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Dependence on a parameter in continuum mechanics

G. GRIOLI

In memory of my dear friend and colleague Gaetano Fichera

RIASSUNTO: *Si mette in evidenza lo speciale significato di taluni parametri presenti nelle relazioni costitutive della Meccanica dei Continui. La loro presenza permette di caratterizzare importanti proprietà, come ad esempio, l'incompressibilità e la rigidità. Inoltre, sembra conveniente di considerare nelle relazioni costitutive la massima dimensione delle particelle elementari dello schema fisico particellare della materia. La sua presenza porta a un più raffinato modello continuo che, tra l'altro, richiede condizioni fisicamente più accettabili di quelle tradizionalmente ammesse sulla frontiera.*

ABSTRACT: *In this paper is pointed out the special meaning of some parameters which are present in the constitutive relations of Continuum Mechanics. They make possible to characterize some important properties such as incompressibility and rigidity. Furthermore, it seems convenient to take into consideration in the constitutive relations the greatest dimension of the elementary particles of the physical corpuscular hypothesis of matter. Its presence leads to a more refined continuum model which, among other things, requires more physically acceptable conditions on the boudary than the usual ones.*

As is well known, in the study of the difficult problems of Continuum Mechanics often is advantageous to pay attention to a special parameter

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which is present in the mathematical model. Remarkable are the quest of Antonio Signorini who, assuming the external loads to be dead and proportional to a parameter, h, studied the problem of the equilibrium of an elastic body with large deformations. In the hypothesis of the existence of a Taylor's series for the solution, SIGNORINI, among other things, showed that the infinitesimal rigid displacement present in each term of the series is, in general, determined, but there are special cases in which the meaning of the linear elasticity is doubtful [1], [2]. F. STOPPELLI [3] studied the difficult analytical questions related to Signorini's theory. Others Authors continued his researches, considering also the case of live loads [4]-[9].

The parameter, h, considered by Signorini, has a mere analytical meaning, with the aim to change a non linear problem in a set of linear ones. Yet, in my opinion, in the constitutive relations of Continuum Mechanics there are some parameters that have physical meaning and influence in a special manner the behaviour of a body.

It is general opinion that the stress for a rigid body is undetermined. This opinion is due to the fact that the motion of a rigid body is determined without the need to know the constitutive relations connecting the stress to the motion. But the indetermination of the stress ceases if one considers a rigid material as the limit of a deformable one, as it seems natural. Further, since the behaviour of a deformable body, in general, may be identified with that of an elastic one when the deformation is very small, it is convenient to consider a rigid material as the limit of a linear elastic one. The theory gives useful informations about the elastic potential energy for finite deformations: it must contain a parameter such that the deformable body becomes rigid when that parameter tends to a certain limit.

I studied the problem some years ago and will give a short account in what follow [10], [11].

As is well known, Cauchy's theory of materials is inadequate for the study of many questions. Often, it is necessary to resolt to the theory of the microstructures. In that case the analytical problem requires a greater number of conditions on the boundary than the Cauchy's theory. The simplest continuum model of microstructure is that of Cosserat in which the basic fields are two: the displacement, u, and the rotation, R (see [12]-[20]). A very subtle question is that of the boundary conditions. In fact, it is very difficult (or impossible) – by example, in Cosserat theory – to define known functions of the points of the boundary that express the couples that the external world exerts on the body through its boundary. The mathematical problem is very interesting but has little physical concreteness. Lately I am studying the problem [21], [22]. The basic points of the theory are the previous consideration of the "elementary particles" of which the materials is constituted according to the corpuscular hypothesis of Physics and the consequent evaluation of the deformation, assuming as parameter, h, the greatest dimension of the elementary particles. The theory represents a refinement of the Cauchy's one. Its limit for h tending to zero coincides just with that theory. I will give a short account in the following, with reference to the linear case. The finite deformations one is much complex but equivalent from the conceptual point of view.

1 – The stress in rigid bodies

I denote by: C and C' the reference configuration and the actual one of an elastic body; P and P' two corresponding points of C and C' X_i and x_i their coordinates with respect to a rectangular cartesian coordinate system, $u = PP'$ the displacement of P, $T_{rs} = T_{sr}$ the components of the stress, $\varepsilon = (\varepsilon_{rs})$ the strain. Assuming that C is a natural equilibrium configuration, in the linear elasticity of a isotropic body, one has

$$T_{rs} = -\lambda I(\varepsilon)\delta_{rs} - 2\mu\varepsilon_{rs}\,, \qquad \varepsilon_{rs} = \frac{1}{2}(u_{r,s} + u_{s,r})\,, \tag{1}$$

where the comma denotes derivation with respect to X_s, δ_{rs} the Krönecker delta, $I(\varepsilon)$ the linear invariant of ε and λ, μ the Lamé coefficients, satisfying the conditions

$$\mu > 0 \qquad 3\lambda + 2\mu > 0 \tag{2}$$

From (1) it follows

$$I(\varepsilon) = -\frac{I(T)}{3\lambda + 2\mu}\,, \qquad \varepsilon_{rs} = -\frac{1}{2\mu}\Big[T_{rs} + \frac{\lambda}{3\lambda + 2\mu} I(T)\delta_{rs}\Big]\,. \tag{3}$$

From (3) one deduces: *every solution of the field and boundary equations* whose stress has a finite limit when λ goes to infinity satisfies the incompressibility property:

$$\lim_{\lambda\to\infty} I(\varepsilon) = 0\,. \tag{4}$$

Further, when μ tends to infinity the strain tends to zero for every finite stress:

$$\lim_{\mu\to\infty} \varepsilon_{rs} = 0\,, \quad \text{(rigidity property)}. \tag{5}$$

That means that the body becomes rigid when μ tends to infinity. Therefore, assuming $h = \mu^{-1}$ $(0 \le h < \infty)$, one may put

$$u_r = a_r + e_{rls} b_l X_s + h u'_r\,, \tag{6}$$

where e_{rls} denotes the Ricci's tensor in the tridimensional euclidean space and a_r, b_r are constant parameters.

According to (1):

$$T'_{rs} = \lim_{h\to 0} T_{rs} = -2\lim_{h\to 0}(h^{-1}\varepsilon_{rs}) = -(u'_{r,s} + u'_{s,r})\,. \tag{7}$$

The relations (7) may be interpreted as *constitutive relations of a rigid body* According to (6), the field and boundary equations deducted as limits from the general equations of the linear elasticity are

$$\begin{cases} (u'_{r,s} + u'_{s,r})_{,s} = -F_r + \gamma(\ddot{a}_r + e_{rls}\ddot{b}_l X_s)\,, & \text{(in } C\text{)} \\ (u'_{r,s} + u'_{s,r})N_s = -f_r\,, & \text{(on } \sigma\text{)}\,, \end{cases} \tag{8}$$

where F_r, f_r denote the densities of the external body and surface forces, γ the density of mass and N_r the inner perpendicular to the boundary, σ, of C.

The integrability conditions for equations (8) coincide with the dynamical equations of a rigid body and determine the functions $a_r(t)$, $b_r(t)$, depending on assigned initial conditions. I remark that the analytical problem (8) is of static type: the time is present as a parameter, contained in the functions $a_r(t)$, $b_r(t)$. The above properties of incompressibility and rigidity are verified by the solutions reported in the treatises of Mechanics.

From the preceeding considerations one deduces a general property for the potential energy, W, of an elastic body with finite deformations: the analytic structure of W must contain a parameter h such that when h tends to zero the strain goes to zero for every finite stress.

2 – Microstructure – Cosserat's Continua

As is well known, the mathematical model of Cosserat's Continua is based on two fundamental fields: the displacement, u, and the rotations $R(Q) = (R_{rs}$, where $Q = (Q_r)$ is a parameter which characterizes the rotation R. In the linear case the density of the work of the internal forces has the expression

$$dl^{(i)} = T_{rs} du_{r,s} + 2P_r dQ_r + 2P_{rs} dQ_{r,s}\,, \tag{9}$$

where T_{rs}, P_r, P_{rs} are parameters. In the usual interpretation, T_{rs} and P_{rs} characterize the stress and the couple stress. Due to the condition $dl^{(i)} = 0$ for every rigid displacement, one has

$$P_r = e_{rlm} T_{lm}\,. \tag{10}$$

The isothermal equilibrium equations are

$$\begin{cases} T_{rs}, s = F_r\,, \qquad P_{rs}, s + e_{rlm} T_{ml} = M_r \quad (\text{in } C)\,, \\ T_{rs} N_s = f_r\,, \qquad P_{rs} N_s = m_r\,, \qquad\qquad (\text{on } \sigma)\,, \end{cases} \tag{11}$$

where M_r denotes the density of the external body couples, while m_r is that of the surface ones. A part from variables as the temperature, the entropy, and so on, the density of the potential energy, W, depends on the functions $u_{r,s}$ and

$$g_{rs} = u_{s,r} + 2e_{rls} Q_l\,. \tag{12}$$

The constitutive equations are

$$
\left\{
\begin{aligned}
T_{rs} &= -\frac{\partial W}{\partial u_{r,s}}\,, \\
P_{rs} &= -\frac{\partial W}{\partial Q_{r,s}}\,.,
\end{aligned}
\right. \tag{13}
$$

The preceding equations correspond to the usual Cosserat's theory in which the size of the elementary particles of the corpuscular hypothesis of the modern Physics is absent. In my opinion, this lessens the theory; I think that the mathematical model is more realistic if one takes into account that size. One gains the end on the basis of the following procedure.

Let c and c_1 be the positions of two very close elementary rigid particles in the reference configuration and c' and c_1' their correspondent in C'. Let P and P_1 be two material points of c and c_1 and P' and P_1' their correspondent in c' and c_1'. Denoting by G and G_1 the center of mass of c and c_1, let us assume $u = GG'$ and suppose that R denotes the rotation of c. The change of distance between the points P and P_1 in the transformation from C to C' is expressible by the matrices

$$
\nu = a^T R\,, \quad \nu^i = R^T R_{,i} \quad (i = 1, 2, 3)\,, \tag{14}
$$

where a denotes the displacement gradient ($a_{rs} = x_{r,s}$). It is convenient to remark that is

$$
\nu\nu^T = 1 + 2\varepsilon\,. \tag{15}
$$

Denoting by h the greatest dimension of the elementary particles, let us assume

$$
GG_1 = n\,, \quad GP = z = hz'\,, \quad G_1P_1 = z^{(1)} = hz^{(1)'}\,. \tag{16}
$$

The dilatation coefficient, d, in the transformation CC' is expressed by

$$
(1 + d)^2 = (e + he' + h^2 e'')(e^0 + e^{0\prime} h + h^2 e^{0\prime\prime})^{-1} = \frac{|P'P_1'|^2}{|PP_1|^2}\,, \tag{17}
$$

where

$$
(18)\qquad \begin{cases} e = (1+2\varepsilon) n \cdot n\,, \\ e' = 2\nu^T n \cdot [z^{(1)'} - z' + \nu^i z^{(1)'} dX_i]\,, \\ e'' = (z^{(1)'} - z')^2 + \nu^i z^{(1)'} dX_i \cdot [2(z^{(1)'} - z') + \nu^l z^{(1)'} dX_l]\,, \end{cases}
$$

$$
(19)\qquad e^0 = n^2\,, \quad e^{0'} = 2n \cdot (z^{(1)'} - z')\,, \quad e^{0''} = (z^{(1)'} - z')^2\,.
$$

Therefore, it seems natural to assume that the potential energy, W, depends on the parameter h, besides the matrices ν, ν^i, that characterize the deformation:

$$
(20)\qquad W = W(\nu, \nu^i; h)\,.
$$

In the linear case of small deformations one has

$$
(21)\qquad \begin{cases} \nu_{rs} = \delta_{rs} + u_{s,r} + 2e_{rls} Q_l = \delta_{rs} + g_{rs}\,, \\ \nu^i_{rs} = 2e_{rls} Q_{l,i}\,. \end{cases}
$$

It is easy to show that for $h = 0$ the partial derivatives of any order of d with respect to h exist. Further, according to (17), (18), (19), the deformation for $h = 0$ is characterized only by the strain, ε. Therefore, it seems natural to admit the following property of the elastic potential energy:

$$
(22)\qquad W(\nu, \nu^i; 0) = W'(\varepsilon)\,.
$$

The equalities (20), (22) are to be considered as constitutive properties.

Keeping in mind the smallness of the parameter h, according to (20), (22), I will consider the simple constitutive hypothesis

$$
(23)\qquad W(\nu, \nu^i; h) = W'(\varepsilon) + h w''(g_{rs}, Q_{i,l})\,.
$$

In the developments of Cosserat's theory various options are possible:

a) No condition is imposed to the rotation R; that is the fields u and R are independent.

b) R coincides with the local rotation, r, present in the polar decomposition of the gradient a $[a = rU$, $U =$ pure deformation$]$.

c) R is such that in the transformation from C' to a very close configuration C'' the rotation of an elementary particle, c', coincides with the local rotation present in the polar decomposition of the gradient of displacement $du = P'P''$.

The case b) is rather complex in the hypothesis of finite deformations. The options b), c) coincide in the linear case of small deformations.

In the following I will consider the case c) that, in my opinion, is more interesting from the physical point of view.

In that hypothesis one has

$$Q_r = \frac{1}{4} e_{rls} u_{s,l} \,, \tag{24}$$

while, according to (10), it is

$$dl^{(i)} = T_{(rs)} d\varepsilon_{rs} + 2 P_{rs} dQ_{r,s} \,. \tag{25}$$

Therefore, the constitutive equations determine only the symmetrical part, $T_{(rs)}$, of the stress, while the asymmetrical one, $T_{[rs]}$, has the meaning of internal constraint reaction.

Concluding, in the case c) one has

$$\nu_{rs} = \delta_{rs} + \varepsilon_{rs} \,, \qquad Q_{r,s} = \frac{1}{4} e_{rlm} u_{m,ls} \tag{26}$$

and, according to (23) the constitutive equations are

$$\begin{cases} T_{(rs)} = -\dfrac{\partial W'}{\partial \varepsilon_{rs}} - h \dfrac{\partial W''}{\partial \varepsilon_{rs}} \,, & W' = W'(\varepsilon_{rs}) \,, \\[2ex] P_{rs} = -h \dfrac{\partial W''}{\partial Q_{r,s}} \,, & W'' = W''(\varepsilon_{rs}, Q_{r,s}) \,. \end{cases} \tag{27}$$

In the hypothesis of existence of Taylor's series in h for the fields u_r, T_{rs}, P_{rs} it is interesting to assume

$$u_r = u_r^{(0)} + h u_r^{(1)} \,, \quad T_{rs} = T_{rs}^{(0)} + h T_{rs}^{(1)} \,, \quad P_{rs} = P_{rs}^{(0)} + h P_{rs}^{(1)} \,, \tag{28}$$

justified owing the smallness of h.

The first terms of the development (28) satisfies to the field and boundary equations

$$\text{(29)}\qquad \begin{cases} T^{(0)}_{(rs),s} + T^{(0)}_{[rs],s} = F_r(h=0)\,, & (\text{in } C)\,,\\ (T^{(0)}_{(rs)} + T^{(0)}_{[rs]})N_s = f_r(h=0)\,, & (\text{on } \sigma)\,, \end{cases}$$

$$\text{(30)}\qquad P^{(0)}_{rs} = 0\,,\quad T^{(0)}_{[rs]} = \frac{1}{2}e_{rts}M_t(h=0)\,,$$

$$\text{(31)}\qquad 0 = m^{(0)}_r = m_r(h=0)\,.$$

For $h=0$ one finds again the Cauchy's theory, but the stress may be asymmetrical if M is different from zero for $h=0$. In my opinion, M must be equal to zero when $h=0$, as happens, by example, in the case that the elementary particles are magnetic dipoles in a magnetic field, being h their length.

The equality $m^{(0)}_r = 0$ do not surprise. In fact, the vector m is an unknown parameter whose opposite, $-m$, characterizes a part of the influence that the body exerts on the external world (couples, in the traditional theory). Therefore, $m^{(0)}_r$ goes to zero together with P_{rs}.

Assuming that F and f are independent of h, the equations of the successive approximation are

$$\text{(32)}\qquad \begin{cases} \left[\Big(\dfrac{\partial^2 W'}{\partial\varepsilon_{rs}\partial\varepsilon_{pq}}\Big)^0 \varepsilon^{(1)}_{pq}\right]_{,s} + \Big(\dfrac{\partial W''}{\partial\varepsilon_{rs}}\Big)^0_{,s} - T^{(1)}_{[rs],s} = 0\,, & (\text{in } C)\,,\\ \Big(\dfrac{\partial^2 W}{\partial\varepsilon_{rs}\partial\varepsilon_{pq}}\Big)^0 \varepsilon^{(1)}_{pq}N_s + \Big(\dfrac{\partial W''}{\partial\varepsilon_{rs}}\Big)^0 N_s - T^{(1)}_{[rs]}N_s = 0\,, & (\text{on } \sigma)\,, \end{cases}$$

$$\text{(33)}\qquad T^{(1)}_{[rs]} = \frac{1}{2}e_{trs}\left[\Big(\frac{\partial W''}{\partial Q_{t,m}}\Big)_{,m} + M^{(1)}_t\right]\,,$$

$$\text{(34)}\qquad m^{(1)}_r = -\Big(\frac{\partial W''}{\partial Q_{r,s}}\Big)^0 N_s\,.$$

The equations (32) are of Cauchy's type depending on the solution $u^{(0)}$. It should be noted that the stress may be asymmetric also in the Cauchy's problem ($M=0, m=0$).

Final Remarks

a) The presence of the parameter h allows to consider the Cosserat theory (and, in general, the theory of the microstructures) as a refinement of the Cauchy's theory.

b) The refinement depends on the geometrical nature that we assume for the elementary particles (rigid, deformable with homogeneous deformations, ...) and on the expression assumed for the density of the internal forces.

c) To take into consideration the parameter h may be valid for more general continua than the elastic ones and in the dynamical case.

d) The stress may be asymmetric also in the traditional theory of Cauchy ($M = 0, m = 0$).

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Indirizzo dell'Autore:

G. Grioli – Via 27 luglio, 54 – 98100 Messina (Italy)