Rendiconti di Matematica, Serie VII
Volume 20, Roma (2000), 263-279

# On the eigenvalues of the rhombical membrane

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*Dedicated to the memory of Prof. G. Fichera*

RIASSUNTO: *Vengono calcolate approssimazioni per difetto e per eccesso dei primi autovalori di una membrana rombica libera sul bordo. Le approssimazioni per eccesso vengono ottenute per mezzo del metodo di Rayleigh-Ritz. Quelle per difetto sono calcolate con un metodo che si fonda sulla teoria degli invarianti ortogonali e che si ispira, data la mancanza di una funzione di Green esplicitamente nota, a quello proposto da G. Fichera in un suo lavoro sugli autovalori nel problema di Neumann.*

ABSTRACT: *In this paper both upper bounds and lower bounds are given for the first eigenvalues of a rhombical membrane with free boundary. The upper bounds are obtained through an application of the Rayleigh-Ritz method. The lower bounds are calculated by means of a procedure relying on the Orthogonal Invariants method, which is inspired, in view of the lack of a known Green function, by that one developed by G. Fichera in a work on the Neumann eigenvalue problem.*

## 1 – Introduction

In the work [3] G. FICHERA addressed the problem of estimating the eigenvalues of the Neumann problem for the Laplace operator in a domain $A$ for which the Green function is not known. The hypotheses on the domain $A$ are quite general: indeed, it is assumed that there exists

KEY WORDS AND PHRASES: *Eigenvalue problems – Orthogonal Invariants.*
A.M.S. CLASSIFICATION: 65N25

a bi-lipschitzian homeomorphism between $A$ and the unitn disc $D$. The theory developed is easily extendable to handle the case of domains $D$ different from discs, for which however the Green function is known.

The upper bounds are obtained by means of the celebrated Rayleigh-Ritz method. The lower bounds are found through an argument inspired by previous work by Weinsteien and Aronszajn (see e.g. [10] and [2]), which involves the lower estimate of the eigenvalues of a sequence of suitable "intermediate" operators. An extension of these results for the case of the elasticity operator with stress null conditions on the boundary is contained in [1].

In spite of the fact that several years have passed since the appearance of [3] and in spite of the interest both theoric and applicative of his results, as far as the present note's authors know, no computational applications of these results have been performed.

In this paper the theory developed in [3] is applied with some slight modifications to study the case of a rhombical membrane. The domain $D$ is assumed to be a square. In such a case one could apply as well the theory exposed in [10], but the aim of the paper is to show the efficiency of the methods of [3].

## 2 – The Neumann problem

Let $A \subset \mathbf{R}^2$ be a rhombus centered at the origin of the coordinate axes and having sides of lenght 2. $A$ is defined through the formulae:

$$-\sin\vartheta < y < \sin\vartheta$$
$$\frac{y}{\tan\vartheta} - 1 < x < \frac{y}{\tan\vartheta} + 1$$

where $\vartheta \in (0, \pi/2)$.

Define in the Sobolev space $H_1(A)$ the bilinear form

$$B(u,v) = \iint_A (u_x v_x + u_y v_y)\; dx\, dy \tag{1}$$

and set

$$(u,v)_{0,A} = \iint_A u\; v\; dx\, dy\,.$$

Now, consider the eigenvalue problem

$$(2)\qquad \begin{cases} \Delta u + \lambda u = 0 & (x,y) \in A \\ \dfrac{\partial u}{\partial \nu} = 0 & (x,y) \in \partial A \end{cases}$$

where $u \in H_1(A)$.

As known, the eigenvalues of (2) constitue a sequence

$$0 = \lambda_0 < \lambda_1 \leq \ldots \leq \lambda_k \leq \ldots$$

where each eigenvalue has finite multiplicity. The eigenvalue problem (2) is equivalent to the following functional problem:

$$(3)\qquad B(u,v) = \lambda\ (u,v)_{0,A}\ , \qquad u \in H_1(A)\,, \quad \forall\, v \in H_1(A)\,.$$

Let $\mathcal{H}_1(A)$ denote the space of the functions $v \in H_1(A)$ determined by the condition

$$\iint_A v(x,y)\ dx\, dy = 0\,.$$

The problem (3), when considered in $\mathcal{H}_1(A)$, admits only the positive eigenvalues and precisely those are the eigenvalues we will try and calculate.

Let $D$ be the square $(-1,1) \times (-1,1)$ of the plane $(\xi,\eta)$ and let us consider the following map, which maps $\overline{A}$ in $\overline{D}$:

$$(4)\qquad \begin{cases} x = & \xi + \eta\ \cos\vartheta \\ y = & \eta\ \sin\vartheta\,. \end{cases}$$

We can as well give the inverse functions

$$(5)\qquad \begin{cases} \xi = & x - \dfrac{y}{\tan\vartheta} \\ \eta = & \dfrac{y}{\sin\vartheta}\,. \end{cases}$$

The jacobian determinants of (4) and (5) are

$$(6)\qquad \begin{aligned} I(x,y) &= \frac{1}{\sin\vartheta} \\ J(\xi,\eta) &= \sin\vartheta\,. \end{aligned}$$

In view of (4) and (5) we can say that every function which is an element of $H_1(A)$ also belongs to the space $\mathcal{H}_1(A)^{(1)}$. Because of that, we will denote functions by the same letter, without specifying their dependence on the variables $(x, y)$ or $(\xi, \eta)$.

Notice that from (1), (4), (5) and (6) one has:

$$B(u,u) = \frac{1}{\sin\vartheta} \iint_A (u_\xi^2 + u_\eta^2 - 2\cos\vartheta\ u_\xi\ u_\eta)\ d\xi\, d\eta\,. \tag{7}$$

## 3 – Upper bounds of the eigenvalues

In order to get upper bounds of the eigenvalues of (2), the Rayleigh-Ritz method has been employed.

It amounts to choosing a system of functions, $\{v_k\}$ with $k = 1, 2, \dots,$ complete in the space $\mathcal{H}_1(A)$, and calculating, for a fixed index $\nu$, the solutions of the secular equation

$$\det((B(v_h, v_k) - \Lambda\ (v_h, v_k)_{0,A})) = 0, \qquad k = 1, \dots, \nu\,. \tag{8}$$

We denote the roots of the equation (8) by

$$\Lambda_1^\nu \le \Lambda_2^\nu \le \dots \le \Lambda_\nu^\nu\ . \tag{9}$$

For every integer $k = 1, \dots, \nu$, one has:

$$\lambda_k \le \Lambda_k^{\nu+1} \le \Lambda_k^\nu\,.$$

For the problem at hand we have choosen the functions

$$v_{nh}(x,y) = x^{n-h} y^h + \gamma_{nh} \quad, \ n = 1, 2, \dots, \quad h = 0, \dots, n \tag{10}$$

(1) In a similar way as for $H_1(A)$, we will denote $\mathcal{H}_1(D)$ the set of functions $v \in H_1(D)$ such that

$$\iint_D v(\xi,\eta)\ J(\xi,\eta)\ d\xi\, d\eta = 0\,.$$

with

$$\gamma_{nh} = \begin{cases} -\sin^h \vartheta \displaystyle\sum_{i=0}^{[\frac{n-h}{2}]} \binom{n-h}{2i} \frac{\cos^{n-h-2i} \vartheta}{(2i+1)(n-2i+1)} & n \text{ even} \\ 0 & n \text{ odd}. \end{cases}$$

For a fixed index $\overline{n}$, it is $n = 1, \dots, \overline{n}$ and $h = 0, 1, \dots, \overline{n}$ e $\nu = \overline{n}(\overline{n}+3)$.

Also, the formula (10) implies:

$$\begin{aligned} B(v_{nh}, v_{mk}) = 2\sin^{h+k} \vartheta \Big( & \sin\vartheta \ (n-h)(m-k)\times \\ & \times \sum_{l=0}^{[\frac{n-h+m-k-2}{2}]} \binom{n-h+m-k-2}{2l} \times \\ & \times \frac{\cos^{n-h+m-k-2-2l} \vartheta}{(2l+1)(n+m-1-2l)} (1-(-1)^{n+m-2l}) + \\ & + \frac{hk}{\cos\vartheta} \sum_{l=0}^{[\frac{n-h+m-k}{2}]} \binom{n-h+m-k}{2l} \times \\ & \times \frac{\cos^{n-h+m-k-2l} \vartheta}{(2l+1)(n+m-1-2l)} (1-(-1)^{n+m-2l}) \Big), \end{aligned}$$

$$\begin{aligned} (v_{nh}, v_{mk})_{0,A} = 2\sin^{h+k+1} \vartheta & \sum_{l=0}^{[\frac{n-h+m-k}{2}]} \binom{n-h+m-k}{2l} \times \\ & \times \frac{\cos^{n+m-h-k-2l} \vartheta}{(2l+1)(n+m-1-2l)} (1+(-1)^{n+m-2l}) + \\ & - 4\sin\vartheta \ \gamma_{nh}\gamma_{mk} . \end{aligned}$$

The values of the upper bound approximations $\Lambda_k^\nu$ obtained by means of the Rayleigh-Ritz method are given in the Section 6.

## 4 – Intermediate problems

To get lower bounds for the eigenvalues of the problem (3), we are next going to introduce certain suitable intermediate problems, whose eigenvalues approximate from below those of the given problem. Subsequently, we will search for lower bounds of the eigenvalues of these intermediate problems.

We recall that the Rayleigh-Ritz functional associated to (3) takes the form

$$(11)\qquad \begin{aligned} F(u)=\frac{B(u,u)}{(u,u)_{0,A}}&=\frac{\iint_D (u_\xi^2+u_\eta^2-2u_\xi u_\eta\cos\vartheta)d\xi\,d\eta}{\sin^2\vartheta\iint_D u d\xi\,d\eta}=\\ &=\frac{(1-\cos\vartheta)\iint_D (u_\xi^2+u_\eta^2)d\xi\,d\eta+\cos\vartheta\iint_D (u_\xi-u_\eta)^2 d\xi\,d\eta}{\sin^2\vartheta\iint_D u d\xi\,d\eta}\,. \end{aligned}$$

Consider now $n$ functions[(2)]

$$(12)\qquad w_1,\dots,w_n\in H_1^0(D)\,.$$

Letting $W_n$ denote their generated subspace, define $P_n$ as the orthogonal projector of $H_1(D)$ on $W_n$.

If

$$\begin{aligned} c_h=(u_\xi-u_\eta,w_h)&=\iint_D (u_\xi-u_\eta)w_h\;d\xi\,d\eta=\\ &=-\iint_D u(u,w_{h,\xi}-w_{h,\eta})\;d\xi\,d\eta=-(u,w_{h,\xi}-w_{h,\eta})\,, \end{aligned}$$

the orthogonal projector $P_n$ may be represented as

$$(13)\qquad P_n(u_\xi-u_\eta)=\sum_{i,j}^{1,n}\alpha_{ij}(u_\xi-u_\eta,w_i)w_j=-\sum_{i,j}^{1,n}\alpha_{ij}c_iw_j\,,$$

where the matrix

$$(14)\qquad A=((\alpha_{ij}))$$

is the inverse of the Gram matrix of the system $\{w_1,\dots,w_n\}$.

(2) $H_1^0(D)$ denotes the functions in $H_1(D)$ whose trace on $\partial D$ vanishes.

Moreover, one has

$$\iint_D (u_\xi - u_\eta)^2 \, d\xi \, d\eta \geq \iint_D |P_n(u_\xi - u_\eta)|^2 \, d\xi \, d\eta \sum_{i,j}^{1,n} \alpha_{ij} \, c_i \, c_j \, . \tag{15}$$

By taking the constants

$$q_0 = \frac{\cos\vartheta}{1-\cos\vartheta} \, ,$$
$$p_0 = 1 + \cos\vartheta \, ,$$

we define the functional

$$F_n(u) = \frac{\displaystyle\iint_D (u_\xi^2 + u_\eta^2) \, d\xi \, d\eta + q_0 \sum_{i,j}^{1,n} \alpha_{ij} \, c_i \, c_j}{\displaystyle p_0 \iint_D u \, d\xi \, d\eta} \, . \tag{16}$$

From (13) and (16) it plainly follows that

$$F_n(u) \leq F_{n+1}(u) \leq F(u) \qquad \forall \, u \in \mathcal{H}_1(D) \, . \tag{17}$$

Straightforward calculations show that the eulerian problem associated to the functional (16) is given by

$$\begin{cases} \dfrac{1}{p_0} \, \triangle u + \dfrac{q_0}{p_0} \displaystyle\sum_{i,j}^{1,n} \alpha_{ij} \, c_i \, (w_{j,\xi} - w_{j,\eta}) + \lambda u = 0 \\ \dfrac{\partial u}{\partial \nu} = 0 \\ \displaystyle\iint_D u \, d\xi \, d\eta = 0 \end{cases} \tag{18}$$

where $u \in \mathcal{H}_1(D)$. And the eigenvalues of the problem(18) form a sequence

$$0 < \lambda_{1,n} \leq \ldots \leq \lambda_{k,n} \leq \ldots$$

which, according to (17), satisfies the inequalities

$$\lambda_{k,n} \leq \lambda_{k,n+1} \leq \lambda_k \qquad \forall \, k = 1, \ldots \tag{19}$$

In order to determine lower bounds for the eigenvalues $\lambda_{k,n}$ we will employ the Orthogonal Invariants Method, which is described e.g. in [2].

Denote by the symbol $\lambda^{\nu}_{k,n}$ these lower approximations:

$$\lambda^{\nu}_{k,n} \leq \lambda_{k,n}\,. \tag{20}$$

At this step, the Green function of the problem (18) has to be known. So, we will devote the final part of this section to the Green function construction.

To start with, consider the problem

$$\begin{cases} \triangle u + f = 0 & (\xi,\eta)\in D \\ \dfrac{\partial u}{\partial \nu} = 0 & (\xi,\eta)\in \partial D \\ \displaystyle\iint_D u \; d\xi\, d\eta = 0 \end{cases} \tag{21}$$

where $f \in L_2(D)$ satisfies

$$\iint_D f \; d\xi\, d\eta = 0\,.$$

It is wellknown that there exists a unique solution $u \in H_2(D)$ of (21), which is given by

$$u(\xi,\eta) = (\Gamma_0 f)(\xi,\eta) = \iint_D H_0(\xi,\eta,\tau_1,\tau_2) f(\tau_1,\tau_1)\; d\tau_1\, d\tau_2 \tag{22}$$

where

$$H_0(\xi,\eta,\tau_1,\tau_2) = \frac{1}{\pi^2} \sum_{\substack{h,k\\ h+k>0}}^{0,+\infty} \frac{1}{h^2+k^2} \cos h\xi\, \cos k\eta \cos h\tau_1 \cos k\tau_2\,. \tag{23}$$

The differential problem associated to the eigenvalue problem (18) is the following:

$$\begin{cases} \dfrac{1}{p_0}\, \triangle u + \dfrac{q_0}{p_0} \displaystyle\sum_{i,j}^{1,n} \alpha_{ij}\; c_i\; (w_{j,\xi} - w_{j,\eta}) + f = 0 \\ \dfrac{\partial u}{\partial \nu} = 0 \\ \displaystyle\iint_D u \; d\xi\, d\eta = 0 \end{cases} \tag{24}$$

where $f \in \mathcal{H}_1(D)$.

The solution of the differential problem (24) is given by

$$u = \Gamma_0 f + \frac{q_0}{p_0} \sum_{h,k}^{1,n} \alpha_{hk}\ c_h\ \Gamma_0(w_{k,\xi} - w_{k,\eta})\,. \tag{25}$$

Having now introduced the matrix

$$\tilde{A} = ((q_0 \Gamma_0(w_{k,\xi} - w_{k,\eta}), (w_{h,\xi} - w_{h,\eta})))$$

and the vector $\phi = (\phi_1, \ldots, \phi_n)$ with

$$\phi_k = (f, \Gamma_0(w_{k,\xi} - w_{k,\eta}))\,, \tag{26}$$

call $B$ the product matrix $B = A\tilde{A}$. Let the matrix $\tilde{B}$ be defined as

$$\tilde{B} = (B + p_0 I)^{-T}\ ^{(3)}\,. \tag{27}$$

Then, the vector $c = (c_1, \ldots, c_n)$ appearing in (25) can be written as

$$c = -p_0\ \ \tilde{B}\ \phi\,. \tag{28}$$

If $\tilde{b}_{ij}$ are the elements of the matrix $\tilde{B}$, in view of (14) and (22) one gets the following final expression of (25):

$$\begin{aligned} u =& \Gamma_n f = \\ =& \Gamma_0 f - q_0 \sum_{h,k}^{1,n} \sum_{j}^{1,n} \alpha_{hk} \tilde{b}_{hj} (f, \Gamma_0(w_{k,\xi} - w_{k,\eta}))\Gamma_0(w_{j,\xi} - w_{j,\eta}) = \\ =& \iint_D \Big( H_0(\xi, \eta, \tau_1, \tau_2) - q_0 \sum_{h,k,j}^{1,n} \alpha_{hk} \tilde{b}_{hj} \Gamma_0(w_{k,\xi} - w_{k,\eta}) \Gamma_0(w_{j,\xi} - w_{j,\eta}) \Big) \cdot \\ & \cdot f(\tau_1, \tau_2) d\tau_1 d\tau_2\,. \end{aligned} \tag{29}$$

(3) The invertibility of the matrix $B^T + p_0 I$ is guaranteed by the uniqueness of the solution of the problem (24).

Finally, set

$$\beta_{kj} = q_0 \sum_{h}^{1,n} \alpha_{kh}\ \tilde{b}_{hj}, \tag{30}$$

$$g_i(\xi,\eta) = \Gamma_n(w_{i,\xi} - w_{i,\eta})\,. \tag{31}$$

The Green operator (29) of the differential problem (24) has an Hermitian kernel given by

$$H_n(\xi,\eta,\tau_1,\tau_2) = H_0(\xi,\eta,\tau_1,\tau_2) - \sum_{k,j}^{1,n} \beta_{kj}\ g_j(\xi,\eta)\ g_k(\tau_1,\tau_2)\,. \tag{32}$$

We are now in the position to apply the Orthogonal Invariants theory, which allows to find lower approximations as in (20).

Denoting by $\mathcal{I}_1^m(\Gamma_n)$ $(m \geq 2)$ the Orthogonal Invariants of the problem (24) and by $\Lambda_k^\nu$ the upper approximations of its eigenvalues appearing in (9), one has

$$\lambda_{k,n}^\nu = \Big(\mathcal{I}_1^m(\Gamma_n) - \sum_{i}^{1,\nu}(\Lambda_{k,n}^\nu)^{-m}\Big)^{-1/m} \leq \lambda_{k,n} \leq \lambda_k\,. \tag{33}$$

In the problem at hand it hasn't been possible finding explicitely the value of the invariants $\mathcal{I}_1^m(\Gamma_n)$. Neverthelees, they have been substituted by some upper bounds and the error (introduced at this step) has been explicitely evaluated.

## 5 – The invariants $\mathcal{I}_1^2(\Gamma_n)$ and $\mathcal{I}_1^3(\Gamma_n)$

Introducing the following iterates of the Green function (32)

$$H_n^{(2)}(\xi,\eta,\tau_1,\tau_2) = \iint_D H_n(\xi,\eta,\sigma_1,\sigma_2)\ H_n(\sigma_1,\sigma_2,\tau_1,\tau_2))\ d\sigma_1\,d\sigma_2\,,$$
$$H_n^{(3)}(\xi,\eta,\tau_1,\tau_2) = \iint_D H_n^{(2)}(\xi,\eta,\sigma_1,\sigma_2)\ H_n(\sigma_1,\sigma_2,\tau_1,\tau_2))\ d\sigma_1\,d\sigma_2\,,$$

the Orthogonal Invariants theory gives

$$\mathcal{I}_1^2(\Gamma_n) = \iint_D H_n^{(2)}(\xi,\eta,\xi,\eta)\ d\xi\,d\eta\,,$$
$$\mathcal{I}_1^3(\Gamma_n) = \iint_D H_n^{(3)}(\xi,\eta,\xi,\eta)\ d\xi\,d\eta\,.$$

Performing now starightforward calculations, setting

$$\varphi(\xi,\eta)=\iint_D H_n(\xi,\eta,\tau_1,\tau_2)\ d\tau_1\,d\tau_2$$

and keeping into account (32), we get:

$$\begin{aligned}\mathcal{I}_1^2(\Gamma_n)&=\iint_D d\xi\,d\eta\iint_D (H_n(\xi,\eta,\tau_1,\tau_2))^2\ d\tau_1\,d\tau_2=\\&=\mathcal{I}_1^2(\Gamma_0)-2\sum_{i,j}^{1,n}\beta_{hk}(\varphi_h,g_k)+\sum_{h,k,l,m}^{1,n}\beta_{ij}\beta_{hk}(g_i,g_j)(g_h,g_k)\,.\end{aligned}\tag{34}$$

Analogously we get for the invariant $\mathcal{I}_1^3(\Gamma_n)$:

$$\begin{aligned}\mathcal{I}_1^3(\Gamma_n)&=\mathcal{I}_1^3(\Gamma_0)-3\sum_{i,j}^{1,n}\beta_{ij}(\varphi_i,\varphi_j)+3\sum_{i,j,h,k}^{1,n}\beta_{ij}\beta_{hk}(\varphi_h,g_j)(g_j,g_k)+\\&-\sum_{i,j,h,k,l,m}^{1,n}\beta_{ij}\beta_{lm}(g_j,g_k)(g_i,g_l)(g_h,g_m)\,.\end{aligned}\tag{35}$$

We have now to calculate, approximately but with a rigorous estimate of the error, the expressions (34) and (35).

### 5.1 – Numerical calculus of the invariant $\mathcal{I}_1^2(\Gamma_n)$

We notice from (34) that the quantities we need to calculate in order to know the value of the invariant $\mathcal{I}_1^2(\Gamma_n)$ are $\mathcal{I}_1^2(\Gamma_0)$, $\beta_{ij}$, $(g_k,\varphi_h)$ and $(g_k,g_i)$.

The value of the invariant $\mathcal{I}_1^2(\Gamma_0)$ can be calculated. It turns out to be

$$\mathcal{I}_1^2(\Gamma_0)=\sum_{\substack{h,k\\h+k>0}}^{0,+\infty}\frac{1}{(h^2+k^2)^2}\,.\tag{36}$$

For a sufficiently large (and explicitely calculable) index $\mu$, the following estimate can be derived regarding the rest of the series (36):

$$|r_\mu|\le\frac{1}{4}\Big(\pi-\ 2\ \mathrm{arctg}\mu-\frac{2}{\mu}+\frac{2+\pi}{\mu^2}+\frac{2}{\mu^3}-\frac{2}{\mu^2}\ \mathrm{arctg}\frac{1}{\mu}\Big)\,.\tag{37}$$

We move now our attention to $\beta_{ij}$, $(g_k, \varphi_h)$ and $(g_k, g_i)$. In order to calculate these quantities, the functions $w_1, \dots, w_n$ introduced in (12) have been choosed as

$$w_{hk} = \sin h\pi\xi \ \sin k\pi\eta \,, \qquad h,k = 1,2,\dots \tag{38}$$

The function $g_i$ defined in (31) becomes then[4]

$$\begin{aligned} g_{hk}(\xi,\eta) &= \Gamma_n(w_{hk,\xi} - w_{hk,\eta}) = \\ &= hk \sum_i^{0,\infty}{}^{(h)} \frac{1}{i^2+k^2} \frac{(1-(-1)^{i+h})}{h^2-i^2} \cos k\pi\xi \ \cos i\pi\eta + \\ &\quad + hk \sum_j^{0,\infty}{}^{(k)} \frac{1}{j^2+h^2} \frac{(1-(-1)^{j+k})}{k^2-j^2} \cos k\pi\xi \ \cos j\pi\eta \,. \end{aligned}$$

The elements of the matrix $B + p_0 I$, defined in (27), take the form:

$$\begin{aligned} ((B+p_0I))_{lm} &= ((q_0(\Gamma_n(w_{hk,\xi} - w_{hk,\eta}), (w_{ij,\xi} - w_{ij,\eta})) + p_0 I))_{lm} = \\ &= p_0\delta_{lm} + \Big(\delta_{ih} \sum_r^{0,\infty}{}^{(kj)} \frac{(1-(-1)^{j+r})(1-(-1)^{k+r})}{(r^2+h^2)(k^2-r^2)(j^2-r^2)} + \\ &\quad + \delta_{jk} \sum_r^{0,\infty}{}^{(hi)} \frac{(1-(-1)^{i+r})(1-(-1)^{h+r})}{(r^2+i^2)(h^2-r^2)(i^2-r^2)} + \\ &\quad + (1-\delta_{ih})(1-\delta_{jk}) \frac{1-(-1)^{i+h}}{h^2-i^2} \frac{1-(-1)^{j+k}}{k^2-j^2} \times \\ &\quad \times \Big(\frac{1}{i^2+k^2} + \frac{1}{j^2+h^2}\Big)\Big) \, q_0 \ h \ k \ i \ j \,. \end{aligned} \tag{39}$$

If $r_\mu^1$ denotes the rest in the series appearing in the formula (39), by easy calculations one gets, for $\mu > \max\{j,k\}$, the following estimate:

$$\begin{aligned} |r_\mu^1| &\le \sum_l^{\mu,+\infty} \frac{1}{(l^2+h^2)(l^2-j^2)(l^2-k^2)} \le \\ &\le 4\Big(\frac{\pi}{2} - \operatorname{arctg} \mu + \frac{\mu}{\mu+1} + \log \frac{\mu-1}{\mu+1}\Big) \,. \end{aligned} \tag{40}$$

[4] The symbol $\sum_i^{0,\infty}{}^{(h)}$ means that the index $i$ assumes all values between 0 and $+\infty$ but the value $h$. Analogously in the subsequent cases.

By other easy calculations, the following formulae can be derived in calculating the quantities $(g_k, \varphi_h)$:

$$(41)\qquad \begin{aligned}(\varphi_{ij}, g_{hk}) = \Big(\delta_{ih} \sum_r^{0,\infty}{}^{(jk)} \frac{(1-(-1)^{j+r})(1-(-1)^{k+r})}{(r^2+h^2)^3(k^2-r^2)(j^2-r^2)} + \\ + \delta_{jk} \sum_r^{0,\infty}{}^{(hi)} \frac{(1-(-1)^{i+r})(1-(-1)^{h+r})}{(r^2+i^2)^3(h^2-r^2)(i^2-r^2)} + \\ + (1-\delta_{ih})(1-\delta_{jk}) \frac{1-(-1)^{i+h}}{h^2-i^2} \, \frac{1-(-1)^{j+k}}{k^2-j^2} \times \\ \times \Big(\frac{1}{(i^2+k^2)^3} + \frac{1}{(j^2+h^2)^3}\Big)\Big) \; 4\,h\,k\,i\,j\,.\end{aligned}$$

Also, an estimate for the rest is given by:

$$(42)\qquad |r^2_\mu| \le \frac{13}{2}\Big(\frac{\pi}{2} - \text{arctg}\;\mu\Big) + \frac{\mu(9\mu^4+2\mu^2-3)}{2(\mu^2+1)^2(\mu^2-1)} + 2\log\,\frac{\mu-1}{\mu+1}\,.$$

Finally, as for the calculus of $(g_i, g_h)$, one gets:

$$(43)\qquad \begin{aligned}(g_{ij}, g_{hk}) = \Big(\delta_{ih} \sum_r^{0,\infty}{}^{(jk)} \frac{(1-(-1)^{j+r})(1-(-1)^{k+r})}{(r^2+h^2)^2(k^2-r^2)(j^2-r^2)} + \\ + \delta_{jk} \sum_r^{0,\infty}{}^{(hi)} \frac{(1-(-1)^{i+r})(1-(-1)^{h+r})}{(r^2+i^2)^2(h^2-r^2)(i^2-r^2)} + \\ + (1-\delta_{ih})(1-\delta_{jk}) \frac{1-(-1)^{i+h}}{h^2-i^2} \, \frac{1-(-1)^{j+k}}{k^2-j^2} \times \\ \times \Big(\frac{1}{(i^2+k^2)^2} + \frac{1}{(j^2+h^2)^2}\Big)\Big) \; h\,k\,i\,j\end{aligned}$$

and an estimate for the rest is given by:

$$(44)\qquad |r^2_\mu| \le 3(\pi - 2\text{arctg}\;\mu) + \frac{4\mu}{\mu^4-1} + 3\log\,\frac{\mu-1}{\mu+1}\,.$$

At this point, the invariant $\mathcal{I}^2_1(\Gamma_n)$ can be approximately calculated and a rigorous estimate of the involved error can be exhibited. In the following table we indicate some values of the rest.

| $\mu$ | $R_\mu$ |
|---|---|
| 1000 | 6.4219908199 E-07 |
| 2000 | 1.6061227046 E-07 |
| 3000 | 7.1392490590 E-08 |
| 4000 | 4.0160880137 E-08 |
| 5000 | 2.5703932985 E-08 |
| 6000 | 1.7850437485 E-08 |
| 7000 | 1.3114850094 E-08 |
| 8000 | 1.0041196619 E-08 |

### 5.2 – Numerical calculus of the invariant $\mathcal{I}_1^3(\Gamma_n)$

The formula (35) suggests that the quantities we need to calculate in order to know the value of the invariant $\mathcal{I}_1^3(\Gamma_n)$ are $\mathcal{I}_1^3(\Gamma_0)$, $\beta_{ij}$, $(g_k,\varphi_h)$, $(\varphi_k,\varphi_h)$ and $(g_k,g_i)$.

The value of the invariant $\mathcal{I}_1^2(\Gamma_0)$ can be calculated and turns out to be

$$\mathcal{I}_1^3(\Gamma_0) = \sum_{\substack{h,k\\ h+k>0}}^{0,+\infty} \frac{1}{(h^2+k^2)^3}\,. \tag{45}$$

Finally, the following formulae are obtained as for $(\varphi_i,\varphi_h)$:

$$\begin{aligned}(\varphi_{ij},\varphi_{hk}) = \Big(\delta_{ih} &\sum_r^{0,\infty}{}^{(jk)} \frac{(1-(-1)^{j+r})(1-(-1)^{k+r})}{(r^2+h^2)^4(k^2-r^2)(j^2-r^2)}+\\ &+\ \delta_{jk}\sum_r^{0,\infty}{}^{(hi)} \frac{(1-(-1)^{i+r})(1-(-1)^{h+r}t)}{(r^2+i^2)^4(h^2-r^2)(i^2-r^2)}+\\ &+(1-\delta_{ih})(1-\delta_{jk})\frac{1-(-1)^{i+h}}{h^2-i^2}\ \frac{1-(-1)^{j+k}}{k^2-j^2}\times\\ &\times\Big(\frac{1}{(i^2+k^2)^4}+\frac{1}{(j^2+h^2)^4}\Big)\Big)\ h\ k\ i\ j\,.\end{aligned}$$

For the calculus of the quantities $\beta_{ij}$, $(g_k,\varphi_h)$, $(g_k,g_i)$ one proceeds as in the previous case and the estimate of the rests are similar too.

The following table provides some values of the rest obtained in the calculus of the invariant $\mathcal{I}_1^3(\Gamma_n)$.

| $\mu$ | $R_\mu$ |
|---|---|
| 10 | 6.6878814010 E-07 |
| 50 | 2.1606547084 E-10 |
| 100 | 6.7541089806 E-12 |
| 120 | 2.7144308230 E-12 |
| 140 | 1.2559145980 E-12 |

## 6 – Tables with the final results

| | $\vartheta = \pi/6$ | $n = 3$ | $\overline{n} = 20$ |
|---|---|---|---|
| | $\mathcal{I}_1^2(\Gamma_n)$ | $\mathcal{I}_1^3(\Gamma_n)$ | |
| k | $\lambda_{k,n}^\nu$ | $\lambda_{k,n}^\nu$ | $\Lambda_k^\nu$ |
| 1 | 1.4136 | 1.5148 | 1.5212 |
| 2 | 2.2526 | 3.6171 | 3.8497 |
| 3 | 2.4053 | 5.6634 | 8.0706 |
| 4 | 2.4228 | 6.3192 | 14.0550 |
| 5 | 2.4254 | 6.4465 | 19.7418 |

| | $\vartheta = \pi/6$ | $n = 4$ | $\overline{n} = 20$ |
|---|---|---|---|
| | $\mathcal{I}_1^2(\Gamma_n)$ | $\mathcal{I}_1^3(\Gamma_n)$ | |
| k | $\lambda_{k,n}^\nu$ | $\lambda_{k,n}^\nu$ | $\Lambda_k^\nu$ |
| 1 | 1.4419 | 1.5148 | 1.5212 |
| 2 | 2.4617 | 3.6173 | 3.8497 |
| 3 | 2.6894 | 5.6646 | 8.0706 |
| 4 | 2.7168 | 6.3210 | 14.0550 |
| 5 | 2.7210 | 6.4485 | 19.7418 |

| | $\vartheta = \pi/4$ | $n = 3$ | $\overline{n} = 20$ |
|---|---|---|---|
| | $\mathcal{I}_1^2(\Gamma_n)$ | $\mathcal{I}_1^3(\Gamma_n)$ | |
| k | $\lambda_{k,n}^\nu$ | $\lambda_{k,n}^\nu$ | $\Lambda_k^\nu$ |
| 1 | 1.4865 | 1.6116 | 1.6236 |
| 2 | 2.2697 | 3.6851 | 4.0766 |
| 3 | 2.3847 | 4.9602 | 6.9545 |
| 4 | 2.4001 | 5.2829 | 8.6201 |
| 5 | 2.4154 | 5.6934 | 17.1677 |

| | $\vartheta = \pi/4$ | $n = 4$ | $\overline{n} = 20$ |
|---|---|---|---|
| | $\mathcal{I}_1^2(\Gamma_n)$ | $\mathcal{I}_1^3(\Gamma_n)$ | |
| k | $\lambda_{k,n}^{\nu}$ | $\lambda_{k,n}^{\nu}$ | $\Lambda_k^{\nu}$ |
| 1 | 1.4930 | 1.6119 | 1.6236 |
| 2 | 2.3055 | 3.6936 | 4.0766 |
| 3 | 2.4286 | 4.9881 | 6.9545 |
| 4 | 2.4452 | 5.3188 | 8.6201 |
| 5 | 2.4617 | 5.7420 | 17.1677 |

| | $\vartheta = \pi/3$ | $n = 3$ | $\overline{n} = 20$ |
|---|---|---|---|
| | $\mathcal{I}_1^2(\Gamma_n)$ | $\mathcal{I}_1^3(\Gamma_n)$ | |
| k | $\lambda_{k,n}^{\nu}$ | $\lambda_{k,n}^{\nu}$ | $\Lambda_k^{\nu}$ |
| 1 | 1.6468 | 1.7895 | 1.7900 |
| 2 | 2.5348 | 4.3695 | 4.3871 |
| 3 | 2.5360 | 4.3796 | 4.3974 |
| 4 | 2.7013 | 9.1154 | 9.4718 |
| 5 | 2.7128 | 11.3915 | 12.3336 |

| | $\vartheta = \pi/3$ | $n = 4$ | $\overline{n} = 20$ |
|---|---|---|---|
| | $\mathcal{I}_1^2(\Gamma_n)$ | $\mathcal{I}_1^3(\Gamma_n)$ | |
| k | $\lambda_{k,n}^{\nu}$ | $\lambda_{k,n}^{\nu}$ | $\Lambda_k^{\nu}$ |
| 1 | 1.6468 | 1.7898 | 1.7900 |
| 2 | 2.5348 | 4.3792 | 4.3871 |
| 3 | 2.5360 | 4.3894 | 4.3974 |
| 4 | 2.7013 | 9.3064 | 9.4718 |
| 5 | 2.7128 | 11.8772 | 12.3336 |

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*Lavoro pervenuto alla redazione il 7 giugno 2000*
*Bozze licenziate il 9 novembre 2000*

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