Rendiconti di Matematica, Serie VII
Volume 20, Roma (2000), 323-336

Comonotone aggregation operators

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Dedicated to the memory of Gaetano Fichera, whose friendship honoured the first author

RIASSUNTO: *Si studiano gli operatori di aggregazione che sono $\oplus$-additivi sulle funzioni comonotone. La operazione $\oplus$ é una qualunque pseudo-addizione. Il risultato principale é un teorema generale di rappresentazione, che esprime gli operatori usando una classe di misure fuzzy (funzioni monotone di insiemi). In particolare la classe di misure fuzzy puó essere ottenuta a partire da una misura "discriminante" mediante una operazione di pseudo-moltiplicazione; in tal caso gli operatori sono espressi esattamente da integrali monotoni generali (integrale di Choquet, integrale di Sugeno, integrale fuzzy generale). Il teorema di rappresentazione fornisce un'ampia classe di operatori di aggregazione e generalizza sensibilmente il ben noto teorema di caratterizzazione dell'integrale di Choquet.*

ABSTRACT: *We consider the aggregation operators which are comonotone-$\oplus$-additive, i.e., $\oplus$-additive for comonotone functions; $\oplus$ is any pseudo-addition. The main result is a representation theorem which expresses any operator by means of a kind of general fuzzy integral. This expression uses a family of fuzzy measures linked by a $\oplus$-Cauchy equation. In particular, the family of fuzzy measures can be obtained from a "discriminant" fuzzy measure by a pseudo-multiplication. In this case, the aggregation operator is exactly expressed by a general fuzzy integral. The main result provides a large class of aggregation operators and gives a wide generalization of the well known characterization theorem of the Choquet's integral.*

KEY WORDS AND PHRASES: *Aggregation operators – Comonotone additivity – Fuzzy integral – Cauchy's equation*
A.M.S. CLASSIFICATION: 28E10

– Introduction

An aggregation operator is a procedure by which a unique value can be associated to the results obtained through different tests or different values of a data base. This unique value is a kind of mean value or average. Many aggregation operators of different type have been considered in connection with different situations [9], [13], [14]. In an axiomatic theory of aggregation operators the results of the tests or the values in a data base are described by a function f defined on the set Ω of the tests or the locations in the data base. The range of the function f is the set V of the possible values as results. The properties which can be recognized for the aggregation operator are suggested from the existence of any meaning structure in V which the operator must preserve.

We propose in this paper an axiomatic definition for the aggregation operators with arbitrarily many inputs (finite or infinite) when the set V is an interval $[0, F]\,(0 < F \leq +\infty)$. We request for the operators the natural properties of idempotence, monotonicity and continuity from below. Moreover, we assume that a structure of I-semigroup is defined on the interval $[0, F]$ by means of a pseudo-addition $\oplus$ and we request a property of weak (conditional) $\oplus$-additivity. The unconditioned $\oplus$-additivity is too strong: it is not true, fort instance, for the Choquet integral, which is the most known aggregation operator in the elementary case, when the structure of I-semigroup is given by means of the common addition. In order to obtain a wide class of operators we request only the comonotone-$\oplus$-additivity, i.e., the $\oplus$-additivity for function which are in relation of comonotonicity [1], [5].

The aim of the paper is the characterization of the $\oplus$-comonotone aggregation operators. First we note that to every operator of this kind a family of fuzzy measures $\{\mu_a,\, a \in [0, F]\}$ defined on Ω is associated. The element μ_a of the family is linked to the values of the operator for the basic functions at level a. For this family we give the characteristic properties and we prove that the aggregation operator is uniquely determined by the associated family of fuzzy measures. In fact we construct explicitly the operator, starting from the family of fuzzy measures and using the comonotone $\oplus$-additivity and the monotonicity.

We obtain the representation theorem (*main theorem*) in any I-semigroup, i.e., for any pseudo-addition. In the mathematical literature this result is never known in the particular and easier case of comonotone-additive operators, i.e. when $\oplus$ is the ordinary addition in $[0,+\infty]$. The representation theorem allows to study the further properties of the operators by analysing the behavior of the operators on the subset of the basic functions [4].

Finally, we consider a special form for the family $\{\mu_a\}$ assuming that it is built from a given measure or fuzzy measure by means of a pseudo-multiplication. We obtain the special aggregation operators defined by a general fuzzy integral [1], [2]. This is a wide generalization of the aggregation operators defined by means of the Choquet integral or the Sugeno integral [9].

1 – Preliminary

Let Ω be an abstract space, $\mathcal{A}$ a σ-algebra of subsets of Ω and $\mathcal{F}$ the family of all $\mathcal{A}$-measurable functions $f:\Omega\to[0,F]$, with $0<F\le+\infty$. A chain $\mathcal{M}_f\supset\mathcal{A}$ is associated to any $f\in\mathcal{F}$: it is constituted by the sets

$$C_f(x) =: \{\omega\in\Omega \mid f(\omega)>x\} \qquad x\in[0,F]\,.$$

We suppose that the interval $[0,F]$ has a structure of I-semigroup defined by means of a *pseudo-addition* $\oplus$. The binary operation $\oplus : [\,0,F\,]^2 \to [\,0,F\,]$ is called pseudo-addition if it is commutative, associative, monotone non decreasing, continuous, and if 0 is its neutral element.

The structure of the pseudo-additions is known; see [10] or [1], [8] for the general form of the operation $\oplus$. We shall use, as well, a *pseudo-difference* [12] defined for $a\le b$ by

$$b\ominus a = \inf\{x\in[0,F] \mid a\oplus x = b\}\,. \tag{1}$$

For any $a\in[0,F]$ and $A\in\mathcal{A}$, a basic function $b(a,A)$ is defined by:

$$b(a,A)(\omega)=\begin{cases} a & \text{if } \omega\in A\\ 0 & \text{if } \omega\notin A.\end{cases}$$

A function $s:\Omega\to[0,F]$ is called *simple function* if its range is finite. The simple functions have several representations by means of basic functions; the well-known classical-standard representation is:

(2) $$s = \bigvee_{i=1}^{n} b(a_i, A_i)$$

with $0 < a_1 < a_2 < \ldots < a_n$ and $A_i = \{\omega \in \Omega \,|\, s(\omega) = a_i\}$.

Moreover, any simple function admits many $\oplus$-*step representations*:

(3) $$s = \bigoplus_{i=1}^{m} b(c_i, C_i)$$

with $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_m$. The standard $\oplus$-step representation [1] is obtained from (2) with $m = n$ assuming:

$$c_1 = a_1\,,\ c_2 = a_2 \ominus a_1, \ldots\,,\ c_n = a_n \ominus a_{n-1}\,,\quad C_i = \bigcup_{j=i}^{n} A_j = \{\omega \in \Omega \mid s(\omega) \geq a_i\}.$$

The standard $\oplus$-step representation is minimal for the number of steps and for the height of the single steps.

The family $\mathcal{F}$ of the $\mathcal{A}$-measurable functions is generated by the set $\mathcal{B}$ of the *basic functions*. Let $\mathcal{S}$ be the set of all simple functions, we put: $\mathcal{S}_f =: \{s \in \mathcal{S} \mid s \leq f\}$. For every $f \in \mathcal{F}$ it is: $f(\omega) = \sup\{s(\omega) \mid s \in \mathcal{S}_f\}$ because every $f \in \mathcal{F}$ is the limit of a increasing sequence of simple functions ([7] Ch. IV).

It is useful to recall the definition of comonotonicity for functions in $\mathcal{F}$ (see [5]). Any function $f : \Omega \to [0, F]$ introduces on Ω a semi-order relation:

$$\omega_1 <_f \omega_2 \Longleftrightarrow f(\omega_1) < f(\omega_2)\,.$$

The functions f and g are called *comonotone* ($f \sim g$) if the two corresponding semi-orders are not contradictory, i.e., there exists no pair ω_1, ω_2 in Ω such that $\omega_1 <_f \omega_2$ and $\omega_2 <_g \omega_1$. An equivalent definition of comonotonicity is expressed by means of the chains associated to the functions:

(4) $$f \sim g \Longleftrightarrow \mathcal{M}_f \cup \mathcal{M}_g = \mathcal{M}_{f+g}.$$

Later we denote *fuzzy measure* any non decreasing set function $\mu : \mathcal{A} \to \mathbb{R}^+$ with $\mu(\emptyset) = 0$ (see [1], [5], [11]).

2 – Aggregation Operators

In this section we present an axiomatic approach of the aggregation operators which are $\oplus$-comonotone-additive, i.e., $\oplus$-additive for comonotone functions.

DEFINITION 2.1. A $\oplus$-comonotone aggregation operator is a functional $L : \mathcal{F} \to [0, F]$ satisfying the following properties:

(L1) $f(\omega) = c \quad \forall\, \omega \in \Omega \Longrightarrow L(f) = c$ (idempotence)

(L2) $f(\omega) \leq f'(\omega) \quad \forall\, \omega \in \Omega \Longrightarrow L(f) \leq L(f')$ (monotonicity)

(L3) $f_k(\omega) \nearrow f(\omega) \quad \forall\, \omega \in \Omega \Longrightarrow L(f_k) \nearrow L(f)$ (continuity from below)

(L4) $f_1 \sim f_2 \Longrightarrow L(f_1 \oplus f_2) = L(f_1) \oplus L(f_2)$ (comonotone $\oplus$-additivity).

The properties (L1-3) seem us to be natural for all aggregation operators ([6], [9], [13], [14]): we want that they model a kind of monotone and continuous averaging operation. The comonotone $\oplus$-additivity (L4) fits the operators to the structure of the semigroup acting on the range of values that shall be aggregated.

For these operators we shall recognize that they are completely determined by their values on the set $\mathcal{B}$ of basic functions.

PROPOSITION 2.2. *Given a $\oplus$-comonotone aggregation operator L*

– *for any $a \in]0, F]$ the map $\mu_a : \mathcal{A} \to [0, F]$ defined by*

$$\mu_a(A) =: L[b(a, A)] \tag{5}$$

is a fuzzy measure continuous from below;

– *the family $\{\mu_a | a \in]0, F]\}$ satisfies the following properties:*

(F1) $\mu_a(\Omega) = a \quad \forall\, a \in]0, F]$

(F2) $a < a' \Longrightarrow \mu_a \leq \mu_{a'}$

(F3) $a \nearrow a' \Longrightarrow \mu_a \nearrow \mu_{a'}$

(F4) $\mu_{a \oplus a'} = \mu_a \oplus \mu_{a'} \quad \forall\, a, a' \in]0, F]$.

The properties above are the restriction on $\mathcal{B}$ of the properties (L1-4). The last one assigns a link between the measures of every fixed element $A \in \mathcal{A}$. It is expressed by a Cauchy equation on the I-semigroup $([0,F],\oplus)$, (see [3]).

Proposition 2.2 characterizes the families of fuzzy measures which correspond to the comonotone-$\oplus$-additive aggregation operators. This is shown by the following representation theorem.

Main Theorem. *Let $\{\mu_a | a \in]0,F]\}$ be a family of fuzzy measures continuous from below. If it satisfies* (F1-4) *there exists a unique $\oplus$-comonotone aggregation operator L_μ engendered from it by means of* (5)*; i.e., for which it is:*

$$L_\mu[b(a,A)] = \mu_a(A) \quad \forall a \in]0,F], \quad \forall A \in \mathcal{A}. \tag{6}$$

The proof of this theorem is given in the next section. It consists in the construction of the operator $L_\mu : \mathcal{F} \to [0,F]$ in a quite natural way which is univocally imposed by (L3) and (L4). The check, however, of correctness of the construction and the verification of properties (L1-4) are really non trivial and laborious because we assume no restriction on the general form of the pseudo-addition $\oplus$.

3 – Proof of the Main Theorem: Construction of the comonotone aggregation operator from a given family of fuzzy measures

In this section we assume that a family $\{\mu_a | a \in]0,F]\}$ of fuzzy measures continuous from below and satisfying (F1-4) is given. We construct explicitly the associated $\oplus$-comonotone aggregation operator. First we give the form of the operator on the family $\mathcal{S}$ of the simple functions. Later the functional is extended on the family $\mathcal{F}$ of all measurable functions. Finally we prove that the so defined operator is a comonotone-$\oplus$-additive operator (properties L1-4) and it is the only one which, satisfying (6), corresponds to the given family of fuzzy measures.

For every simple function $s \in \mathcal{S}$ we consider any $\oplus$-step representation (3). We observe that the $\oplus$-step representation is built adding at various time comonotone functions. So, if the functional $L : \mathcal{F} \to [0, F]$ satisfies (L4) and (6), then its value $L(s)$ is necessarily given by

$$L(s) = \bigoplus_{i=1}^{m} L[b(c_i, C_i)] = \bigoplus_{i=1}^{m} \mu_{c_i}(C_i). \tag{7}$$

LEMMA 3.1. *The expression* (7) *has the same value for all* $\oplus$-*step representations of the same simple function; therefore it defines a functional* $L : \mathcal{S} \to [0, F]$.

PROOF. 1 - First we consider the possible presence in (3) of an inessential step. If $c_1 \oplus c_2 \oplus \ldots \oplus c_{j-1} = a_i$ and $a_i \oplus c_j = a_i$, the step $b(c_j, C_j)$ is inessential in the representation (3). We prove that the corresponding addendum $\mu_{c_j}(C_j)$ in (7) is also negligible. Let

$$S_j = \mu_{c_1}(C_1) \oplus \mu_{c_2}(C_2) \oplus \ldots \oplus \mu_{c_{j-1}}(C_{j-1}),$$

from (F2) and (F3) we obtain:

$$S_j \geq \mu_{c_1}(C_{j-1}) \oplus \mu_{c_2}(C_{j-1}) \oplus \cdots \oplus \mu_{c_{j-1}}(C_{j-1}) = \mu_{a_i}(C_{j-1})\,.$$

From the continuity of the operation $\oplus$ there exists $w \in [0, S_j]$ such that $S_j = w \oplus \mu_{a_i}(C_{j-1})$. Again from (F2) and (F3) it holds:

$$S_j \leq S_{j+1} = S_j \oplus \mu_{c_j}(C_j) \leq S_j \oplus \mu_{c_j}(C_{j-1}) = w \oplus \mu_{a_i \oplus c_j}(C_{j-1}) =$$

$$= w \oplus \mu_{a_i}(C_{j-1}) = S_j\,.$$

2 - We consider now the presence in (3) of a step subdivided in two steps. Let $C_{j+1} = C_j$, from (F3) it is: $\mu_{c_j}(C_j) \oplus \mu_{c_{j+1}}(C_j) = \mu_{c_j \oplus c_{j+1}}(C_j)$. The two basic functions $b(c_j, C_j)$ and $b(c_{j+1}, C_j)$ can be replaced by the unique one $b(c_j \oplus c_{j+1}, C_j)$ without modifying the result of the expression (7).

3 - By elimination of all negligible steps and all subdivisions of a single step in two or many steps, without changing the value of the expression (7), we obtain from any $\oplus$-step representation (3) a *quasi-standard* one, i.e., with $m = n$ and $C_i = \cup_{j=i}^{n} A_j$.

4 - We consider finally the possible presence in the quasi-standard $\oplus$-step representation of a non minimal step. Let $a_i = a_{i-1} \oplus c_i$ $(1 < i \leq n)$ and $c_i > a_i \ominus a_{i-1}$ then there exists d_i so that $c_i = (a_i \ominus a_{i-1}) \oplus d_i$ and $b(c_i, C_i) = b(a_i \ominus a_{i-1}, C_i) \oplus b(d_i, C_i)$. The presence in a quasi-standard $\oplus$-representation of the non minimal step $b(c_i, C_i)$ is equivalent to the presence of the inessential step $b(d_i, C_i)$. This one can be eliminated and the non minimal step $b(c_i, C_i)$ can be replaced by the minimal step $b(a_i \ominus a_{i-1}, C_i)$.

So we recognize that any $\oplus$-step representation of a given simple function s can be modified until to reach the standard $\oplus$-step representation without changing the value of the corresponding expression (7). Therefore, for any $\oplus$-step representation of s the value of the expression (7) is equal to that corresponding to the standard one. □

LEMMA 3.2. *The functional* $L : \mathcal{S} \to [0, F]$, *defined by* (7) *is monotone.*

PROOF. We consider the representations of two simple function s and s' by means of a common partition $\{A_1, A_2, \dots, A_n\}$

$$s = \bigvee_{i=1}^{n} b(a_i, A_i), \qquad s' = \bigvee_{i=1}^{n} b(a'_i, A_i)$$

with $a_1 \leq a_2 \leq \dots \leq a_n$ and $a'_1 \leq a'_2 \leq \dots \leq a'_n$. The relation $s \leq s'$ involves $a_i \leq a'_i$ for all i. For evaluating the functional L on s and s' we consider the $\oplus$-step representations:

$$s = \bigoplus_{i=1}^{n} b(c_i, C_i), \quad s' = \bigoplus_{i=1}^{n} b(c'_i, C_i), \tag{8}$$

with $C_i = \bigcup_{j=i}^{n} A_j$ and $\begin{cases} c_1 = a_1,\ c_2 = a_2 \ominus a_1, \dots, c_n = a_n \ominus a_{n-1}, \\ c'_1 = a'_1,\ c'_2 = a'_2 \ominus a'_1, \dots, c'_n = a'_n \ominus a'_{n-1}. \end{cases}$

It is sufficient to prove that

$$L(s) = \bigoplus_{i=1}^{n} \mu_{c_i}(C_i) \leq L(s') = \bigoplus_{i=1}^{n} \mu_{c'_i}(C_i) \tag{9}$$

if $a_k < a'_k$ and $a_i = a'_i$ for $i \neq k$.

Let $k = 1$, i.e., $a'_1 = c'_1 > a_1 = c_1$ and $a'_i = a_i$ for $i > 1$; putting $a'_1 \ominus a_1 = d$, it is $a_2 = a'_2 = a'_1 \oplus c'_2 = a_1 \oplus d \oplus c'_2$ and from (1) we deduce $c_2 = a_2 \ominus a_1 \leq d \oplus c'_2$. The two sides of (9) have the same addenda except $\mu_{c_1}(C_1) \oplus \mu_{c_2}(C_2)$ in the left side and $\mu_{c'_1}(C_1) \oplus \mu_{c'_2}(C_2)$ in the right one; for these terms we have:

$$\mu_{c_1}(C_1) \oplus \mu_{c_2}(C_2) \leq \mu_{c_1}(C_1) \oplus \mu_{d \oplus c'_2}(C_2) = \mu_{c_1}(C_1) \oplus \mu_d(C_2) \oplus \mu_{c'_2}(C_2) \leq$$
$$\leq \mu_{c_1}(C_1) \oplus \mu_d(C_1) \oplus \mu_{c'_2}(C_2) = \mu_{c'_1}(C_1) \oplus \mu_{c'_2}(C_2)\,.$$

We obtain the inequality (9) from the associativity and monotonicity of the operation $\oplus$.

Let $1 < k < n$, i.e., $a'_k > a_k$ and $a'_i = a_i$ for $i \neq k$; it is as well $c'_k > c_k$ and putting $c'_k \ominus c_k = d$, it is $a'_k = a_k \oplus d$, too. Similarly to the previous case we obtain $a_{k+1} = a'_{k+1} = a'_k \oplus c'_{k+1} = a_k \oplus d \oplus c'_{k+1}$; we deduce $c_{k+1} = a_{k+1} \ominus a_k \leq d \oplus c'_{k+1}$ and the inequality (9) holds likewise.

Finally, let $k = n$, i.e., $a'_i = a_i$ for $i < n$ and $a'_n > a_n$. The two sides of (9) have the same addenda except $\mu_{c_n}(C_n)$ in the left side and $\mu_{c'_n}(C_n) \geq \mu_{c_n}(C_n)$ in the right one. The inequality (9) is immediate. ∎

Lemma 3.3. *The functional L defined in $\mathcal{S}$ by* (7) *is comonotone-$\oplus$-additive.*

Proof. Let s_1, s_2 be two simple functions. If $s_1 \sim s_2$ the corresponding chains $\mathcal{M}_{s_1}$ and $\mathcal{M}_{s_2}$ belong to the chain $\mathcal{M}_{s_1+s_2} =: \{\Omega, C_1, C_2, ..., C_m, \emptyset\}$ and the functions s_1, s_2, $s_1 \oplus s_2$, admit the following $\oplus$-step representations:

$$s_1 = \bigoplus_{i=1}^{m} b(c_i, C_i)\,, \quad s_2 = \bigoplus_{i=1}^{m} b(c'_i, C_i)\,, \quad s_1 \oplus s_2 = \bigoplus_{i=1}^{m} b(c_i \oplus c'_i, C_i)\,.$$

By (F3) and associativity of $\oplus$ we obtain from (7): $L(s_1 \oplus s_2) = L(s_1) \oplus L(s_2)$. ∎

The Lemmas 3.1 and 3.2 suggest to assume the following definition for the functional L_μ. This definition is the only one which is in agreement with the properties (L3) and (L4) which are requested for the functional L_μ.

DEFINITION 3.4. The functional $L_\mu : \mathcal{F} \to [0, F]$ is defined by:

$$(10)\quad L_\mu(f) = \sup\{L(s) \,|\, s \in \mathcal{S}_f\} = \sup\{\bigoplus_{i=1}^{m} \mu_{c_i}(C_i) \;|\; \bigoplus_{i=1}^{m} b(c_i, C_i) \le f\}.$$

LEMMA 3.5. *The functional $L_\mu : \mathcal{F} \to [0, F]$ defined by* (10) *is non decreasing, continuous from below and it is the extension to $\mathcal{F}$ of the functional L defined on $\mathcal{S}$ by means of* (7)*:*

$$(11)\qquad L_\mu(s) = L(s) \quad \forall s \in \mathcal{S}.$$

PROOF. The monotonicity of the functional L_μ is evident from the definition itself; the equality (11) is consequence of the Lemma 3.2.

We must give only the proof of the continuity. Let $\{f_k\}$ be an increasing sequence of functions in $\mathcal{F}$ which converges pointwise to a simple function s:

$$f_k \nearrow s = \bigvee_{i=1}^{n} b(a_i, A_i) = \bigoplus_{i=1}^{n} b(c_i, C_i)\,.$$

Let $c'_1, \ldots, c'_n$ be real numbers such that:

$$c'_i < c_i \quad \text{and} \quad a_{i-1} < \bigoplus_{j=1}^{i} c'_j = a'_i < a_i \quad \forall i\,.$$

Taking $C_i^k = \{\omega \in \Omega \,|\, f_k(\omega) > a'_i\}$, it is $C_i^k \subseteq C_i$. It is furthermore $C_i^k \nearrow C_i$, because $f_k(\omega) \nearrow f(\omega)$ $\forall \omega \in \Omega$. Therefore, we have:

$$L_\mu(f_k) \ge \bigoplus_{i=1}^{n} \mu_{c'_i}(C_i^k) \qquad \text{and} \qquad \lim_{k\to+\infty} L_\mu(f_k) \ge \bigoplus_{i=1}^{n} \mu_{c'_i}(C_i).$$

From the arbitrariness of c'_i, we obtain $L_\mu(f_k) \nearrow L(s) = L_\mu(s)$.

Now we assume $f_k \nearrow f$, with $f \in \mathcal{F}$. Let $s \in \mathcal{S}_f$, it is $f_k \wedge s \nearrow s$ and therefore

$$\lim_{k\to+\infty} L_\mu(f_k) \ge \lim_{k\to+\infty} L_\mu(f_k \wedge s) = L(s)\,.$$

As s is arbitrary in $\mathcal{S}_f$, we obtain the thesis: $L_\mu(f_k) \nearrow L_\mu(f)$. □

Lemma 3.6. *The functional L_μ is comonotone $\oplus$-additive.*

Proof. Given two functions f and f' in $\mathcal{F}$, we consider the two families of simple functions:

$$\mathcal{S}^*_f=\{s=\bigoplus_{i=1}^{n} b(c_i,C_i)\mid n\in\mathbb{N},\ c_i\in]0,F],\ C_i=\{\omega\in\Omega|f(\omega)\geq c_i\}\}\subset\mathcal{S}_f\,,$$

$$\mathcal{S}^*_{f'}=\{s'=\bigoplus_{i=1}^{n} b(c'_i,C'_i)\mid n\in\mathbb{N},\ c'_i\in]0,F],\ C'_i=\{\omega\in\Omega|f'(\omega)\geq c'_i\}\}\subset\mathcal{S}_{f'}.$$

It is: $f=\sup\{s\,|\,s\in\mathcal{S}^*_f\}$, $f'=\sup\{s\,|\,s\in\mathcal{S}^*_{f'}\}$. Moreover, if $f\sim f'$, the sets $C_i=\{\omega\in\Omega\,|\,f(\omega)\geq c\}$ and $C_i=\{\omega\in\Omega\,|\,f'(\omega)\geq c\}$ belong to the chain $\mathcal{M}_{f+f'}$ for all $c\in[0,F]$. Therefore any function $s\in\mathcal{S}^*_f$ is comonotone with every function $s'\in\mathcal{S}^*_{f'}$.

Given, now, two sequences $s_k\in\mathcal{S}^*_f$ and $s'_k\in\mathcal{S}^*_{f'}$, with $s_k\nearrow f$ and $s'_k\nearrow f'$, we have $s_k\oplus s'_k\nearrow f\oplus f'$ and from Lemma 3.3 we obtain $L(s_k\oplus s'_k)=L(s_k)\oplus L(s'_k)$. From continuity from below of the functional L_μ and from the continuity of the operation $\oplus$, it is : $L_\mu(f\oplus f')=L_\mu(f)\oplus L_\mu(f')$. □

The Lemmas 3.5 and 3.6 show that the functional L_μ, defined by (10), verifies the properties (L1), (L3) and (L4). The equality (11) gives in particular (6) and from (F1) we obtain (L1), too.

Moreover the construction which leads to the definitions (7) and (10) is the only possible in agreement with (6) and (L1-4), so the functional L_μ is the unique $\oplus$-comonotone aggregation operator satisfying (6). This observation ends the proof of the Main Theorem.

4 – Aggregation Operators as General Integrals

A significative example of family $\{\mu_a|a\in]0,F]\}$ can be build, by means of a $\oplus$-fitting pseudo-multiplication [1], from a given measure or, as well, a given fuzzy measure $m:\mathcal{A}\to[0,M]\,,(0<M\leq+\infty)$.

Definition 4.1. A binary operation $\odot$: $[0,F]\times[0,M]\to[0,F]$ is called a $\oplus$-fitting pseudo-multiplication if the following properties are satisfied:

(M1) $a \odot 0 = 0 \odot m = 0$ (zero element)

(M2) $a \le a',\, m \le m' \Longrightarrow a \odot m \le a' \odot m'$ (monotonicity)

(M3) $(\sup a_k) \odot (\sup m_h) = \sup(a_k \odot m_h)$ (left continuity)

(M4) $(a \oplus b) \odot m = (a \odot m) \oplus (b \odot m)$ (left distributivity).

Let $m : \mathcal{A} \to [0, M]$ $(0 < M \le +\infty)$ be a fuzzy measure continuous from below, we define a family of fuzzy measures setting:

$$\mu_a(A) = a \odot m(A). \tag{12}$$

PROPOSITION 4.2. *If* $m(\Omega)$ *is right unitary element for the pseudo-multiplication* $\odot$*, the family* (12) *of fuzzy measures satisfies the properties* (F1-4)*. The corresponding aggregation operator is exactly the general integral* (*see* [1]*, and* [2])*:*

$$L_\mu(f) = \int^{\oplus} f \odot dm\,. \tag{13}$$

PROOF. From (M1-3) and from the continuity from below of m we obtain directly that every μ_a defined by means of (12) is a fuzzy measure continuous from below. From (M1) we obtain (F2) too; the property (F3) is equivalent to (M4) and, if $m(\Omega)$ is right unitary element, (F1) is true. Moreover if the family $\{\mu_a | a \in]0, F]\}$ is given by (12), the definitions (7) and (10) repeat the construction of the general integral given in [2] and [1]. □

The possibility of building the family of fuzzy measures from a given fuzzy measure using a pseudo-multiplication happens since the given fuzzy measure m is *discriminant* for the family $\{\mu_a\}$, according to the following definition.

DEFINITION 4.3. The fuzzy measure m is discriminant for the family $\{\mu_a | a \in]0, F]\}$ if

$$m(A) = m(A') \Longrightarrow \mu_a(A) = \mu_a(A') \quad \forall\, a \in\,]0, F].$$

Proposition 4.4. *Let $m : \mathcal{A} \to [0, M]$ be a fuzzy measure with $\mathrm{Ran}\, m = [0, M]$. If m is discriminant for the family $\{\mu_a | a \in]0, F]\}$, then there exists a $\oplus$-fitting pseudo-multiplication $\odot$ so that*

$$\mu_a(A) = a \odot m(A) , \quad L_\mu(f) = \int^{\oplus} f \odot dm .$$

Proof. From Definition 4.3 we obtain that the set $\{\mu_a(A) \,|\, A \in \mathcal{A},\, m(A) = x\}$ is a singleton in $[0, F]$ for all $a \in]0, F]$ and $x \in [0, M]$. So a function $\varphi :]0, F] \times [0, M] \to [0, F]$ is defined putting:

$$\{\varphi(a, m)\} =: \{\mu_a(A) \,|\, A \in \mathcal{A},\, m(A) = x\}.$$

We recognize that $\mu_a(A)$ depends really from A only by means of $m(A)$:

$$\mu_a(A) = \varphi[a, m(A)].$$

The pseudo-multiplication $\odot$ is defined by putting for $x \in [0, M]$

$$0 \odot x = 0 , \quad \text{and} \quad a \odot x = \varphi[a, x] \quad \forall\, a \in]0, F] .$$

This operation verifies the conditions (F1-4) and $M = m(\Omega)$ is right unitary element for it. ∎

Remark. If the discriminant measure m belongs to the family $\{\mu_a\}$, i.e., $m = \mu_u$ $u \in]0, F]$, then $M = u$ is as well a left unitary element for $\odot$.

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Lavoro pervenuto alla redazione il 5 ottobre 2000
Bozze licenziate il 3 novembre 2000

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Research supported by G.N.F.M. and MURST