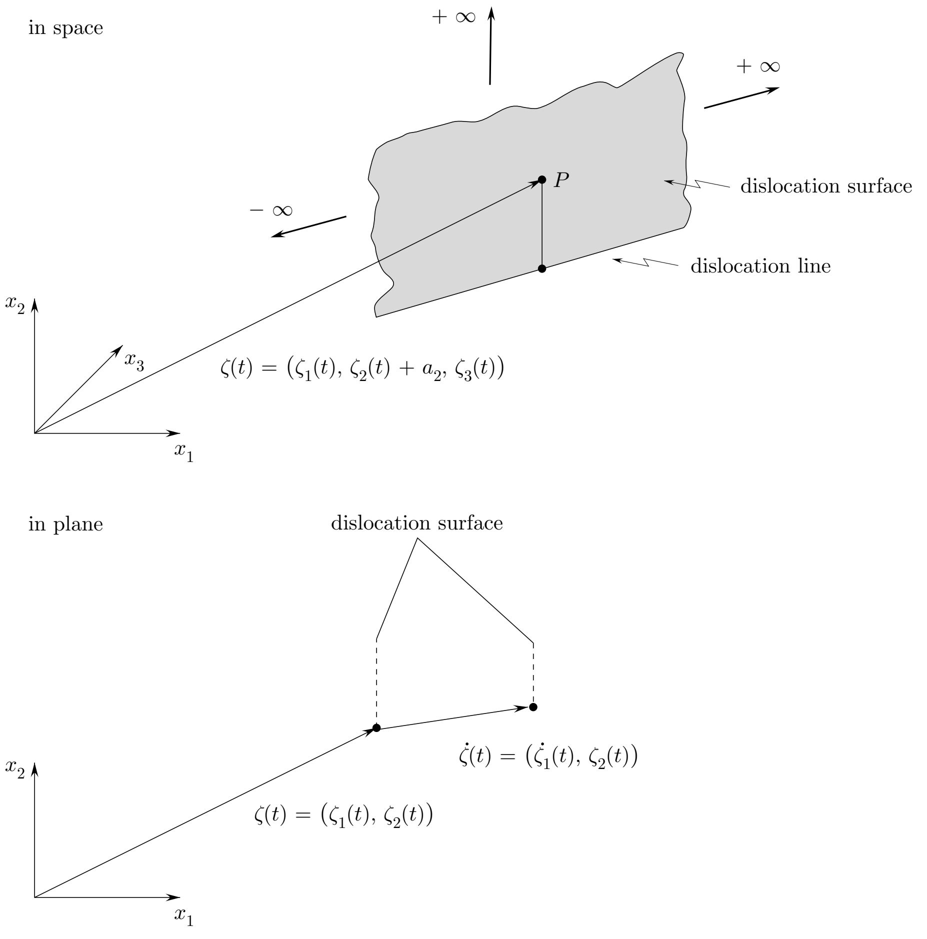

Fig. 1. – Dislocation surface and its rigid motion. $\dot{\zeta}(t)$ is the velocity of climbing.

Denote by $G_{ij}(\overset{A}{\zeta}_1(t)-x_1,\overset{A}{\zeta}_2(t)-x_2)$ the two-dimensional dynamic Green function of the linear elasticity. Then displacement $u_i(x_\alpha,t)(i=1,2,3,\alpha=1,2)$ produced by the dislocation is given by the sum of the static and dynamic parts ([2], [1])

$$-\frac{1}{\mu}\overset{A}{u}_i(x_\alpha,t)=\overset{A}{A}_i(x_\alpha,t)+\overset{A}{B}_i(x_\alpha,t) \tag{1.1}$$

where

$$\begin{aligned}\overset{A}{A}_i(x_\alpha,t)&=b_j\int_0^\infty da_2\int d\tau\sigma_{j1pq}G_{iq,p}\ ,\\ \overset{A}{B}_i(x_\alpha,t)&=\frac{1}{c_2^2}b_j\int_0^\infty da_2\int d\tau v_1(\tau)\frac{\partial}{\partial\tau}G_{ij}\ .\end{aligned} \tag{1.2}$$

$\mu\sigma_{ijpq}$ is the Hookean tensor of elastic constants. For an isotropic body $\sigma_{ijpq}=\frac{\lambda}{\mu}\delta_{ij}\delta_{pq}+\delta_{ip}\delta_{jq}+\delta_{iq}\delta_{jp}$. Time integration is extended from $-\infty$ to $t+0$. $\frac{\partial}{\partial\tau}G_{ij}$ is time derivative with $\overset{A}{\zeta}(\tau)$ held constant. Applying to $\overset{A}{A}_i(x_\alpha,t)$ the Lamé operator L_{ir} and taking into account that $L_{ir}G_{rq}=\delta_{iq}\delta(\overset{A}{\zeta}_1(\tau)-x_1)\delta(\overset{A}{\zeta}_2(\tau)-x_2)\delta(t-\tau)$, we obtain

$$\begin{aligned}L_{ir}\overset{A}{A}_r&=b_j\int_0^\infty da_2\int d\tau\sigma_{j1\beta i}\frac{\partial}{\partial\zeta_\beta}\Big[\delta\Big(\overset{A}{\zeta}_1(\tau)-x_1)\delta(\overset{A}{\zeta}_2(\tau)-x_2+a_2\Big)\delta(t-\tau)\Big]=\\ &=-b_j\frac{\partial}{\partial x_\beta}\int_0^\infty da_2\sigma_{j1\beta i}\delta\Big(\overset{A}{\zeta}_1(t)-x_1\Big)\delta\Big(\overset{A}{\zeta}_2(t)-x_2+a_2\Big)=\\ &=-b_j\sigma_{j1\beta i}\frac{\partial}{\partial x_\beta}\Big[\delta\Big(\overset{A}{\zeta}_1(t)-x_1\Big)\eta\Big(x_2-\overset{A}{\zeta}_2(t)\Big)\Big]\ ,\end{aligned}$$

for $\int_0^\infty da_2\delta(\overset{A}{\zeta}_2(t)-x_2+a_2)=\eta(x_2-\overset{A}{\zeta}_2(t))$, the Heaviside function.

Similarly

$$\begin{aligned}L_{ir}\overset{A}{B}_r&=\frac{1}{c_2^2}b_i\int_0^\infty da_2\int d\tau v_1(\tau)\Big[\delta\Big(\overset{A}{\zeta}_1(\tau)-x_1\Big)\delta\Big(\overset{A}{\zeta}_2(\tau)-x_2+a_2\Big)\frac{\partial}{\partial\tau}\delta(t-\tau)\Big]=\\ &=-\frac{1}{c_2^2}b_i\frac{\partial}{\partial t}\Big[\overset{A}{v}_1(t)\delta\Big(\overset{A}{\zeta}_1(t)-x_1\Big)\eta\Big(x_2-\overset{A}{\zeta}_2(t)\Big)\Big]\end{aligned}$$

where $\overset{A}{v}_1(t)=\dot{\overset{A}{\zeta}}_1(t)$, $(\cdot)$ denoting time differentiation. Hence

$$L_{ir}\overset{A}{u}_r(x_\alpha,t)=b_j\Big[\mu\sigma_{j1\beta i}\frac{\partial}{\partial x_\beta}+\rho\delta_{ij}\frac{\partial}{\partial t}\overset{A}{v}_1(t)\Big]\delta\Big(\overset{A}{\zeta}_1(t)-x_1\Big)\eta\Big(x_2-\overset{A}{\zeta}_2(t)\Big). \tag{1.3}$$

The right-hand side of the above relation represents the (singular) body forces producing the considered dislocation. We note that these forces are distributed over the surface of the dislocation, i.e. in our two-dimensional case over the line $0 \leq a_2 \leq \infty$.

The coordinates of the dislocation line $\overset{A}{\zeta}_\alpha(t)\alpha = 1, 2$ and its velocities are assumed to be random variables. Representation (1.3) naturally suggests an application of the Kirkwood statistical formalism based on generalised functions (see e.g. [5], [4]). The necessary definitions and formulae are given in App. 1.

Prior to deriving the equations of motion for the mixture of the elastic and dislocation continua consider the density of dislocations. The natural definition following from equation (1.3) is

$$\epsilon(\mathbf{x}, t) = \Big\langle \sum_A \delta\Big(\overset{A}{\zeta}_1(t) - x_1\Big)\eta\Big(x_2 - \overset{A}{\zeta}_2(t)\Big)\Big\rangle . \tag{1.4}$$

Substituting the local expresion $\sum_A \delta(\overset{A}{\zeta}_1(t) - x_1)\eta(x_2 - \overset{A}{\zeta}_2(t))$ into the transport law A1.3 App. 1, we obtain the continuity equation

$$\dot{\epsilon} + (\epsilon v_\alpha)_{,\alpha} = 0 \tag{1.5}$$

where

$$\epsilon v_\alpha(\mathbf{x}, t) = \Big\langle \sum_A \overset{A}{v}_\alpha(t)\delta\Big(\overset{A}{\zeta}_1(t) - x_1\Big)\eta\Big(x_2 - \overset{A}{\zeta}_2(t)\Big)\Big\rangle . \tag{1.6}$$

The dimension of ϵ is cm^{-1} and since

$$\eta\Big(x_2 - \overset{A}{\zeta}_2(t)\Big) = \int_0^\infty da_2 \delta\Big(\overset{A}{\zeta}_2(t) - x_2 + a_2\Big) = \int_{-\infty}^{x_2 - \overset{A}{\zeta}_2(t)} db\delta(b) \tag{1.7}$$

the quantity

$$\nu(\mathbf{x}, t) = \Big\langle \sum_\alpha \delta\Big(\overset{A}{\zeta}_1(t) - x_1\Big)\delta\Big(x_2 - \overset{A}{\zeta}_2(t)\Big)\Big\rangle \tag{1.8}$$

the dimension of which is cm^{-2} , is the average number of dislocation lines per unit area. We have

$$\epsilon(\mathbf{x}, t) \geq 0, \qquad \nu(\mathbf{x}, t) \geq 0 . \tag{1.9}$$

For the average of the total displacement at point (x_α, t) including a displacement $\overset{\circ}{u}_i$, which we call the complementary field, such that $L_{ir}\overset{\circ}{u}_r = 0$,

$$u_i = \overset{\circ}{u}_i + \sum_A \overset{A}{u}_i \tag{1.10}$$

averaging (1.3) and writing for simplicity $\langle u_1 \rangle = u_i$, we obtain

$$L_{ir}u_r = \rho\ddot{u}_i - \sigma_{i\beta,\beta} = \mu b_p \sigma_{p1i\beta}\epsilon_{,\beta} + \rho b_1 \dot{\overline{\epsilon v_1}} \tag{1.11}$$

where now $(\cdot) = \frac{\partial}{\partial t}, \epsilon_{,\beta} = \frac{\partial}{\partial x_\beta}$ are partial derivatives. $\epsilon(x_\alpha, t)$ is the average density of dislocation gas and $v_\alpha(x_\alpha, t)$ is its average velocity (eq. 1.10 contains v_1 only). $\sigma_{i\beta}$ is the Hookean stress tensor generated by displacement u_i. Except for the term $\rho b_i \dot{\overline{\epsilon v_i}}$ the above equation resembles that of linear thermoelasticity. In fact, it can be written in the form

$$\rho(\dot{u}_i - b_i\epsilon v_1)^{\cdot} = \overline{\sigma}_{i\beta,\beta} \tag{1.12}$$

where

$$\overline{\sigma}_{ij} = \sigma_{ij} + \mu b_q \sigma_{q1ij}\epsilon = \mu\sigma_{ijmn}\overline{e}_{mn}, \quad \overline{e}_{mn} = u_{(m,n)} + \delta_{1(m}b_{n)}\epsilon \tag{1.13}$$

is the total stress tensor. Its first part is the usual Hookean stress, while the second part is proportional to the dislocation density (temperature in thermoelasticity).

Let us now proceed to the equations of motion of the dislocation gas. Before calculating the averages, we shall make clear the meaning of the force per unit area on a single dislocation A (in gcm^{-2}), derived in App. 2, due to the displacement field $\overset{\circ}{u}_i(x_\alpha, t)$. It is easier to start with the three-dimensional case. We have (App. 2)

$$\overset{Ao}{f}_i\Big(\overset{A}{\zeta}_\alpha, t\Big) = -b_p\Big[\Big(n_q\overset{\circ}{\sigma}_{pq,i} - n_i\rho\ddot{\overset{\circ}{u}}_p\Big) + \rho\overset{A}{v}_q\Big(n_q\dot{\overset{\circ}{u}}_{p,i} - n_i\dot{\overset{\circ}{u}}_{p,q}\Big)\Big] . \tag{1.14}$$

$\overset{\circ}{\sigma}_{ij}$ denotes the stress tensor due to displacement $\overset{\circ}{u}_i$, at a surface point $x_\alpha = \overset{A}{\zeta}_\alpha(a^\Delta, t), \Delta = 1, 2$. This force consists of two parts, both linear in displacement. The static part is the known Peach-Koehler force, per unit area. The dynamic part of $\overset{Ao}{f}_i$ has the following property: its rate of work on velocity $\overset{A}{v}_i$ vanishes; in fact

$$\overset{A}{v}_i\overset{A}{v}_q\Big(n_q\dot{\overline{\overset{\circ}{u}}}_{p,i}-n_i\dot{\overline{\overset{\circ}{u}}}_{p,q}\Big)=0\ .$$

In general, the dynamic part of the above force is strictly a surface density, in the particular case however, when both n_α and $\overset{A}{v}_\alpha$ are independent of a^Δ applying Stokes' formulae we find that the total line density is

$$-\Big[\mathbf{b}\cdot\Big(\overset{\circ}{\boldsymbol{\sigma}}+\rho\dot{\overline{\overset{\circ}{\mathbf{u}}}}\overset{A}{\mathbf{v}}\Big)\Big]\times\boldsymbol{\ell} \tag{1.15}$$

rather than $-(\mathbf{b}\cdot\overset{\circ}{\boldsymbol{\sigma}})\times\boldsymbol{\ell}$.

The force on dislocation A, due to displacement $\overset{B}{u}$, produced by dislocation B has the form identical to (1.13)

$$\overset{AB}{f}_i=-b_p\Big[\Big(n_q\overset{B}{\sigma}_{pq,i}-n_i\rho\ddot{\overline{\overset{B}{u}}}_p\Big)+\rho\overset{A}{v}_q\Big(n_q\dot{\overline{\overset{B}{u}}}_{p,i}-n_i\dot{\overline{\overset{B}{u}}}_{p,q}\Big)\Big]\,. \tag{1.16}$$

We are now in a position to calculate the averages. In view of (A1.1) and (A1.10), denoting the average force at (x_α,t), due to the complementary field $\overset{\circ}{u}_i(x_\alpha,t)$, by $\epsilon(x_\alpha,t)\overset{\circ}{f}(x_\alpha,t)$ (in gcm^{-3}) we have

$$\epsilon\overset{\circ}{f}_i(x_\alpha,t)=-b_p\epsilon\Big[\Big(n_q\overset{\circ}{\sigma}_{pq,i}-n_i\rho\ddot{\overline{\overset{\circ}{u}}}_p\Big)+\rho v_q\Big(n_q\dot{\overline{\overset{\circ}{u}}}_{p,i}-n_i\dot{\overline{\overset{\circ}{u}}}_{p,q}\Big)\Big]\,. \tag{1.17}$$

Similarly, although $\overset{AB}{f}_i$ depends also on $\overset{B}{\zeta}_\alpha(\tau),\tau<t$,

$$\epsilon\overset{B}{f}_i(x_\alpha,t)=-b_p\epsilon\Big[\Big(n_q\overset{B}{\sigma}_{pq,i}-n_i\rho\ddot{\overline{\overset{B}{u}}}_p\Big)+\rho v_q\Big(n_q\dot{\overline{\overset{B}{u}}}_{p,i}-n_i\dot{\overline{\overset{B}{u}}}_{p,q}\Big)\Big] \tag{1.18}$$

and the total external average force at (x_α,t) is

$$\epsilon f_i(x_\alpha,t)=\epsilon\overset{\circ}{f}_i(x_\alpha,t)+\sum_{B\neq A}\epsilon\overset{B}{f}_i(x_\alpha,t)\,. \tag{1.19}$$

In absence of a self-force, $f_i(x_\alpha, t) = 0$. Denoting the self-force by $\epsilon \overset{AA}{f}_i(x_\alpha, t)$ and assuming that its dependence on its own displacement $\overset{A}{u}(\overset{A}{\zeta}_\alpha(t), t)$ is (1.16), we obtain again by linearity in displacement

$$\epsilon \overset{A}{f}_i(x_\alpha, t) = -b_p \epsilon[(n_q \sigma_{pq,i} - n_i \rho \ddot{u}_p) + \rho v_q (n_q \dot{u}_{p,i} - n_i \dot{u}_{p,q})] \tag{1.20}$$

where the stress tensor in the right-hand side is based on $u_i = \overset{\circ}{u} + \sum_B \overset{B}{u}$, the total displacement at point (x_α, t).

Prior to averaging we recall ([6], [3]) that the self-force is infinite and to represent it in terms of the dislocation velocity a special normalization procedure is required. For point defects (small open or closed surfaces in space) it was shown in [6] on the basis of the Wheeler-Feynman method that the required expresion has the form

$$\overset{A}{f}_\alpha = \overset{A}{m}_{\alpha\beta} \overset{\ddot{\overline{A}}}{\zeta}_\beta = \overset{A}{m}_{\alpha\beta} \overset{\dot{\overline{A}}}{v}_\beta \,. \tag{1.21}$$

Since all our dislocations are identical $\overset{A}{m}_{\alpha\beta} = m_{\alpha\beta}$. The tensorial mass in $\mathrm{gcm}^{-3}\ \mathrm{sec}^2$ in the case of isotropic elasticity was shown to be

$$\begin{aligned} m_{ij} = \mu c_2^{-5} \Delta^1 [\delta_{ij}(m_1 |b|^2 + m_2 (b \cdot n)^2) + m_3 b \cdot n n_{(i} b_{j)} + \\ + m_4 b_i b_j + m_5 |b|^2 n_i n_j] \,. \end{aligned} \tag{1.22}$$

Δ^1 is an infinite constant, independent of the motion and properties of the solid or the dislocation

$$\Delta^1 = \int \frac{\delta(y)}{y} dz$$

where $\delta(z)$ is the one-dimensional Dirac function. $m_1, \ldots, m_5$ depend on the Lamé constants only. The matrix m_{ij} is positive-definite and this is the only property of m_{ij} used in this paper.

Averaging (1.20) is standard in the kinetic theory (App. 1). Namely, in two dimensions

$$\begin{aligned} \langle \overset{A}{f}_i \rangle (x_\alpha, t) = m_{\alpha\beta} [\dot{\overline{\epsilon v_\beta}} + (\epsilon v_\beta v_\gamma)_{,\gamma}] - m_{\alpha\beta} \overset{K}{\sigma}_{\beta\gamma,\gamma} = \\ = m_{\alpha\beta} \epsilon \frac{D v_\beta}{Dt} - m_{\alpha\beta} \overset{K}{\sigma}_{\beta\gamma,\gamma} \,. \end{aligned} \tag{1.23}$$

$\overset{K}{\sigma}_{\alpha\beta}$ is the kinetic stress tensor

$$\overset{K}{\sigma}_{\alpha\beta} = \langle (v_\alpha - v)(v_\beta - v) \rangle \tag{1.24}$$

which requires a constitutive relation. Substituting (1.22) into (1.20) we arrive at the required equation of motion of the dislocation gas. Thus our system has the form $(i, p, \ldots = 1, 2, 3, \alpha, \beta, \ldots = 1, 2)$

$$\begin{aligned} &\rho \ddot{u}_i - \sigma_{i\beta,\beta} = \mu b_p \Big(\sigma_{p1i\beta} \epsilon_{,\beta} + \delta_{ip} \frac{1}{c_2^2} \dot{\overline{\epsilon v_1}} \Big) \\ &\epsilon m_{\alpha\beta} \frac{Dv_\beta}{Dt} = m_{\alpha\beta} \overset{K}{\sigma}_{\beta\gamma,\gamma} - b_p \epsilon [(\sigma_{1p,\alpha} - \delta_{\alpha 1} \rho \ddot{u}_p) + \\ &\qquad\qquad + \rho (v_1 \dot{u}_{p,\alpha} - \delta_{\alpha 1} v_\beta \dot{u}_{p,\beta})] \\ &\dot{\epsilon} + (\epsilon v_\alpha)_{,\alpha} = 0 \ . \end{aligned} \tag{1.25}$$

The second of the above equations can be transformed as follows. Since $m_{\alpha\beta}$ is invertible, we multiply the equation by $m^{-1}_{\gamma\alpha}$, denote $\frac{1}{\rho} m_{\alpha\beta}$ by $\bar{m}_{\alpha\beta}$ and substitute for $\ddot{u}_i$ from the first equation (1.24); then

$$\begin{aligned} \epsilon \frac{Dv_\alpha}{Dt} = &\overset{K}{\sigma}_{\alpha\gamma,\gamma} - b_p \epsilon \bar{m}^{-1}_{\alpha\gamma} \Big[\frac{1}{\rho} (\sigma_{ip,\gamma} - \delta_{\gamma 1} \sigma_{p\beta,\beta}) + (v_1 \dot{u}_{p,\gamma} - \delta_{\gamma 1} v_\beta \dot{u}_{p,\beta}) \Big] \\ &- b_p b_q \epsilon \bar{m}^{-1}_{\alpha 1} (c_2^2 \sigma_{q1p\beta} \epsilon_{,\beta} + \delta_{pq} \dot{\overline{\epsilon v_1}}) \ . \end{aligned} \tag{1.26}$$

The first term in the square bracket is the static part of the Peach-Koehler force. The last term is proportional to the dimensionless parameter $\beta = \frac{|b|^2 \bar{\epsilon}}{\bar{m}}$ where $\bar{\epsilon}$ and $\bar{m}$ are some typical values of the density of dislocations and its mass, respectively. (1.24) is a system of 6 partial differential equations for u_i, ϵ and v_α, provided $\overset{K}{\sigma}_{\alpha\beta}$ is given in terms of these quantities. This constitutive relation in what follows will be assumed to have the form

$$\overset{K}{\sigma} = \overset{K}{\sigma}(\epsilon, \epsilon_{,\alpha}) \ . \tag{1.27}$$

System (1.24) is conservative and implies the energy balance. To derive it we multiply $(1.25)^1$ by $\dot{u}_i$, $(1.25)^2$ by v_α. Adding the results, using $(1.25)^3$ and taking into account the identities

$$\epsilon_{,\beta}\dot{u}_i = (\epsilon\dot{u}_i)_{,\beta} - \dot{\overline{\epsilon u_{i,\beta}}} - (\epsilon v_\gamma)_{,\gamma} u_{i,\beta}$$

$$\epsilon u_{i,\beta\gamma} v_\gamma = (\epsilon u_{i,\beta} v_\gamma)_{,\gamma} - u_{i,\beta}(\epsilon v_\gamma)_{,\gamma}$$

we arrive at the energy balance

$$\dot{E} + P_{\alpha,\alpha} = -m_{\alpha\beta} v_{\alpha,\gamma} \overset{K}{\sigma}_{\beta\gamma} \tag{1.28}$$

where

$$\begin{aligned} E &= K^{el} + \phi^{el} + K^d + b_p\epsilon(\sigma_{p1} - \rho v_1 \dot{u}_p) \\ P_\alpha &= -\sigma_{p\alpha}\dot{u}_p + K^d v_\alpha - m_{\beta\gamma} v_\beta \overset{K}{\sigma}_{\gamma\alpha} + b_p\epsilon(v_\alpha \sigma_{p1} - \mu\sigma_{p1m\alpha}\dot{u}_m)\ . \end{aligned} \tag{1.29}$$

K^{el} and K^d are the kinetic energies of the elastic body and the dislocation gas, respectively. $\phi^{el} = \frac{1}{2}\sigma_{p\alpha} u_{p,\alpha}$ is the Hookean elastic energy. The right-hand side in (1.28) is the rate of work of the kinetic dislocational stress tensor. Both, the total energy density E and the energy flux vector P_α contain terms due to the interaction between the two continua, all proportional to the dimensionless vector ϵb_i.

2 – Screw dislocations

We assume in this section that $b_1 = b_2 = 0$ and we write $b_3 = b$. To simplify the problem we neglect the dynamic part of the force on a dislocation (the second term in the square bracket in (1.26)) and the term proportional to β. The system of equations of motion and the continuity equation $(1.25)^1$, $(1.25)^3$, (1.26) is split into the homogeneous purely elastic plane problem for u_1, u_2 and the system of four equations

$$\begin{aligned} &\ddot{u}_3 - c_2^2\nabla^2 u_3 = b(c_2^2\epsilon_{,1} + \dot{\overline{\epsilon v_1}}) \\ &\bar{m}_{1\beta}\epsilon[\dot{v}_\beta + (v_\gamma v_\beta)_{,\gamma}] = \bar{m}_{1\beta}\overset{K}{\sigma}_{\beta\gamma,\gamma} + \frac{1}{\rho} b\epsilon\sigma_{32,2} \\ &\bar{m}_{2\beta}\epsilon[\dot{v}_\beta + (v_\gamma v_\beta)_{,\gamma}] = \bar{m}_{2\beta}\overset{K}{\sigma}_{\beta\gamma,\gamma} - \frac{1}{\rho} b\epsilon\sigma_{31,2} \\ &\dot{\epsilon} + (\epsilon v_\gamma)_{,\gamma} = 0 \end{aligned} \tag{2.1}$$

where $\sigma_{3\alpha} = \mu u_{3,\alpha}, \alpha = 1, 2$.

To close the system we shall employ the following constitutive relation for the kinetic stress tensor:

$$\overset{K}{\sigma}_{\alpha\beta} = -\delta_{\alpha\beta}(R\epsilon + r_\gamma \epsilon_{,\gamma}) \tag{2.2}$$

where R and r_γ are constant, R is the customary gas constant equal to the square of the sound velocity in the dislocation gas. Vector r_α is the first nonlocality coefficient; its meaning will be clarified later in this section.

First, consider the hyperbolicity of system (2.1) with $r_\alpha = 0$. To this end (see e.g. [7]) we rewrite our system in the form of a first order system. Denote $u_{3,1} = y_1, u_{3,2} = y_2, \dot{u}_3 = y_3$, $v_1 = y_4, v_2 = y_5, \epsilon = y_6$ and complete the system by the compatibility conditions $\dot{y}_1 = y_{3,1}, \dot{y}_2 = y_{3,2}$. Now

$$A^0_{KL}\dot{y}_L + A^\alpha_{KL} u_{L,\alpha} = 0 \qquad K, L = 1, \ldots, 6 \tag{2.3}$$

is the system (2.1) and the compatibility conditions, in the required form. Here the matrices $\mathbf{A}^0, \mathbf{A}^\alpha$ are

$$\mathbf{A}^0 = \begin{pmatrix} 0 & 0 & 1 & -by_6 & 0 & -by_4 \\ 0 & 0 & 0 & y_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.4}$$

$$\mathbf{A}^1 = \begin{pmatrix} -c_2^2 & 0 & 0 & 0 & 0 & -bc_2^2 \\ 0 & \alpha_{12}y_6 & 0 & y_6y_4 & 0 & R \\ 0 & \alpha_{22}y_6 & 0 & 0 & y_6y_4 & 0 \\ 0 & 0 & 0 & y_6 & 0 & y_4 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A}^2 = \begin{pmatrix} 0 & -c_2^2 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_{11}y_6 & 0 & y_6y_5 & 0 & 0 \\ 0 & -\alpha_{21}y_6 & 0 & 0 & y_6y_5 & R \\ 0 & 0 & 0 & 0 & y_6 & y_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.4}$$

where $\alpha_{\alpha\beta} = c_2^2 b\bar{m}_{\alpha\beta}^{-1}$. Equating to zero the determinant

$$\det(\omega \mathbf{A}^0 + \omega_\alpha \mathbf{A}^\alpha) \tag{2.5}$$

we obtain after some calculations

$$\det(\omega \mathbf{A}^0 + \omega_\alpha \mathbf{A}^\alpha) = \omega D(\omega, \omega_\alpha) = 0 \tag{2.6}$$

where

$$\begin{aligned} D(\omega, \omega_\alpha) = {} & \dot{\omega}[\dot{\omega}^2 - R|\boldsymbol{\omega}|^2][\omega^2 - c_2^2|\boldsymbol{\omega}|^2] + \\ & + b^2 c_2^2 \epsilon \omega_2 \{\alpha_1[\omega(\dot{\omega}^2 - R\omega_2^2) - \dot{\omega}\omega_1(v_1\omega + \omega_1)] + \\ & + \alpha_2 \omega_2 [R\omega\omega_1 - (v_1\omega + c_2^2\omega_1)\dot{\omega}]\} \end{aligned} \tag{2.7}$$

and we have introduced the notation $\dot{\omega} = \omega + v_\alpha \omega_\alpha, \alpha_1 = \bar{m}_{12}^{-1}\omega_1 - \bar{m}_{11}^{-1}\omega_2$, $\alpha_2 = \bar{m}_{22}^{-1}\omega_1 - \bar{m}_{21}^{-1}\omega_2, |\boldsymbol{\omega}|^2 = \omega_1^2 + \omega_2^2$. Thus, there is a solution $\omega = 0$, due to the introduction of the compatibility conditions. The first part in $D(\omega, \omega_\alpha)$ is independent of the coupling between the two continua and clearly corresponds to their independent characteristics. The second part is proportional to $\beta = \frac{b^2\bar{\epsilon}}{\bar{m}}$. The whole $D(\omega, \omega_\alpha)$ is homogeneous of degree 5 in (ω, ω_α). The second part vanishes when $\omega_2 = 0$.

The dispersion equation for $\omega(\mathbf{k})$ of the problem linearized around a constant solution $(\mathbf{u}_0 = 0, \mathbf{v}_0, \epsilon_0)$ is obtained from (2.6) by setting $\omega_\alpha = -k_\alpha, \epsilon = \epsilon_0, \mathbf{v} = \mathbf{v}_0$. In the particular case when $\mathbf{v}_0 = 0$ we have for $D(\omega_, k_\alpha)$

$$\begin{aligned} & \omega\{(\omega^2 - R|\mathbf{k}|^2)(\omega^2 - c_2^2|\mathbf{k}|^2) + \\ & \quad - b^2 c_2^2 \epsilon_0 k_2[\alpha_1(\omega^2 - Rk_2^2 - c_2^2 k_1^2) + \alpha_2 k_1 k_2 (R - c_2^2)]\} = 0\,. \end{aligned} \tag{2.8}$$

To see the role played by some of the constants in the problem let us examine (2.8) assuming that $\bar{m}_{12}^{-1} = 0$, i.e. $\alpha_1 = \bar{m}_{11}^{-1}\omega_2, \alpha_2 = \bar{m}_{22}^{-1}\omega$. Denoting $n_1 = \bar{m}_{11}^{-1}\bar{m}, n_2 = \bar{m}_{22}^{-1}\bar{m}$ where $\bar{m}$ is the typical mass entering β and $\bar{\epsilon}$ is a typical density, besides the solution $\omega = 0$ we obtain a quadratic equation for ω^2 namely

$$\omega^n + B\omega^2 + C = 0 \tag{2.9}$$

where

$$B = -[(R + c_2^2)|\mathbf{k}|^2 + \beta c_2^2 n_1 k_2^2] < 0$$

$$C = Rc_2^2(k_1^4 + bk_1^2k_2^2 + ck_2^4)\ ,$$

$$b = 2 + \frac{\beta}{R}[c_2^2 n_1 + (R - c_2^2)n_2]\ ,$$

$$c = 1 + \beta n_1 > 0\,.$$

For $C = 0, \omega_{1,2} = 0, \omega_{3,4} = \pm\sqrt{B}.C > 0$ for arbitrary $\mathbf{k}, |k| > 0$ iff either $b^2 - 4c < 0$ or $b^2 - 4c \geq 0$ and $b > 0$. Then

$$\omega^2 = \frac{1}{2}(-B \pm \sqrt{B^2 - 4C})$$

and both solutions for ω^2 are real and positive iff $B^2 - 4C > 0$. Now, $B^2 - 4C = a'k_1^4 + b'k_1^2k_2^2 + c'k_2^4$ where $a' = (R - c_2^2)^2 \geq 0, b' = -2(R - c_2^2)[(R - c_2^2) - c_2^2\beta(n_1 - 2n_2)], c' = [(R - c_2^2) - c_2^2\beta n_1]^2 \geq 0, b'^2 - 4a'c' = 16(R - c_2^2)^2 c_2^2 \beta n_2[(R - c_2^2) - c_2^2\beta(n_2 - n_1)]$.

To examine travelling waves we assume that all unknown functions depend on one variable $\xi = k_\alpha x_\alpha - Vt, |k|^2 = k_1^2 + k_2^2 = 1$. Then the system (2.1) takes the form (after integrating once equation $(2.1)^1$) of a system of four ordinary differential equations

$$\begin{aligned}
&(V^2 - c_2^2)u_3' = b\epsilon(c_2^2 k_1 - Vv_1) + c_3, \quad c_3 = \text{const.}\\
&K\bar{m}_{1\beta}v_\beta' = \bar{m}_{1\beta}k_\gamma \overset{K}{\sigma}{}'_{\beta\gamma} + c_2^2 b k_2^2 \epsilon u_3''\\
&K\bar{m}_{2\beta}v_\beta' = \bar{m}_{2\beta}k_\gamma \overset{K}{\sigma}{}'_{\beta\gamma} - c_2^2 b k_1 k_2 \epsilon u_3''\\
&\epsilon(k_\gamma v_\gamma - V) = K\,.
\end{aligned} \tag{2.10}$$

The last equation is algebraic. It should be born in mind that $\operatorname{sign} K = \operatorname{sign}(k_\gamma v_\gamma - V)$; otherwise K is constant. We observe after multiplying equations $(2.10)^{2,3}$ by $\bar{m}_{\alpha\beta}^{-1}$ that the terms containing the displacement u_3'' are proportional to the parameter $\beta = \frac{b^2\epsilon}{\bar{m}}$ and in this analysis they will be neglected. The constitutive relation for the kinetic stress tensor $\overset{K}{\sigma}_{\alpha\beta}$ is taken in the form

$$\overset{K}{\sigma}_{\alpha\beta} = -\delta_{\alpha\beta}(R\epsilon + r_\gamma \epsilon_{,\gamma}) = -\delta_{\alpha\beta}R(\epsilon + \bar{r}_\gamma \epsilon_{,\gamma}) \tag{2.11}$$

R is here as before the square of the sound velocity in the dislocation gas and r_γ or $\bar{r}_\gamma = \frac{r_\gamma}{R}$ is the coefficient of nonlocality. In the variable ξ we have

$$\overset{K}{\sigma}_{\alpha\beta} = -\delta_{\alpha\beta} R(\epsilon + \bar{r}_\gamma k_\gamma \epsilon') \; . \tag{2.12}$$

Now the system is separated: we first solve equations $(2.10)^{2,3,4}$ for the dislocation gas and then the displacement u_3 is determined from $(2.10)^1$. Thus, integrating once (note that $\bar{m}_{\alpha\beta}$ is eliminated)

$$K v_\alpha = -k_\alpha R(\epsilon + \bar{r}_\gamma k_\gamma \epsilon') + C_\alpha \tag{2.13}$$

and substituting into $(2.10)^4$ we obtain a first order differential equation for the density

$$\bar{r}\epsilon\epsilon' + (\epsilon^2 + 2\bar{D}\epsilon + \bar{K}^2) = 0 \tag{2.14}$$

where the new constants are $\bar{D} = \frac{1}{2R}(KV - C_\alpha K_\alpha), \bar{K}^2 = \frac{1}{R}K^2$. We are interested in a (physical) solution $\epsilon > 0$, bounded on the whole line $\xi \in (-\infty, +\infty)$, we assume therefore that $\bar{D} < 0$ and $\bar{D}^2 > \bar{K}^2$. Then the required solution for $\epsilon(\xi)$ can be written in the form (fig. 2)

$$\epsilon_1 \ln \frac{\epsilon_1 - \epsilon}{\epsilon_1 - \epsilon_0} - \epsilon_2 \ln \frac{\epsilon - \epsilon_2}{\epsilon_0 - \epsilon_2} = -\frac{\epsilon_1 - \epsilon_2}{\bar{r}}(\xi - \xi_0), \qquad \epsilon(\xi_0) = \epsilon_0 \tag{2.15}$$

where for $\bar{r} > 0$, $\epsilon_1 = \epsilon(\infty) = |\bar{D}| + \sqrt{\bar{D}^2 - \bar{K}^2}, \epsilon_2 = \epsilon(-\infty) = |\bar{D}| - \sqrt{\bar{D}^2 - \bar{K}^2}$. We note that for $\bar{r} < 0, \epsilon(-\infty) = \epsilon_1, \epsilon(\infty) = \epsilon_2$. Figure 2 shows that we are faced with a kink.

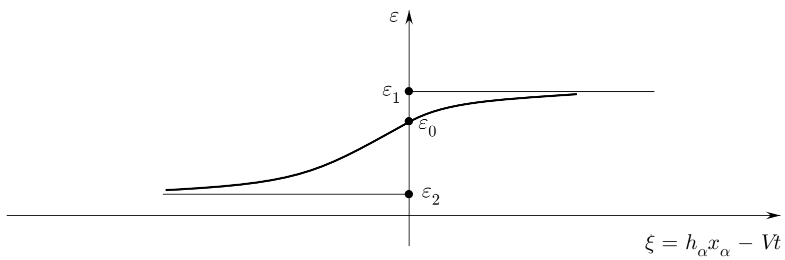

Fig. 2. – Density kink in traveling waves, screw dislocations.

The constants appearing in the above solution in terms of the values of $\epsilon(-\infty)$ and $v_\alpha(-\infty)$ follow from equations (2.10), (2.13), namely

$$
\begin{aligned}
&K = \epsilon(-\infty)[k_\alpha v_\alpha(-\infty) - V]\\
&C_\alpha = \epsilon(-\infty)[Rk_\alpha + v_\alpha(-\infty)(k_\beta v_\beta(-\infty) - V)^2]\\
&2R\bar{D} = -\epsilon(-\infty)[R + (k_\beta v_\beta(-\infty) - V)^2] < 0\\
&4R^2(\bar{D}^2 - \bar{K}^2) = \epsilon^2(-\infty)[R - (k_\beta v_\beta(-\infty) - V)^2]^2 \geq 0\,.
\end{aligned}
\tag{2.16}
$$

To avoid trivial or nonphysical solutions we assume that $k_\alpha v_\alpha(-\infty) - V \neq 0$ and $k_\alpha v_\alpha(-\infty) - V \neq R$. Furthermore to solve for $u_3(\xi), V^2 \neq c_2^2$.

The Hookean stress $\frac{1}{\mu}\sigma_{\alpha 3} = u_3' k_\alpha$ has the form

$$
\frac{1}{\mu}\sigma_{\alpha 3} = \frac{1}{V^2 - c_2^2} k_\alpha (A + B\epsilon)
\tag{2.17}
$$

where

$$
A = c_3 - VKk_1 b, \quad B = b\{c_2^2 k_1 - V(k_1(k_\beta v_\beta(-\infty) - V) - v_1(-\infty))]\}\,.
\tag{2.18}
$$

If $k_2 = 0$, $k_1 = 1$ i.e. $\xi = x_1 - Vt$ the terms containing u_3'' (2.10) vanish and our solution is valid for arbitrary β. Now the constants K, C_α, etc. have a simpler form, e.g.

$$
A = c_3 - V\epsilon(-\infty)[v_1(-\infty) - V]b\,, \qquad B = b(c_2^2 + V^2)\,.
$$

Observe that since the terms containing u_3'' have been neglected, (2.15) is the solution of a pure gas dynamics problem with the constitutive relation (2.12). However, the more general problem without the above assumption of $\beta \ll 1$, can be solved in similar manner.

– Appendix 1. Statistical mechanics ([5], [4])

The phase space is the space of pairs $\Gamma = \{\overset{A}{\zeta}_\alpha(t), \overset{A}{v}_\alpha(t)\}, \alpha = 1, 2; A = 1, 2, \ldots, N_A$ (the number of dislocation lines). $F(\{\overset{A}{\zeta}_\alpha(t)\}, \{\overset{A}{v}_\alpha(t)\})$ is the distribution function obeying the Liouville equation, so that

$$
\frac{d}{dt}(F d\Gamma) = 0\,.
\tag{A1.1}
$$

We define the local average at point (x,t) as follows:

$$P(x,t) = \langle P \rangle (x,t) = \int d\Gamma F \widehat{P}\Big(\Big\{\overset{A}{\zeta}_\alpha(t), \overset{A}{v}_\alpha(t)\Big\}; x, t\Big) . \tag{A1.2}$$

Then, in view of (A1.1) we have the transport theorem

$$\frac{\partial}{\partial t} P(x,t) = \int d\Gamma F \frac{d\widehat{P}}{dt} . \tag{A1.3}$$

In particular, if

$$\widehat{P} = \sum_A \delta\Big(\overset{A}{\zeta}_1(t) - x_1)\delta(\overset{A}{\zeta}_2(t) - x_2\Big) \tag{A1.4}$$

we define $P(x,t)$ to be the density $\nu(x,t)(cm^{-2})$, i.e. the average number of points in the plane or, the average number of dislocation lines. The transport theorem (A1.3) now yields the continuity equation (we denote $\frac{\partial}{\partial t}$ by $(\cdot)$)

$$\dot{\nu} + (v_\alpha \nu)_{,\alpha} = 0 \tag{A1.5}$$

where

$$\nu(x,t) v_\alpha(x,t) = \int d\Gamma F \sum_A \overset{A}{v}_\alpha(t) \delta\Big(\overset{A}{\zeta}_1(t) - x_1\Big)\delta\Big(\overset{A}{\zeta}_2(t) - x_2\Big) . \tag{A1.6}$$

Similarly, since

$$\int_0^\infty da_2 \delta(\zeta_2 + a_2 - x_2) = \eta(x_2 - \zeta_2)$$

replacing in (A1.4) $\delta(\overset{A}{\zeta}_2(t) - x_2)$ by $\delta(\overset{A}{\zeta}_2(t) + a_2 - x_2)$ and integrating over a_2 we have a new density

$$\epsilon(x,t) = \Big\langle \sum_A \delta\Big(\overset{A}{\zeta}_1(t) - x_1\Big)\eta\Big(x_2 - \overset{A}{\zeta}_2(t)\Big)\Big\rangle \tag{A1.7}$$

and bearing in mind taht $\overset{A}{v}_\alpha(t)$ is independent of a_2 we obtain the new continuity equation

$$\dot{\epsilon} + (v_\alpha \epsilon)_{,\alpha} = 0 \tag{A1.8}$$

where

$$\epsilon(x,t) v_\alpha(x,t) = \int d\Gamma F \sum_A \overset{A}{v}_\alpha(t) \delta\Big(\overset{A}{\zeta}_1(t) - x_1\Big) \eta\Big(x_2 - \overset{A}{\zeta}_2(t)\Big) \ . \tag{A1.9}$$

Finally we note that if

$$\widehat{P} = \sum_A \delta\Big(\overset{A}{\zeta}_1(t) - x_1\Big) \eta\Big(x_2 - \overset{A}{\zeta}_2(t)\Big) \cdot f(x_\alpha, t)$$

then

$$\langle f(x_\alpha, t)\rangle = \epsilon(x,t) f(x_\alpha, t) \tag{A1.10}$$

for an arbitrary function $f(x_\alpha, t)$.

– Appendix 2. Force on a dislocation

In order to simplify the procedure, bearing in mind that in this paper the dislocation surface is plane, we assume from the begining that the normal n_i is constant and the surface element $da = da^1 da^2$ where a^Δ, $\Delta = 1, 2$, are Cartesian coordinates on the surface.

Consider the Lagrangian containing a displacement field $\mathbf{u}(\mathbf{x}, t)$ external to the dislocation

$$L\{u_i\} = \int_0^t \int_s da b_p t_{(n)p}(u_i) \qquad i = 1, 2, 3 \tag{A2.1}$$

where $t_{(n)p}\mathbf{u} = \sigma_{(n)p}\mathbf{u} + \rho n_q \dot{\zeta}_q \frac{\partial}{\partial \tau} u_p$ is the dynamic stress vector on the surface, $\sigma_{(n)p}\mathbf{u} = n_r \sigma_{rp}(\mathbf{u})$. Thus, the Lagrangian density is the work done by the dynamic stress vector on the displacement discontinuity, i.e. Burgers vector b_i. The displacement $u_i(\zeta_p, t) = \lim_{\mathbf{x}\to\boldsymbol{\zeta}} \mathbf{u}(x,t)$ and its derivatives are assumed to be smooth and given in the vicinity of the surface. Consequently, the Lagrangian density depends on $\boldsymbol{\zeta}(a^b, t)$ and $\dot{\boldsymbol{\zeta}}(a^b, t)$, derivatives with respect to a^b being absent, and on point $(\mathbf{x}, t)$ through $\mathbf{u}(\mathbf{x}, t)$.

We define the density of the force, i.e. the force per unit area on the dislocation due to the displacement field $\mathbf{u}$ as the variational derivative

$$f_i(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}) = -\left(\frac{\partial}{\partial \zeta_i} - \frac{d}{dt}\frac{\partial}{\partial \dot{\zeta}_i}\right)(b_p t_{(n)p}\mathbf{u}) \tag{A2.2}$$

where $\frac{d}{dt}$ denotes the total derivative. Hence

$$f_i(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}) = -b_p[(n_q\sigma_{pq,i} - n_i\rho\ddot{u}_p) + \rho\dot{\zeta}_q(n_q\dot{u}_{p,i} - n_i\dot{u}_{p,q})] \,. \tag{A2.3}$$

Since in our simplified model of the dislocation in a two-dimensional solid $n_i = (1, 0, 0)$, $\frac{\partial}{\partial x_3} = 0$

$$f_\alpha(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}) = -b_p[(\sigma_{p1,\alpha} - \delta_{1\alpha}\rho\ddot{u}_p) + \rho\dot{\zeta}_\beta(\delta_{\beta 1}\dot{u}_{p,\alpha} - \delta_{\alpha 1}\dot{u}_{p,\beta})] \quad \alpha = 1, 2 \,. \tag{A2.4}$$

Two remarks are in order here. First, since the field $\mathbf{u}(\mathbf{x}, t)$ satisfies homogeneous Lamé equation in the vinicity of s, $\rho\ddot{u}_p = \sigma_{pq,q}$ and the first part of the force takes the "static" form

$$-b_p(\sigma_{p1,\alpha} - \delta_{\alpha 1}\sigma_{p\beta,\beta}) \,. \tag{A2.5}$$

Secondly, the second, dynamic part of the forcr does not contribute to the rate of work on velocity $\dot{\zeta}_\alpha$ (similarly to the magnetic part of the Lorentz force on a charged particle). Thus

$$f_\alpha\dot{\zeta}_\alpha = -b_p(\dot{\zeta}_\alpha\sigma_{p1,\alpha} - \dot{\zeta}_1\rho\ddot{u}_p) \,. \tag{A2.6}$$

The formally static part of f_α (A2.5) is the Peach-Koehler force per unit area. In fact, let us integrate it over the dislocation surface s setting in the Stokes' formula

$$\int_s ds n_r \varepsilon_{rqs} T_{iq,s} = \int_l dl l_q T_{iq} \,,$$

$T_{iq} = \varepsilon_{ipq}\sigma_{mp}b_m$. Then

$$b_p\int_s ds(n_q\sigma_{pq,i} - n_i\sigma_{pq,q}) = b_m\int_l dl l_q\varepsilon_{ipq}\sigma_{mp} \,. \tag{A2.7}$$

The line density in the right-hand side $(\sigma \cdot b) \times l$ is the Peach-Koehler force per unit boundary of the dislocation surface ([1]).

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Lavoro pervenuto alla redazione il 1 marzo 2000
Bozze licenziate il 9 novembre 2000

Indirizzo dell'Autore:

H. Zorski – Institute of Fundamental Technological Research – Polish Academy of Sciences – 00-049 Warsaw – Świętokrzyska 21