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A class of totally positive refinable functions

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Dedicated to the memory of Gaetano Fichera

RIASSUNTO: *Le funzioni di raffinamento giocano un ruolo fondamentale nella costruzione di ondine; in questo lavoro vogliamo presentare una ampia classe di funzioni di raffinamento che possono essere identificate attraverso l'espressione esplicita delle loro maschere. Queste funzioni di raffinamento possono essere considerate come una generalizzazione delle B-splines cardinali, cosa che è di fondamentale interesse in molte applicazioni. Inoltre, ciascuna delle funzioni di raffinamneto qui considerate genera una analisi multirisoluzione. Vengono infine costruite le corrispondenti basi di ondine e basi biortogonali.*

ABSTRACT: *Refinable functions play a main role in the construction of wavelets; here we present a large class of refinable functions which can be identified through the explicit expression of their masks. These refinable functions can be considered as a generalization of cardinal B-splines, from which they borrow a few nice properties, like symmetry and total positivity, that are of relevant interest in most filtering applications. Moreover, each of the refinable functions here considered generates a multiresolution analysis. The associated wavelet bases and the corresponding biorthogonal bases are constructed, too.*

1 – Introduction

The impact of the multiresolution analysis in several fields, from approximation theory to digital signal processing, is well known. In recent

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years, several important papers and monographs have been devoted to multiresolution analyses and wavelets; we mention here only a few books which are particularly relevant, in that several different aspects of the wavelets are there treated and pointed out (see, for instance, [2], [3], [6], [7], [14], [22] and references therein).

We recall that the construction of both multiresolution analyses and wavelet bases are based on the properties of the underlying *refinable functions*, which are solutions of the *refinement* (or *scaling) equation*

$$\varphi(x) = \sum_{j \in \mathbb{Z}} a_j \varphi(2x - j), \quad x \in \mathbb{R} \tag{1.1}$$

The main purpose of this paper is to characterize some particular sets of refinable functions which have certain smoothness properties, and, at the same time, generate multiresolution analyses. The paper is organized as follows. In Section 2, we recall some basic definitions; in Section 3, we introduce the above quoted sets of refinable functions and show that, as a consequence of certain assumptions, they are completely determined by the explicit expression of their masks and that they can be considered as a generalization of the cardinal B-splines; their properties are pointed out in Section 4. Section 5 is addressed to the construction of the associated pre-wavelets and of the corresponding biorthogonal bases.

2 – Preliminaries

A *multiresolution analysis* (MRA) is a sequence $\{V_j\}$ of embedded approximation spaces, which are closed subspaces of $L^2(\mathbb{R})$ such that (i) $V_j \subset V_{j+1}$, $\forall j \in \mathbb{Z}$; (ii) $f \in V_j \iff f(2\cdot) \in V_{j+1}, \forall j \in \mathbb{Z}$; (iii) $f \in V_j \iff f(\cdot - 2^{-j} i) \in V_j, \forall j \in \mathbb{Z}$; (iv) there is an unconditional basis in V_0; (v) $\cap V_j = \{0\}$ (separation); (vi) $\overline{\cup V_j} = L^2(\mathbb{R})$ (density).

There are three ways to construct a multiresolution analysis: by the spaces V_j; by the refinable function φ; by the coefficients $\{a_j\}$ of (1.1) which form the *mask* $\mathbf{a}$ of the refinement equation and are required to fulfill the conditions:

$$\sum_{j \in Z} a_{2j+1} = \sum_{j \in Z} a_{2j} = 1. \tag{2.1}$$

Here, we shall follow the latter way introducing a family of suitable masks; indeed, it is known that the choice of the mask controls the properties of φ, and then it completely influences the efficiency of the associated wavelets and/or the subdivision schemes. On the other hand, the discussion of biorthogonal wavelet bases begins with the coefficients of the mask.

We associate with the mask $\mathbf{a}$ its *symbol*, that is the discrete Fourier transform of the sequence $\{a_j\}$, given by

$$p(z) = \sum_{j\in\mathbb{Z}} a_j z^j. \tag{2.2}$$

The existence and the properties of a solution to the refinement equation can be derived from the fulfillment of suitable conditions on $\mathbf{a}$; in particular, if

$$\textit{Condition } C \begin{cases} \mathbf{a} \text{ is compactly supported on } [0, n+1], a_0 > 0, a_{n+1} \neq 0, \\ p(z) \text{ is a Hurwitz polynomial,}^{(1)} \end{cases}$$

the following results were proved in [10]:

– there exists a unique refinable function, whose support is $[0, n+1]$, such that

$$\varphi(x) = \sum_{j=0}^{n+1} a_j \varphi(2x - j), \quad x \in \mathbb{R},$$

$$\sum_{j\in Z} \varphi(x - j) = 1, \quad x \in \mathbb{R};$$

– φ is a *ripplet*, that is

$$\varphi\begin{pmatrix} x_1 & x_2 & \dots & x_p \\ i_1 & i_2 & \dots & i_p \end{pmatrix} := \det_{l,j=1,\dots,p} \varphi(x_l - i_j) \geq 0 \tag{2.3}$$

for all $x_1 < x_2 < \ldots < x_p$; $i_1 < i_2 < \ldots < i_p, x_l \in \mathbb{R}, i_j \in \mathbb{Z}$, with the strict inequality holding if and only if

$$i_l < x_l < i_l + n, \quad l = 1, \dots, p; \tag{2.4}$$

(1) A polynomial is said to be of Hurwitz type when all its zeros have negative real part.

– the following relation holds:

$$(2.5)\quad \varphi \in C^{n-k}(\mathbb{R}) \Longleftrightarrow p(z) = (1+z)^{n-k+1} q_k(z),\ q_k(1) = 2^{-n+k},\ 1 \le k \le n;$$

– the function φ enjoys the *variation diminishing property*:

$$S^-\left(\sum_{j\in\mathbb{Z}} c_j \varphi(\cdot - j)\right) \le S^-(\mathbf{c}),$$

(here, $S^-(\mathbf{c})$ denotes the number of sign changes of the sequence $\mathbf{c}$), which is relevant for the study of the planar curve

$$F(x) = \sum_{j\in\mathbb{Z}} c_j \varphi(x - j), \quad c_j \in \mathbb{R}^2,\ j \in \mathbb{Z};$$

– the functions $\{\varphi(\cdot - j)\}_{j\in\mathbb{Z}}$ are linearly independent although not orthonormal.

Finally, it can be shown that φ generates a multiresolution analysis [17].

Thus, these refinable functions, whose masks satisfy *Condition C*, are endowed with many nice features; but it is possible to get more, for instance, a certain symmetry property.

Actually, beyond the role that these refinable functions play in geometric modeling, they turn out to be interesting for other reasons, in particular, for image processing, a field where one often requires to deal with real filters having linear phase or generalized linear phase, in order to avoid distortion of the signal. Symmetry provides a standard tool in digital filter design to achieve general linear-phase filtering. Thus, we shall consider a class $\mathbf{H}$ of masks satisfying the *Condition C* and the central symmetry condition

$$(2.6)\qquad a_{n+1-j} = a_j, \quad j = 0, \dots, \left[\frac{n+1}{2}\right].$$

A simple example of a mask $\mathbf{a} \in \mathbf{H}$ is provided by the mask of the B-spline M_{n+1} of degree n, which is a refinable function $\in C^{n-1}$ and whose mask is given by $\mathbf{a} = \{m_j, 0 \le j \le n+1\}$, where

$$m_j = \frac{1}{2^n}\binom{n+1}{j}.$$

Let us remark that the B-spline M_{n+1} not only provides an example of a refinable functions $\in C^{n-1}$ corresponding to a mask $\in \mathbf{H}$, but, what is more, M_{n+1} is the only refinable function of this type, as immediately follows from (2.5).

3 – On the set H

This section will be devoted to characterize the set $\mathbf{H}$; in particular, we shall give the explicit expression of the coefficients of a generic mask $\mathbf{a} \in \mathbf{H}$, such that the corresponding refinable function belongs to the class $C^{n-k}(\mathbb{R})$, $1 \le k \le n$, and we shall conclude that the refinable function φ associated with $\mathbf{a}$ can be considered as a *perturbation* of a B-spline. In fact, the following theorem holds; observe that, for the reasons better explained in the remark below, k can be assumed to be even.

Theorem 3.1. *Let $\varphi \in C^{n-k}(\mathbb{R})$, $2 \le k \le n$, with* $\operatorname{supp}(\varphi) = [0, n+1]$. *Then $\mathbf{a} \in \mathbf{H}$ if and only if*

$$a_j = \sum_{r=0}^{k/2} b_r^{(r)} \binom{n+1-2r}{j-r}, \quad j = 0, 1, \dots, n+1, \tag{3.1}$$

(assume $\binom{l}{i} = 0$ for $i < 0$ or $i > l$) where

$$\begin{gathered} b_j^{(r)} = b_j^{(r-1)} - \binom{k-2r+2}{j-r+1} b_{r-1}^{(r-1)}, \quad r = 1, \dots, K, \\ K := \frac{k}{2} - 1, \quad j = r+1, \dots, K+1, \end{gathered} \tag{3.2}$$

and $b_j^{(0)}$, $j = 0, \dots, k$, *are such that*

$$b_{k-r}^{(0)} = b_r^{(0)}, \qquad r = 0, 1, \dots, k, \tag{3.3}$$

$$b_{\frac{k}{2}}^{(0)} = 2^{k-n} - 2\sum_{j=0}^{K} b_j^{(0)}, \tag{3.4}$$

$$\det(b_{2j-i}^{(0)}; i, j = 1, \dots, p) > 0, \qquad p = 1, \dots, k \tag{3.5}$$

(*assume* $b_j^{(0)} = 0$ *for* $j < 0$ *and* $j > k$).

PROOF. For $k = 1$, the unique family of refinable functions that we obtain is that one of the B-splines, thus, we can assume $k > 1$, and, in particular, k even (see Remark below).

The proof of the necessary condition is based on the use of (2.5), which, for the polynomial (2.1), gives

$$\begin{aligned} p(z) &= \sum_{j=0}^{n+1} a_j z^j = (z+1)^{n-k+1} q_k(z) := \\ &:= (z+1)^{n-k+1}\left(b_0^{(0)} z^k + b_1^{(0)} z^{k-1} + \dots + b_{\frac{k}{2}}^{(0)} z^{\frac{k}{2}} + \dots + b_{k-1}^{(0)} z + b_k^{(0)}\right); \end{aligned} \tag{3.6}$$

observe now that (2.6) yields (3.3), *Condition C* yields (3.5) [8, p. 221], and (3.4) follows from (2.5) in that $q_k(1) = 2^{-n+k}$.

Moreover, any symmetric polynomial of even degree N can be decomposed as follows

$$\begin{aligned} \Pi_N(z) &:= d_0^{(0)} z^N + d_1^{(0)} z^{N-1} + \dots + d_{\frac{N}{2}}^{(0)} z^{\frac{N}{2}} + \dots + d_1^{(0)} z + d_0^{(0)} = \\ &= d_0^{(0)} (z+1)^N + z\Pi_{N-2}(z) \end{aligned} \tag{3.7}$$

where

$$\begin{cases} \Pi_{N-2}(z) = d_1^{(1)} z^{N-2} + d_2^{(1)} z^{N-3} + \dots + d_2^{(1)} z + d_1^{(1)} \\ d_j^{(1)} = d_j^{(0)} - \binom{N}{j} d_0^{(0)}, \qquad j = 1, \dots, \dfrac{N}{2}. \end{cases}$$

A repeated application of (3.6) to the polynomial $q_k(z)$ gives

$$q_k(z) = \sum_{j=0}^{k/2} b_j^{(j)} z^j (z+1)^{k-2j}.$$

and, consequently, yields (3.1); moreover, the coefficients (3.1) fulfill (2.1).

The converse proposition can be proved by reversing the reasoning above. ∎

REMARK. Observe that if k is odd, then $z = -1$ is necessarily a zero of $q_k(z)$ in (3.6). Thus

$$p(z) = (z+1)^{n-k+1} q_k(z) = (z+1)^{n-J+1} q_{J-2}(z), \quad J = k+1$$

and $\varphi \in C^{n-J}$ with J even.

We shall denote by $\Phi_{n,k}$, $n \in \mathbb{N}^+$ ($\mathbb{N}^+$ is the set of the positive integers), with $1 \leq k \leq n$, the set below

$$\Phi_{n,k} := \{\varphi \in C^{n-k}(\mathbb{R}) : \operatorname{supp}(\varphi) = [0, n+1],\ \mathbf{a} \in \mathbf{H}\};$$

moreover, we set $\Phi := \bigcup \Phi_{n,k}$, $n \in \mathbb{N}^+$.

Below we provide a few examples of masks belonging to $\mathbf{H}$.

For any k fixed, k even, there are $k/2$ free parameters, that is, the arbitrary numbers $b_j^{(0)}$, $j = 0, 1, \ldots, k/2-1$. For computational purposes, it is preferable to express such parameters as dyadic fractions.

For $k = 2$, the mask (3.1) becomes:

$$a_{j,n}^{(h)} = 2^{-h}\left[\binom{n+1}{j} + 4(2^{h-n}-1)\binom{n-1}{j-1}\right], \tag{3.8}$$
$$j = 0, 1, \ldots, n+1,\ n > 2,\ h > n-1,$$

which corresponds to the symbol

$$p_{n,h}(z) = 2^{-h}(1+z)^{n-1}(z^2 + (2^{h-n+2}-2)z + 1). \tag{3.9}$$

The second term in the mask (3.8) can be seen as a perturbation of the mask of the B-spline to which (3.8) reduces when $h = n$.

Observe that the symbol (3.9), in the case $n = 3$, has the form

$$
\begin{aligned}
p_{3,h}(z) &= 2^{-h}(1+z)^2(z^2 + (2^{h-1} - 2)z + 1) = \\
&= \frac{1}{2^h} + \frac{1}{2}z + \left(1 - \frac{1}{2^{h-1}}\right) z^2 + \frac{1}{2}z^3 + \frac{1}{2^h}z^4, \quad h > 2.
\end{aligned} \tag{3.10}
$$

These coefficients are a subset of those ones of the filters exploited by Burt and Adelson in vision analysis [1]. These filters are given explicitly by

$$
\begin{aligned}
p_d(z) &= d(1+z)^2 \left(z^2 + \left(\frac{1}{2d} - 2\right) z + 1\right) = \\
&= d + \frac{1}{2}z + (1-2d)z^2 + \frac{1}{2}z^3 + dz^4, \quad d \in \mathbb{R},
\end{aligned} \tag{3.11}
$$

d being a free parameter.

Obviously, setting $d = 1/2^h$, $h \geq 3$, we find the coefficients $a_{j,3}^{(h)}$, but for the Burt-Adelson filters also negative coefficients of the mask are allowed. For instance, for applications in vision, the choice $d = -0.1$ is especially popular, because it seems to lead to results that are better from the point of view of visual perception; this is probably because, in this case, the Burt-Adelson filter is very close to an orthonormal wavelet filter [7].

The refinable functions $\varphi_{n,2}$, associated with the mask (3.8), have been used in [9] as weight functions of Gaussian quadrature rules, which turn out to be useful in some applications, such as the wavelet Galerkin method.

For $k = 4$, the symbol $p(z)$ depends on two parameters, that is $b_0^{(0)}$ and $b_1^{(0)}$, which again, for computational convenience, we choose as dyadic fractions:

$$
b_0^{(0)} = 2^{-h}, \quad b_1^{(0)} = 2^{l-h}
$$

where $h, l \in \mathbb{R}$ are arbitrary numbers such that $h > n - 2 + \log_2(1 + 2^{l-1})$, in order to fulfill (3.5). Thus, the symbol has the form

$$
\begin{aligned}
p_{n,h,l}(z) =& 2^{-h}(1+z)^{n-3} \\
& (z^4 + 2^l z^3 + (2^{-n+4+h} - 2 - 2^{l+1})z^2 + 2^l z + 1), \quad n > 3,
\end{aligned} \tag{3.12}
$$

and the coefficients of the corresponding mask are

$$a_{j,n}^{(h,l)} = \frac{1}{2^h}\left[\binom{n+1}{j} + (2^l-4)\binom{n-1}{j-1} + (2^{-n+4+h} - 2^{l+2})\binom{n-3}{j-2}\right], \quad 0 \le j \le n+1. \tag{3.13}$$

Also in this case, the mask of the B-spline M_{n+1} can be obtained setting suitably the parameters h and l, that is, $h = n$ and $l = 2$.

The explicit form of masks (3.8) and (3.13), and in general of all masks of the type (3.1), shows that, for n and k fixed, the coefficients of any masks in $\mathbf{H}$ are linear combinations of the coefficients of the masks of the B-splines $M_{n+1-2j}(x-j)$, $j = 0, \dots, k/2$ (see also [11]).

4 – Properties of the refinable functions $\in \Phi$

The well-known property of the convolution of the B-splines, that is

$$M_{l+1} = M_l * M_1,$$

has been generalized in [10], where it was shown that the convolution between a refinable function whose symbol is a Hurwitz polynomial, and M_l is a refinable function corresponding to a symbol of Hurwitz type, too. But, for the refinable functions in Φ, even more is true, as the following proposition shows.

PROPOSITION 4.1. *If $\varphi \in \Phi$, then also the convolution $\varphi * M_l \in \Phi$; in particular, if $\varphi \in \Phi_{n,k}$, then $\varphi * M_l \in \Phi_{n+l+1,k}$.*

PROOF. The claim follows observing that the symbol corresponding to $\phi = \varphi_1 * \varphi_2$, where φ_1 and φ_2 are refinable functions, is the product of the symbols corresponding to φ_1 and φ_2 respectively. ∎

Thus, the effect of the convolution of a function in Φ with a B-spline is to increase the regularity by increasing the support, as in the case of the convolution of the B-splines with themselves.

Interestingly enough, also the convolution between refinable functions in Φ gives a refinable function whose symbol is of Hurwitz type and, moreover, the corresponding refinable function is in Φ. In fact, we have the following proposition.

PROPOSITION 4.2. *If* $\varphi_{n_1,k_1} \in \Phi_{n_1,k_1} \subset \Phi$ *and* $\varphi_{n_2,k_2} \in \Phi_{n_2,k_2} \subset \Phi$, *then the convolution* $\varphi_{n_1,k_1} * \varphi_{n_2,k_2} \in \Phi_{n_1+n_2+1,k_1+k_2} \subset \Phi$.

PROOF. Let us denote by $p_{n,k}(z)$ the symbol corresponding to $\varphi_{n,k}$. The symbol corresponding to $\varphi_{N,K} = \varphi_{n_1,k_1} * \varphi_{n_2,k_2}$ is

$$p_{N,K}(z) = \frac{1}{2} p_{n_1,k_1}(z) p_{n_2,k_2}(z) = (z+1)^{n_1+n_2+2-(k_1+k_2)} q_{k_1}(z) q_{k_2}(z).$$

$p_{N,K}(z)$ corresponds to a mask $\in \mathbf{H}$ and compactly supported on $[0, n_1 + n_2 + 2]$, then supp $(\varphi_{N,K}) = [0, n_1 + n_2 + 2]$. Moreover, from (2.5) we conclude that $\varphi_{N,K} \in C^{N-K}$, where $N = n_1 + n_2 + 1$ and $K = k_1 + k_2$. ◻

From Propositions 4.1 and 4.2 we can construct, for any fixed support $[0, n+1]$, symmetric and totally positive refinable functions with a chosen regularity, even (odd) for n even (odd), convolving suitable refinable functions in Φ. This observation suggests a practical method to obtain masks of type (3.1), specifically, it is sufficient to expand the product of two symbols of type (3.6). For instance, if we multiply two symbols of the form (3.9) corresponding to two different choices of the parameter n and h we obtain a symbol of type (3.12). Obviously, the highest regularity for a fixed support is achieved convolving the B-splines.

It is to be noted that φ is the minimally supported function in the family of all functions generating the same multiresolution analysis; indeed, its symbol has no symmetric roots [4].

The properties of symmetry, total positivity and compact support are not the unique properties that the refinable functions in Φ share with the B-splines. In fact, they fulfill some other interesting properties, analogous to certain ones of the B-splines or of the **L**-splines bases, such as those ones proved in [20] and in [15] respectively.

In many wavelet approximation problems, the Euler-Frobenius polynomial, defined by

$$\text{(4.1)} \qquad \Pi(z;\alpha) := \sum_{j=0}^{n} \varphi(\alpha+j) z^j, \quad \alpha \in [0,1), \quad \text{supp } \varphi = [0, n+1],$$

plays an important role [3].

Seen the properties of the functions belonging to Φ, the zeros of $\Pi(z;\alpha)$, say $\{\lambda_j(\alpha)\}$, $j = 1, \dots, n$, are real and negative for any $\alpha \in [0, 1)$ [12].

Now, let us order the sequence $\{\lambda_j(\alpha)\}$, $j = 1, \dots, n$, as follows

$$0 > \lambda_1(\alpha) > \dots > \lambda_n(\alpha).$$

In [19] it has been proved that $\lambda_j(\alpha)$, $j = 1, \dots, n$, are monotone functions of α and there results

$$\lim_{\alpha\to 1} \lambda_j(\alpha) = \lambda_{j+1}(0), \quad j = 1, \dots, n-1.$$

The graphs of Fig. 1 display the behaviour of the functions $\lambda_j(\alpha)$, $j = 1, 2, 3$, for the B-spline M_4.

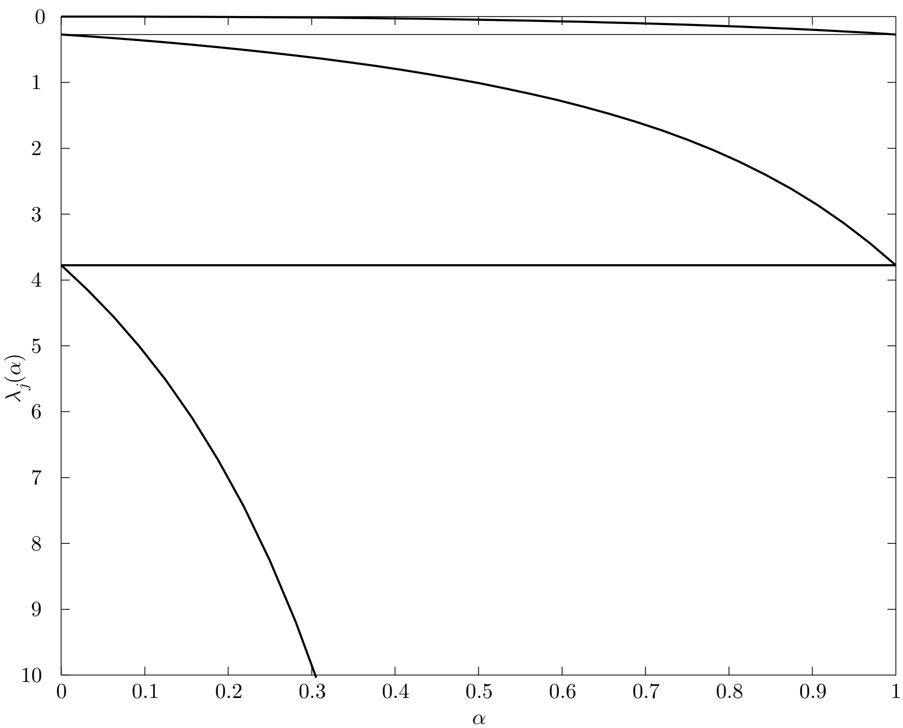

Fig. 1. The zeros $\{\lambda_j(\alpha)\}$, $j = 1, 2, 3$, for M_4.

Here, we shall prove some further results.

LEMMA 4.3. *For $t \in \mathbb{R}$ negative, $\Pi(t;\alpha)$ and $\frac{\partial \Pi}{\partial \alpha}(t;\alpha)$, corresponding to a refinable function $\in \Phi$, cannot vanish for the same value of $\alpha \in [0,1)$.*

PROOF. It is easy to show that, due to the compact support of φ,

$$\Pi(t;\alpha + j) = t^{-j}\Pi(t;\alpha), \quad j > 0.$$

In particular, for $j = 1$ and $\alpha = 0$, one gets

$$\Pi(t;1) = t^{-1}\Pi(t;0),$$

that is $\Pi(t;\alpha)$ has at least one zero for $\alpha \in [0,1)$.

We will first show that $\Pi(t;\alpha)$ cannot have an interval of zeros. Suppose that for $0 < r \le \alpha \le s < 1$, $\Pi(\alpha;t) = 0$. For $l = 0,\dots,n-1$ choose α_l such that $r \le \alpha_l \le s$. Then, for $D := (\varphi(\alpha_l + j))_{l,j=0}^{n-1}$ and $\mathbf{t} := (t^0,\dots,t^{n-1})^T$ we have $D\mathbf{t} = 0$ and so $\det(D) = 0$. This contradicts (2.3), (2.4) since $\alpha_l \in [r,s]$ implies that $-l < \alpha_l < n-l$, $l = 0,\dots,n-1$. Thus $\Pi(t;\alpha)$ can have only isolated zeros.

Suppose now that $\Pi(t;\alpha)$ has $2m+1$ zeros (with their multiplicity) in $[0,1)$. Then, the function

$$g(x) = \sum_{j=-L}^{n} \varphi(x+j)t^j, \quad x \in [0,L)$$

has $(2m+1)L$ zeros in $[0,L)$. On the other hand, φ is a ripplet, then $Z(g,[0,L)) \le n+L+1$ [21, p. 42]. Thus, we obtain the inequality $(2m+1)L \le n+L+1$, which is false for $m > L$ sufficiently large. □

Now, let $\Pi_{n,k}(z;\alpha)$ be the Euler-Frobenius polynomial corresponding to a $\varphi_{n,k} \in \Phi_{n,k}$ with the free parameters $b_j^{(0)}$, $j = 0,\dots,k/2-1$, fixed, and let $\lambda_j^{n,k}(\alpha)$, $j = 1,\dots,n-1$, be the corresponding zeros.

COROLLARY 4.4. *The polynomials $\Pi_{n,k}(t;\alpha)$ and $\Pi_{n-1,k}(t;\alpha)$, where $t \in \mathbb{R}$ is negative, cannot vanish simultaneously for $\alpha \in [0,1)$.*

PROOF. From Proposition 4.1 we have, for $l = 1$, $\varphi_{n,k}(x) = (\varphi_{n-1,k} * M_1)(x)$. Differentiating with respect to x yields

$$\frac{\partial \varphi_{n,k}(x)}{\partial x} = \varphi_{n-1,k}(x) - \varphi_{n-1,k}(x-1).$$

Thus,

$$\frac{\partial \Pi_{n,k}}{\partial \alpha}(t;\alpha) = (1-t)\Pi_{n-1,k}(t;\alpha),$$

and the claim follows from Lemma 4.3. ∎

THEOREM 4.5. *The zeros of* $\Pi_{n,k}(z;\alpha)$ *and* $\Pi_{n-1,k}(z;\alpha)$ *interlace, that is*

$$\begin{cases} \lambda_1^{n,k}(0) = \lambda_1^{n-1,k}(0) \\ \lambda_j^{n,k}(\alpha) > \lambda_j^{n-1,k}(\alpha) > \lambda_{j+1}^{n,k}(\alpha), \quad j = 1, \dots, n-1. \end{cases}$$

PROOF. The proof is *ab absurdo.* Suppose that the zeros of $\Pi_{n,k}$ and $\Pi_{n-1,k}$ do not interlace and let us reach a contradiction.

If the zeros do not interlace, then there exists a value of $\alpha \in [0,1)$, say $\tilde{\alpha}$, such that $\lambda_i^{n,k}(\tilde{\alpha}) = \lambda_j^{n-1,k}(\tilde{\alpha})$; thus, $\Pi_{n,k}(\lambda_i^{n,k}(\tilde{\alpha})) = \Pi_{n-1,k}(\lambda_j^{n-1,k}(\tilde{\alpha})) = 0$. But this contradicts Corollary 4.4. ∎

5 – Pre-wavelets and biorthogonal bases

The multiresolution analysis is the most appropriate framework to construct wavelet bases, that is functions laying in the orthogonal complement of V_j in V_{j+1}. Let us denote by W_j this orthogonal complement, then we have $W_j \perp W_l$, $j \neq l$.

A *pre-wavelet* (or *semi-orthogonal wavelet*) is any function $\psi \in W_0$ having the property that

$$\psi_{lm}(\cdot) \perp \psi_{rs}(\cdot), \quad l \neq r,$$

where $\psi_{lm}(\cdot) := \psi(2^l \cdot - m)$ [16]. A significant problem in multiresolution approximation is to find pre-wavelets $\in W_0$ such that their integer translates form an unconditional basis for W_0.

The first step to construct pre-wavelets associated with a refinable function φ, is to verify if the spaces V_j, generated by $\{\varphi_{jm}(\cdot)\}_{m\in\mathbb{Z}}$, where $\varphi_{jm}(\cdot) := \varphi(2^j \cdot -m)$, constitute a multiresolution analysis.

For the $\varphi \in \Phi$ this property, and other ones useful in constructing wavelet decomposition, follow from some results in [17]. In particular, we record the following facts.

PROPERTY 1. *Any refinable function $\varphi \in \Phi$, generates a multiresolution analysis.*

PROPERTY 2. *For any fixed* $\mathrm{supp}(\mathbf{a}) = [0, n+1]$*, let us denote by $V_{0,k}$ the space generated by the integer translates of $\varphi_{n,k} \in \Phi_{n,k}$, that is*

$$V_{0,k} := span\{\varphi_{n,k}(x-i) : i \in \mathbb{Z}\},$$

and define

$$V_{j,k} := span\{2^{-j/2}\varphi_{n,k}(2^{-j}x - i) : i \in \mathbb{Z}\}.$$

Consider any $\varphi_{n,k} \in \Phi_{n,k} \subset \Phi$ and introduce the function so defined

$$\psi_{0,k}(x) = \sum_{j\in\mathbb{Z}} (-1)^j \mu_{j-1,k} \varphi_{n,k}(2x - j)\,, \tag{5.1}$$

where

$$\mu_{j,k} = \int_{\mathbb{R}} \varphi_{n,k}(x)\varphi_{n,k}(2x+j)dx, \quad j \in \mathbb{Z}\,. \tag{5.2}$$

The function $\psi_{0,k}$ is a pre-wavelet in the sense that it is orthogonal to $V_{0,k}$ and has l^2-stable integer translates; moreover, $\psi_{0,k}$ is the function of minimal support in $V_{1,k}$ and the following relations hold: $W_{j,k} \perp V_{j,k}$; $V_{j+1,k} = V_{j,k} \oplus W_{j,k}$; $W_{l,k} \perp W_{j,k}, l \neq j$; $L^2 = \oplus W_{j,k}$.

From (5.2) it is easy to derive that $\psi_{0,k}$ is compactly supported on $[-n, n+1]$. Moreover, since $\varphi_{n,k}$ has generalized linear phase, so too does $\psi_{0,k}$. In particular, if $\varphi_{n,k}$ is the B-spline of degree n, then $\psi_{0,k}$ is the pre-wavelet constructed in [4].

In order to complete the characterization of the functions $\varphi_{n,k} \in \Phi_{n,k} \subset \Phi$, we also recall that the smoothness of $\varphi_{n,k}$ implies that certain polynomials can be built from the translates of $\varphi_{n,k}$. To this end, denote by $\mathbb{P}_r$ the space of the polynomials of degree at most r. From (3.6), we conclude that $\mathbb{P}_{n-1} \subseteq V_{0,k}$ [18].

A further property of $\psi_{0,k}$ is that it has n vanishing moments. In fact, since $V_{0,k} \supset \mathbb{P}_{n-1}$ and $\psi_{0,k}$ is orthogonal to $V_{0,k}$, then one has

$$\int_R x^j \psi_{0,k}(x) dx = 0, \qquad j = 0, \ldots, n-1.$$

In order to construct a decomposition and reconstruction formula for a function $f \in L^2(\mathbb{R})$ one needs dual bases of φ and ψ_0, that is functions $\widetilde{\varphi}$ and $\widetilde{\psi}_0$ such that

$$\langle \varphi_{rj}, \widetilde{\varphi}_{sl} \rangle = \delta_{rs}\delta_{jl}, \qquad \langle \psi_{0,rj}, \widetilde{\psi}_{0,sl} \rangle = \delta_{rs}\delta_{jl}, \tag{5.3}$$

where, as usual, ζ_{rj} denotes the dyadic dilates and the integer translates of the function $\zeta(x)$, that is $\zeta_{rj}(\cdot) := \zeta(2^r \cdot - j)$.

Thus, we have the following wavelet decomposition formula for f_N, the approximation of $f \in L^2(\mathbb{R})$ in V_N:

$$\begin{aligned} f_N(x) = & \sum_{r=N-M}^{N-1} \sum_{j \in \mathbb{Z}} 2^r \langle f_N, \widetilde{\psi}_{0,rj} \rangle \psi_{0,rj} + \\ & + \sum_{j \in \mathbb{Z}} 2^{N-M} \langle f_N, \widetilde{\varphi}_{N-M,j} \rangle \varphi_{N-M,j}(x) \end{aligned} \tag{5.4}$$

where $\widetilde{\varphi}$ and $\widetilde{\psi}$ are the *analysis functions*, while φ and ψ are the *synthesis functions*. Due to the fact that φ_{jk} and $\widetilde{\varphi}_{jk}$ are not orthogonal at the same level j, while they are orthogonal at different levels, this case is referred to in the literature as the *semi-orthogonal* case.

We can construct the dual bases of $\varphi_{n,k}$ and $\psi_{0,k}$ following the procedure outlined in [4]. An example corresponding to the case $n = 2$ and $k = 2$ can be found in [13].

The dual bases so constructed have infinite support and lay in the same space.

In order to achieve the generalized linear phase property, while maintaining the compact support of the dual bases, Cohen *et al.* [5] introduced the *biorthogonal* bases , i.e., two dual bases ψ_{mn}, $\widetilde{\psi}_{mn}$, each given by the dilates and the translates of a single function, ψ or $\widetilde{\psi}$, such that $\langle \psi(x-l), \widetilde{\psi}(x-l') \rangle = \delta_{ll'}$.

The innovation in the biorthogonal case comes from the considerable flexibility that one has for the choice of filters. This allows one to avoid the principal constraints associated to the use of orthogonal filters with the finite impulse response, namely, asymmetry and the numerical complexity of their coefficients, and semiorthogonal filters, namely, infinite support [5].

The wavelet decomposition formula for the biorthogonal case is analogous to (5.4). Also in this case we have two different wavelets: an analyzing wavelet $\widetilde{\psi}$ and a reconstruction wavelet ψ. This gives us the possibility to choose $\widetilde{\psi}$ enjoying properties useful for analysis, that is oscillations, vanishing moments, while we choose ψ enjoying properties attractive for synthesis, that is regularity, simplicity.

As an example, below we shall give the coefficients $\{\tilde{a}_j\}$ of the mask of one of the biorthogonal wavelets corresponding to $\phi_{3,2}$, namely that one supported on $[0,6]$ and symmetric, constructed following the procedure outlined in [5]:

$$\tilde{a}_0 = \tilde{a}_6 = -\frac{2^{-h}(1+2^{-h+2})}{4(2^{-h+2}-1)}, \qquad \tilde{a}_1 = \tilde{a}_5 = \frac{2^{-h+2}+1}{8(2^{-h+2}-1)},$$

$$\tilde{a}_2 = \tilde{a}_4 = \frac{2^{-2h+2}+5\cdot 2^{-h}-1}{4(2^{-h+2}-1)}, \qquad \tilde{a}_3 = \frac{2^{-h+2}-3}{4(2^{-h+2}-1)}.$$

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