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Orthogonal invariants and the Bell polynomials

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Dedicated to the memory of Gaetano Fichera

Riassunto: *Utilizzando delle formule di rappresentazione che fanno uso dei polinomi di Bell, vengono trovate delle relazioni verificate dagli invarianti ortogonali degli autovalori di operatori hermitiani compatti, che includono dei precedenti risultati di* Didier Robert.

Abstract: *By using representation formulas based on the Bell polynomials we derive some relations verified by the orthogonal invariants of the eigenvalues of hermitian compact operators, including some preceding results of* Didier Robert.

1 – Introduction

It is well known that the eigenvalues μ_k of a positive compact operator (shortly PCO) T in a complex Hilbert space $\mathcal{H}$ can be ordered in a sequence

$$0 \leq \cdots \leq \mu_3 \leq \mu_2 \leq \mu_1, \tag{1.1}$$

s.t. when infinite many eigenvalues exist, they have the zero as an accumulation point.

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A classical example of eigenvalue problem for such an operator (which is strictly positive) is given by

$$T\phi = \mathcal{K}\phi := \int_A K(x,y)\phi(y)dy = \mu\phi(x), \tag{1.2}$$

where the kernel $K(x,y)$ of the second kind Fredholm operator $\mathcal{K}$ belongs to $L^2(A \times A)$, and is such that $K(x,y) = \overline{K(y,x)}$, $(\mathcal{K}\phi, \phi) > 0$ if $\phi \neq 0 \in L^2(A)$ (see S.G. Mikhlin [2]).

The numerical computation of the eigenvalues of T is usually performed by using the Rayleigh-Ritz method for obtaining lower bounds, and the orthogonal invariants method (see G. Fichera [3], [4], [5]) for upper bounds. A short description of such methods will be given in Section 3. In a recent paper [6], an iterative method for computing the above mentioned eigenvalues has been shown.

The orthogonal invariants are, by definition, symmetric functions of the eigenvalues of T:

$$\mathcal{I}_s^n(T) = \sum_{k_1<k_2<\cdots<k_s} [\mu_{k_1}\mu_{k_2}\cdots\mu_{k_s}]^n, \tag{1.3}$$

so that it is natural to expect that (as in the algebraic case) connections with $\mathcal{I}_s^1(T)$ or $\mathcal{I}_1^n(T)$ hold true.

As a matter of fact, such formulas can be found in the classical book of J. Riordan [7], as an application of a standard tool of Combinatorial Analysis: the Bell polynomials.

Some results in this direction have been previously given by D. Robert [1].

In this article, by using the Bell polynomials, we will recover Robert's formulas for operators $\mathcal{T} := T^n$, and we will write explicit expressions of the orthogonal invariant $\mathcal{I}_s^1(\mathcal{T})$ in terms of $\mathcal{I}_1^r(\mathcal{T})$, $r \leq s$. Similar expressions of $\mathcal{I}_1^s(\mathcal{T})$ in terms of $\mathcal{I}_m^1(\mathcal{T})$, $m \leq s$ are also obtained. These last formulas have not been considered in Robert's article.

2 – Recalling the Bell polynomials

The Bell polynomials are a standard mathematical tool for representing the n-th derivative of a composite function (see e.g. [7]).

Denoting by $\Phi(t) := f(g(t))$ the composition of functions $x = g(t)$ and $y = f(x)$, defined in suitable intervals of the real axis, and putting

$$\Phi_m := D_t^m \Phi(t), \qquad f_h := D_x^h f(x)|_{x=g(t)}, \qquad g_k := D_t^k g(t),$$

then the n-th derivative can be represented by

$$\Phi_n = Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n),$$

where the Y_n are the Bell polynomials satisfying the recurrence relation:

$$\text{(2.1)} \qquad \begin{cases} Y_0 := f_1; \\ Y_{n+1}(f_1, g_1; \ldots; f_n, g_n; f_{n+1}, g_{n+1}) = \\ \qquad = \displaystyle\sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(f_2, g_1; f_3, g_2; \ldots; f_{n-k+1}, g_{n-k}) g_{k+1}. \end{cases}$$

A well known explicit expression for the Bell polynomials is given by the Faà di Bruno formula:

$$Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n) = \sum_{\pi(n)} \frac{n!}{r_1! r_2! \ldots r_n!} f_r \left[\frac{g_1}{1!}\right]^{r_1} \left[\frac{g_2}{2!}\right]^{r_2} \cdots \left[\frac{g_n}{n!}\right]^{r_n},$$

where the sum runs over all partitions $\pi(n)$ of the integer n, r_i denotes the number of parts of size i, and $r = r_1 + r_2 + \cdots + r_n$ denotes the number of parts of the considered partition.

3 – The Rayleigh-Ritz and orthogonal invariants methods

Let $\{v_k\}_{k \in \mathbf{N}}$ be a complete system of linearly independent vectors in a Hilbert space $\mathcal{H}$, put $\mathcal{V}_\nu := \operatorname{span}\{v_1, v_2, \ldots, v_\nu\}$, denote by P_ν the orthogonal projector, $P_\nu : \mathcal{H} \to \mathcal{V}_\nu$, and consider a strictly positive compact operator $T : \mathcal{H} \to \mathcal{H}$ and the corresponding eigenvalue problem

$$\text{(3.1)} \qquad T\phi = \mu\phi.$$

We assume here the operator T to be strictly positive, in order to avoid quotient spaces.

PROPOSITION 3.1 (Rayleigh-Ritz). *Consider the positive eigenvalues*

$$\mu_1^{(\nu)} \geq \mu_2^{(\nu)} \geq \cdots \geq \mu_\nu^{(\nu)}, \tag{3.2}$$

of the operator $P_\nu T P_\nu$. *Then*

i) *the positive eigenvalues of* $P_\nu T P_\nu$ *are obtained by solving the equation:*

$$det\{(Tv_j, v_h) - \mu(v_j, v_h)\} = 0, \quad (j, h = 1, ..., \nu),$$

($\mu = 0$ *is always an eigenvalue of* $P_\nu T P_\nu$).

ii) *For any fixed* k *and for any* $\nu \geq k$ *the following inequality hold true*

$$\mu_k^{(\nu)} \leq \mu_k^{(\nu+1)} \leq \mu_k.$$

iii) *Furthermore the limit condition is valid*

$$\lim_{\nu\to\infty} \mu_k^{(\nu)} = \mu_k.$$

i.e. the Rayleigh-Ritz method always gives lower bounds for the first ν eigenvalues of the operator T, and the condition iii) holds true.

The method of the orthogonal invariants have been introduced by G. FICHERA [5], in order to provide upper bounds for the same eigenvalues.

A complete orthogonal invariants system is a complete system of numbers which is invariant under the unitary equivalence for operators. Such a system must depend only on the eigenvalues of the operator. Theoretically we could consider the system

$$\mathcal{I}_s^n(T) = \sum_{k_1<k_2<\cdots<k_s} [\mu_{k_1}\mu_{k_2}\cdots\mu_{k_s}]^n,$$

for any fixed s (order of the invariant) and $n = 1, 2, 3, ...$ (degree of the invariant), provided that all these numbers can be computed independently of the knowledge of the eigenvalues of T.

Let

$$v_1{}^{(\nu)}, v_2{}^{(\nu)}, \ldots, v_\nu{}^{(\nu)},$$

be normalized eigenvectors of $P_\nu T P_\nu$ corresponding to the eigenvalues (3.2) and denote by

$$\mathcal{V}_\nu^{(k)} := \text{span}\{v_1^{(\nu)}, \dots v_{k-1}^{(\nu)}, v_{k+1}^{(\nu)}, \dots, v_\nu^{(\nu)}\}$$

and by $P_\nu^{(k)}$ the orthogonal projector $P_\nu^{(k)} : \mathcal{H} \to \mathcal{V}_\nu^{(k)}$.

PROPOSITION 3.2 (Fichera). *If $\mathcal{I}_s^n(T) < \infty$, for any fixed $n, s \in \mathbf{N}$ and $\forall k : k \leq \nu$, put*

$$\sigma_k^{(\nu)} := \left[\frac{\mathcal{I}_s^n(T) - \mathcal{I}_s^n(P_\nu T P_\nu)}{\mathcal{I}_{s-1}^n(P_\nu^{(k)} T P_\nu^{(k)})} + [\mu_k^{(\nu)}]^n\right]^{\frac{1}{n}} .$$

Then:

$$\sigma_k^{(\nu)} \geq \sigma_k^{(\nu+1)} \geq \mu_k,$$

and the limit condition

$$\lim_{\nu \to \infty} \sigma_k^{(\nu)} = \mu_k$$

holds true.

In the particular case of an Hilbert space $\mathcal{H} = L^2(A)$,

$$\forall n, s \in \mathbf{N},\ \ \mathcal{I}_s^n(T) < \infty \ \text{ iff } \ T^n \varphi = \int_A K(x,y)\varphi(y)dy,$$

$$\text{where } \ K(x,y) = \int_A H(x,z)H(z,y)dz, \ \ H(x,y) = \overline{H(y,x)} \in L^2(A \times A).$$

Then the orthogonal invariants can be expressed (see [5]) by the multiple integral

$$\mathcal{I}_s^n(T) = \frac{1}{s!} \int_A \cdots \int_A f(x_1, \dots, x_s) dx_1 \cdots dx_s, \tag{3.3}$$

where $f(x_1, \cdots, x_s)$ denotes the Fredholm determinant

$$f(x_1, x_2, \cdots, x_s) := \begin{vmatrix} K(x_1,x_1) & K(x_1,x_2) & \dots & K(x_1,x_s) \\ K(x_2,x_1) & K(x_2,x_2) & \dots & K(x_2,x_s) \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ K(x_s,x_1) & K(x_s,x_2) & \dots & K(x_s,x_s) \end{vmatrix} .$$

In particular, for $s = 1$:

$$\mathcal{I}_1^n(T) = \int_A K(x,x)dx = \int\int_{A\times A} |H(x,y)|^2\, dxdy,$$

and, for $s = 2$:

$$\mathcal{I}_2^n(T) = \frac{1}{2}\int\int_{A\times A} [K(x,x)K(y,y) - |K(x,y)|^2]\, dx\, dy.$$

4 – Fredholm determinants and Robert's formulas

In the cited book of S.G. Mikhlin [2], explicit and recurrent formulas for the Fredholm determinants are recalled. Namely, putting $\lambda := \frac{1}{\mu}$, the Fredholm resolvent is given by the meromorphic function:

$$\mathcal{H}(x,y;\lambda) = \frac{D(x,y;\lambda)}{D(\lambda)}.$$

In the above formula, by definition (1.3), and recalling (3.3)-(3.4), the *Fredholm first minor* $D(\lambda)$ is expressed by

$$D(\lambda) = \sum_{s=0}^{\infty} (-1)^s\, \mathcal{I}_s^n(T)\, \lambda^s. \tag{4.1}$$

Moreover, the *Fredholm determinant* $D(x,y;\lambda)$ is given by

$$D(x,y;\lambda) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!}\, B_s(x,y)\, \lambda^s, \tag{4.2}$$

where

$$\begin{cases} B_0(x,y) = K(x,y) \\ B_s(x,y) = s!\, \mathcal{I}_s^n(T)\, K(x,y) - s\int_A K(x,z)B_{s-1}(z,y)dz. \end{cases} \tag{4.3}$$

In the above cited article [1], D. Robert has found the following formulas

$$\mathcal{I}_s^n(T) = \frac{1}{s}\sum_{q=1}^{s} (-1)^{q-1}\mathcal{I}_1^{qn}(T)\mathcal{I}_{s-q}^n(T) \tag{4.4}$$

$$(4.5)\qquad \mathcal{I}_s^n(T) = (-1)^s \sum_{k=1}^{s} \frac{(-1)^k}{k!} \sum_{\substack{r_1+\cdots+r_k=s\\ 1\le r_i\le s}} \frac{\mathcal{I}_1^{nr_1}(T)\cdots\mathcal{I}_1^{nr_k}(T)}{r_1\cdots r_k},$$

which allow to reduce the orthogonal invariant $\mathcal{I}_s^n(T)$ to $\mathcal{I}_1^n(T)$.

Since the eigenvalues of T^n are given by μ_i^n, if μ_i are the eigenvalues of T, in the following, denoting by n the smallest integer such that $\mathcal{I}_s^n(T) < \infty$, we will put $\mathcal{T} := T^n$, so that $\mathcal{I}_s^1(\mathcal{T}) = \mathcal{I}_s^1(T^n) = \mathcal{I}_s^n(T)$, and the above eq. (4.4)-(4.5) become:

$$(4.6)\qquad \mathcal{I}_s^1(\mathcal{T}) = \frac{1}{s}\sum_{q=1}^{s}(-1)^{q-1}\mathcal{I}_1^q(\mathcal{T})\mathcal{I}_{s-q}^1(\mathcal{T})$$

$$(4.7)\qquad \mathcal{I}_s^1(\mathcal{T}) = (-1)^s \sum_{k=1}^{s} \frac{(-1)^k}{k!} \sum_{\substack{r_1+\cdots+r_k=s\\ 1\le r_i\le s}} \frac{\mathcal{I}_1^{r_1}(\mathcal{T})\cdots\mathcal{I}_1^{r_k}(\mathcal{T})}{r_1\cdots r_k}.$$

Remark 4.1. It is worth to note that there exist PCO not satisfying the above mentioned condition which requires the existence of an integer n such that $\mathcal{I}_s^n(T) < \infty$ (see [5]), however this condition is satisfied by all the PCO occurring in applications.

5 – Symmetric functions and the Bell polynomials

Consider the (real and nonnegative) eigenvalues μ_k of the PCO $\mathcal{T}$, and put

$$(5.1)\qquad \begin{aligned} \sigma_1 &= \sum_i \mu_i \\ \sigma_2 &= \sum_{i_1<i_2} \mu_{i_1}\mu_{i_2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \sigma_k &= \sum_{i_1<i_2<\cdots<i_k} \mu_{i_1}\mu_{i_2}\cdots\mu_{i_k} = \mathcal{I}_k^1 \end{aligned}$$

$$
\begin{aligned}
s_1 &= \sum_i \mu_i \\
s_2 &= \sum_i \mu_i^2 \\
&\dots\dots\dots\dots\dots \\
s_k &= \sum_i \mu_i^k = \mathcal{I}_1^k
\end{aligned}
\tag{5.2}
$$

Another class of symmetric functions is given by the so called *homogeneous product sum* symmetric functions h_n, $n = 1, 2, \dots$, defined by the equations (see eq. (4.1)):

$$
\begin{aligned}
(1-\mu_1\lambda)(1-\mu_2\lambda)(1-\mu_3\lambda)\cdots &= 1 - \sigma_1\lambda + \sigma_2\lambda^2 - \sigma_3\lambda^3 + \cdots = \\
&= (1 + h_1\lambda + h_2\lambda^2 + h_3\lambda^3 + \dots)^{-1} = \\
&= D(\lambda).
\end{aligned}
\tag{5.3}
$$

Note that these functions represent the coefficients of the power expansion with respect to λ of the first Fredholm minor.

Then (see e.g. [7]), the following representation formulas in terms of the Bell polynomials hold true:

$$\sigma_k = \frac{(-1)^k}{k!} Y_k(1, -s_1; 1, -s_2; 1, -2!s_3; \dots; 1, -(k-1)!s_k) \tag{5.4}$$

$$s_k = -\frac{1}{(k-1)!} Y_k(1, -\sigma_1; -1, 2!\sigma_2; \dots; (-1)^{k-1}(k-1)!, (-1)^k k!\sigma_k) \tag{5.5}$$

$$h_k = \frac{1}{k!} Y_k(1, s_1; 1, s_2; 1, 2!s_3; \dots; 1, (k-1)!s_k) \tag{5.6}$$

$$s_k = \frac{1}{(k-1)!} Y_k(1, h_1; -1, 2!h_2; 2!, 3!h_3; \dots; (-1)^{k-1}(k-1)!, k!h_k). \tag{5.7}$$

The above formulas (5.4)-(5.5) constitute a generalization of the well known Newton-Girard formulas and its inverse, since we have, in particular:

$$
\begin{aligned}
\sigma_1 &= s_1 \\
\sigma_2 &= \frac{1}{2}(s_1^2 - s_2) \\
\sigma_3 &= \frac{1}{6}(s_1^3 - 3s_1 s_2 + 2s_3) \\
&\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
s_1 &= \sigma_1 \\
s_2 &= \sigma_1^2 - 2\sigma_2 \\
s_3 &= \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \\
&\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
\tag{5.9}
$$

6 – Orthogonal invariants' reduction formulas

Writing formulas of the preceding section in terms of orthogonal invariants, we obtain:

$$
\begin{aligned}
&\mathcal{I}_k^1(\mathcal{T}) = \\
&= \tfrac{(-1)^k}{k!} Y_k\big(1, -\mathcal{I}_1^1(\mathcal{T}); 1, -\mathcal{I}_1^2(\mathcal{T}); 1, -2!\mathcal{I}_1^3(\mathcal{T}); \dots; 1, -(k-1)!\mathcal{I}_1^k(\mathcal{T})\big)
\end{aligned}
\tag{6.1}
$$

$$
\begin{aligned}
&\mathcal{I}_1^k(\mathcal{T}) = \\
&= -\tfrac{1}{(k-1)!} Y_k\big(1, -\mathcal{I}_1^1(\mathcal{T}); -1, 2!\mathcal{I}_2^1(\mathcal{T}); \dots; (-1)^{k-1}(k-1)!, (-1)^k k!\mathcal{I}_k^1(\mathcal{T})\big).
\end{aligned}
\tag{6.2}
$$

6.1 – A simple proof of the first Robert formula

We start from the recurrence relation (2.1) written in the form:

$$
\begin{aligned}
&Y_s(f_1, g_1; \dots; f_{s-1}, g_{s-1}; f_s, g_s) = \\
&\qquad = \sum_{q=1}^{s} \binom{s-1}{q-1} Y_{s-q}(f_2, g_1; f_3, g_2; \dots; f_{s-q+1}, g_{s-q}) g_q.
\end{aligned}
\tag{6.3}
$$

Then, by (6.1), it follows

$$
\begin{aligned}
\mathcal{I}_s^1(\mathcal{T}) &= \tfrac{(-1)^s}{s!} Y_s(1, -\mathcal{I}_1^1(\mathcal{T}); 1, -\mathcal{I}_1^2(\mathcal{T}); 1, -2!\mathcal{I}_1^3(\mathcal{T}); \dots; 1, -(s-1)!\mathcal{I}_1^s(\mathcal{T})) = \\
&= \tfrac{(-1)^s}{s!} \sum_{q=1}^{s} \binom{s-1}{q-1} Y_{s-q}(1, -\mathcal{I}_1^1(\mathcal{T}); 1, -\mathcal{I}_1^2(\mathcal{T}); \dots \\
&\dots; 1, -(s-q-1)!\mathcal{I}_1^{s-q}(\mathcal{T}))(-(q-1)!)\mathcal{I}_1^q(\mathcal{T})),
\end{aligned}
$$

and consequenty:

$$\text{(6.4)}\quad \begin{aligned} \mathcal{I}_s^1(\mathcal{T}) &= \frac{(-1)^s}{s!}\sum_{q=1}^{s}\binom{s-1}{q-1}(-1)^{s-q}(s-q)!\mathcal{I}_1^{s-q}(\mathcal{T})(-(q-1)!)\mathcal{I}_1^q(\mathcal{T}) = \\ &= \frac{1}{s}\sum_{q=1}^{s}(-1)^{q-1}\mathcal{I}_1^{s-q}(\mathcal{T})\mathcal{I}_1^q(\mathcal{T}), \end{aligned}$$

which is the first Robert formula.

6.2 – Reduction formulas for the invariants

We first show a representation of $\mathcal{I}_s^1(\mathcal{T})$, $s > 1$ by means of $\mathcal{I}_1^k(\mathcal{T})$, $k = 1, 2, \dots, s$.

By eq. (6.1), and the Faà di Bruno formula immediately follows:

PROPOSITION 6.1. *For any integer $s \geq 1$, the orthogonal invariant $\mathcal{I}_s^1(\mathcal{T})$, is espressed in terms of $\mathcal{I}_1^k(\mathcal{T})$, $k = 1, 2, \dots, s$ by*

$$\text{(6.5)}\quad \mathcal{I}_s^1(\mathcal{T}) = (-1)^s \sum_{\pi(s)} \frac{(-1)^k}{r_1!r_2!\cdots r_s!}\left[\frac{\mathcal{I}_1^1(\mathcal{T})}{1}\right]^{r_1}\left[\frac{\mathcal{I}_1^2(\mathcal{T})}{2}\right]^{r_2}\cdots\left[\frac{\mathcal{I}_1^s(\mathcal{T})}{s}\right]^{r_s},$$

where $\pi(s)$ denotes the sum running on all partitions of $s = r_1 + 2r_2 + \cdots + sr_s$ and $k = r_1 + r_2 + \cdots + r_s$.

In particular:

$$\text{(6.6)}\quad \begin{aligned} \mathcal{I}_2^1(\mathcal{T}) &= \frac{1}{2}((\mathcal{I}_1^1(\mathcal{T}))^2 - \mathcal{I}_1^2(\mathcal{T})) \\ \mathcal{I}_3^1(\mathcal{T}) &= \frac{1}{6}((\mathcal{I}_1^1(\mathcal{T}))^3 - 3\mathcal{I}_1^1(\mathcal{T})\mathcal{I}_1^2(\mathcal{T}) + 2\mathcal{I}_1^3(\mathcal{T})). \end{aligned}$$

REMARK 6.2. The second Robert formula is essentially equivalent to the Faà di Bruno formula. This could be proved verifying that for first indices the two formulas give the same values, and taking into account the recurrence relation (2.1), which is translated into the first Robert formula, as we have shown in the preceding section.

In a similar way, by eq. (6.2) and the Faà di Bruno formula, $\mathcal{I}_1^k(\mathcal{T})$, $k > 1$ can be represented by means of $\mathcal{I}_s^1(\mathcal{T})$, $s = 1, 2, \dots, k$:

PROPOSITION 6.3. *For any integer $k \geq 1$, the orthogonal invariant $\mathcal{I}_1^k(\mathcal{T})$, is expressed in terms of $\mathcal{I}_h^1(\mathcal{T})$, $h = 1, 2, \ldots, k$ by*

$$\mathcal{I}_1^k(\mathcal{T}) = \sum_{\pi(k)} (-1)^{k+s} \frac{k\,(s-1)!}{r_1! r_2! \cdots r_k!} \big(\mathcal{I}_1^1(\mathcal{T})\big)^{r_1} \big(\mathcal{I}_2^1(\mathcal{T})\big)^{r_2} \cdots \big(\mathcal{I}_k^1(\mathcal{T})\big)^{r_k}, \tag{6.7}$$

where $\pi(k)$ denotes the sum running on all partitions of $k = r_1 + 2r_2 + \cdots + kr_k$ and $s = r_1 + r_2 + \cdots + r_k$.

In particular:

$$\begin{aligned} \mathcal{I}_1^2(\mathcal{T}) &= (\mathcal{I}_1^1(\mathcal{T}))^2 - 2\mathcal{I}_2^1(\mathcal{T}) \\ \mathcal{I}_1^3(\mathcal{T}) &= (\mathcal{I}_1^1(\mathcal{T}))^3 - 3\mathcal{I}_1^1(\mathcal{T})\mathcal{I}_2^1(\mathcal{T}) + 3\mathcal{I}_3^1(\mathcal{T}). \end{aligned} \tag{6.8}$$

REFERENCES

[1] D. ROBERT: *Invariants orthogonaux pour certaines classes d'operateurs*, Annales Mathém. pures appl., **52** (1973), 81-114.

[2] S.G. MIKHLIN: *Integral equations and their applications*, 2nd Ed., Pergamon Press, Oxford, 1964.

[3] G. FICHERA: *Linear elliptic differential systems and eigenvalue problems*, Lect. Notes, **8**, Springer, Berlin 1965.

[4] G. FICHERA: *Abstract and Numerical Aspects of Eigenvalue Theory*, Lecture Notes, The University of Alberta, Dept. of Math., Edmonton, 1973.

[5] G. FICHERA: *Metodi e risultati concernenti l'analisi numerica e quantitativa*, Atti Acc. Naz. Lincei, Memorie 1974, Vol. XII, Sez. I.

[6] V. M. ARENA - B. GERMANO - P. E. RICCI: *On a method for computing the eigenvalues of second kind Fredholm operators*, Applied Math. and Informatics, **3** (1998), 12-23.

[7] J. RIORDAN: *An Introduction to Combinatorial Analysis*, J Wiley & Sons, Chichester, 1958.

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