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Universal relations for elastic materials

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In affectionate memory of Gaetano Fichera, admired colleague and friend

RIASSUNTO: *Si ricavano alcune relazioni universali per le deformazioni finite arbitrarie dei materiali elastici isotropi e trasversalmente isotropi. Esse sono adattate a certi casi particolari nei quali la deformazione è composta da due deformazioni di taglio, cui si sovrappone una dilatazione pura omogenea, quando la direzione del taglio non è necessariamente sovrapposta ad una delle direzioni principali della dilatazione.*

ABSTRACT: *Universal relations are derived for arbitrary finite deformations of isotropic and transversely isotropic elastic materials. These are specialized to certain particular cases in which the deformation consists of a simple shear superposed on a pure homogeneous deformation, when the direction of shear does not necessarily lie in a principal direction of the pure homogeneous deformation.*

1 – Introduction

In this paper a *universal relation* means a relation between the stress and deformation for a class of materials that is valid for all materials in the class and does not involve the values of the material parameters which vary from material to material within the class. We shall be concerned with isotropic and transversely isotropic elastic materials.

For an isotropic elastic material subjected to infinitesimal deformations, universal relations are obtained trivially from the constitutive equa-

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A.M.S. CLASSIFICATION: missing

tion by eliminating the Lamé constants from at most three of the expressions for the six components of stress, as is seen in Section 2.

In 1948, RIVLIN [1] presented a simple universal relation for finite simple shear of isotropic elastic materials (see eq. (4.11) below). Since then a number of workers have addressed the problem of determining further universal relations for finite deformations of isotropic elastic materials. HAYES and KNOPS [2] gave a procedure for obtaining such universal relations and concluded that at most three of them can be independent. They derived the relation (4.11) as a special case. WINEMAN and GANDHI [3] obtained a universal relation for simple shear superposed on an arbitrary pure homogeneous deformation when the direction of shear is a principal direction and the plane of shear is a principal plane of the pure homogeneous deformation (see eq. (4.9) below). The relation (4.11) again arises as a special case.

In the present paper we obtain universal relations for arbitrary finite deformations of isotropic and transversely isotropic elastic materials. BEATTY [4] obtained such relations in the isotropic case using a different procedure from that adopted here. The relation of the present work to his is discussed in Section 11. PUCCI and SACCOMANDI [5] have presented a more general procedure for obtaining universal relations than that employed by previous workers. The approach adopted in the present paper, although differently presented, is not essentially different from theirs. Our universal relations are obtained by eliminating the material parameters from the six equations expressing the stress components in terms of the strain. In the case when the material is isotropic this requires at most four of the six equations and the choice of these can be made in many ways leading to a correspondingly large number of universal relations. It is seen, in agreement with the conclusion of Hayes and Knops, that at most three of the universal relations so obtained are independent.

In Sections 4, 5 and 6 the general universal relations are specialized to three types of deformation of an isotropic elastic material. These consist of a simple shear superposed on an arbitrary pure homogeneous deformation, but differ in direction of shear or plane of shear.

If the material considered is transversely isotropic the expressions in the constitutive equation for all six of the stress components must, in general, be used to eliminate the material parameters, leading to a single relation (see eqs. (7.5) and (7.6) below). In Sections 8, 9 and 10 of the present paper this universal relation is specialized for the same three types of deformation as were discussed for an isotropic material. The pure homogeneous deformation has one of its principal directions parallel to the axis of rotational symmetry of the material.

2 – Universal relations for infinitesimal deformations

In classical elasticity theory, the constitutive equation for an isotropic material, referred to a rectangular cartesian coordinate system x, is

$$\boldsymbol{\sigma} = 2\mu\mathbf{e} + \lambda(\operatorname{tr}\mathbf{e})\boldsymbol{\delta}\,, \tag{2.1}$$

where $\boldsymbol{\sigma} = \|\sigma_{ij}\|$ is the stress matrix, $\mathbf{e} = \|e_{ij}\|$ is the (infinitesimal) strain matrix and $\boldsymbol{\delta}$ is the unit matrix. $\mathbf{e}$ is defined by

$$e_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)\,, \tag{2.2}$$

where $\mathbf{u} = (u_i)$ is the displacement vector for a particle in vector position $\mathbf{x} = (x_i)$. μ and λ, the Lamé constants, are material constants for the elastic material.

Equation (2.1) may be regarded as six simultaneous linear equations in μ and λ. We may solve two of these for μ and λ and then, by substitution in the remaining four equations, obtain four independent relations between $\boldsymbol{\sigma}$ and $\mathbf{e}$ which are independent of μ and λ and hence of the particular isotropic elastic material considered. These relations are

$$\frac{\sigma_{11} - \sigma_{22}}{e_{11} - e_{22}} = \frac{\sigma_{22} - \sigma_{33}}{e_{22} - e_{33}} = \frac{\sigma_{23}}{e_{23}} = \frac{\sigma_{31}}{e_{31}} = \frac{\sigma_{12}}{e_{12}}\,. \tag{2.3}$$

These are the *universal relations* for classical elasticity theory.

3 – Finite elasticity theory - isotropic materials

We consider an isotropic elastic material, which undergoes finite elastic deformations in which a particle initially in vector position $\mathbf{X} = (X_A)$, referred to a rectangular cartesian reference system x, moves to vector position $\mathbf{x}=(x_i)$. The deformation gradient matrix $\mathbf{g}=\|g_{ij}\|$ is defined by

$$\mathbf{g} = \partial\mathbf{x}/\partial\mathbf{X} = \|\partial x_i/\partial X_A\| \,. \tag{3.1}$$

The Cauchy strain matrix $\mathbf{C} = \|C_{AB}\|$ and Finger strain matrix $\mathbf{c} = \|c_{ij}\|$ are defined by

$$\mathbf{C} = \mathbf{g}^\dagger\mathbf{g}, \quad \mathbf{c} = \mathbf{g}\mathbf{g}^\dagger \,. \tag{3.2}$$

Let W be the strain-energy function for the material. For an isotropic material W is a function of the three strain invariants I_1, I_2, I_3 defined by

$$\begin{aligned} I_1 &= \operatorname{tr}\mathbf{C} = \operatorname{tr}\mathbf{c}\,, \\ I_2 &= \frac{1}{2}[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\mathbf{C}^2] = \frac{1}{2}[(\operatorname{tr}\mathbf{c})^2 - \operatorname{tr}\mathbf{c}^2]\,, \\ I_3 &= \det\mathbf{C} = \det\mathbf{c}\,. \end{aligned} \tag{3.3}$$

The Cauchy stress matrix $\boldsymbol{\sigma} = \|\sigma_{ij}\|$, referred to the system x, is given by

$$\boldsymbol{\sigma} = \alpha_1\mathbf{c} + \alpha_2\boldsymbol{\gamma} + \alpha_3\boldsymbol{\delta}\,, \tag{3.4}$$

where

$$\alpha_1 = 2I_3^{-1/2}(W_1 + I_1W_2), \quad \alpha_2 = -2I_3^{-1/2}W_2, \quad \alpha_3 = 2I_3^{1/2}W_3\,. \tag{3.5}$$

$\boldsymbol{\delta}$ denotes the unit matrix, $\boldsymbol{\gamma}$ denotes $\mathbf{c}^2$ and W_1, W_2, W_3 denote $\partial W/\partial I_1$, $\partial W/\partial I_2$, $\partial W/\partial I_3$ respectively.

We introduce the notation

$$c_\nu(\nu = 1,\dots,6) = c_{11}, c_{22}, c_{33}, c_{23}, c_{31}, c_{12}\,, \tag{3.6}$$

with analogous meanings for γ_ν, δ_ν and σ_ν. We also introduce the notation

$$L_\nu = \alpha_1 c_\nu + \alpha_2\gamma_\nu + \alpha_3\delta_\nu\,. \tag{3.7}$$

With this notation we may rewrite (3.4) as

$$L_\nu = \sigma_\nu \quad (\nu = 1,\dots,6)\,. \tag{3.8}$$

We may regard (3.8) as six simultaneous linear equations in the unknowns α_1, α_2, α_3.

We suppose that the deformation is such that the six expressions L_ν ($\nu = 1, \dots, 6$) are independent. Then, we may use any four of eqs. (3.8) to eliminate α_1, α_2, α_3 and obtain a universal relation. We may also obtain a universal relation by eliminating α_1 and α_2 from the three equations $L_\nu = \sigma_\nu$ ($\nu = 4, 5, 6$).

Equivalently, we may eliminate α_3 from eqs. (3.7) to obtain the five equations

$$(3.9)\quad L_1 - L_3 = \sigma_1 - \sigma_3, \quad L_2 - L_3 = \sigma_2 - \sigma_3, \quad L_\nu = \sigma_\nu \quad (\nu = 4, 5, 6)\,.$$

Then, we may use any three of these equations to eliminate α_1 and α_2 and obtain a universal relation. In particular, by choosing the first two of eqs. (3.9) together with one of the remaining relations, we obtain the three universal relations

$$(3.10)\qquad \begin{vmatrix} \sigma_1 - \sigma_3 & \sigma_2 - \sigma_3 & \sigma_\nu \\ c_1 - c_3 & c_2 - c_3 & c_\nu \\ \gamma_1 - \gamma_3 & \gamma_2 - \gamma_3 & \gamma_\nu \end{vmatrix} = 0 \quad (\nu = 4, 5, 6)\,.$$

These are evidently independent.

With $\boldsymbol{\gamma} = \mathbf{c}^2$ and the notation

$$(3.11)\quad A = (c_2 - c_3)(c_2c_3 - c_4^2) + (c_3 - c_1)(c_3c_1 - c_5^2) + (c_1 - c_2)(c_1c_2 - c_6^2)\,,$$

we obtain, from (3.10), the three independent universal relations

$$(3.12)\quad \begin{aligned} &(\sigma_1 - \sigma_3)[c_5c_6(c_2 - c_3) + c_4(c_5^2 - c_6^2)] + \\ &\quad - (\sigma_2 - \sigma_3)[c_4(c_1 - c_2)(c_3 - c_1) + c_5c_6(c_1 - c_3) + c_4(c_4^2 - c_6^2)] = \sigma_4 A, \end{aligned}$$

$$(3.13)\quad \begin{aligned} &(\sigma_1 - \sigma_3)[c_5(c_1 - c_2)(c_2 - c_3) + c_4c_6(c_2 - c_3) + c_5(c_5^2 - c_6^2)] + \\ &\quad - (\sigma_2 - \sigma_3)[c_4c_6(c_1 - c_3) + c_5(c_4^2 - c_6^2)] = \sigma_5 A\,, \end{aligned}$$

$$(3.14)\quad \begin{aligned} &(\sigma_1 - \sigma_3)[c_6(c_1 - c_3)(c_2 - c_3) + c_4c_5(c_2 - c_3) + c_6(c_5^2 - c_6^2)] + \\ &\quad - (\sigma_2 - \sigma_3)[c_6(c_1 - c_3)(c_2 - c_3) + c_4c_5(c_1 - c_3) + c_6(c_4^2 - c_6^2)] = \sigma_6 A. \end{aligned}$$

It is easily seen that any universal relation, obtained by some other choice of three equations from (3.9), can be derived from the three relations (3.10). For each choice of $\nu = 4, 5, 6$ the relation (3.10) expresses the condition that the straight line in the $\alpha_1\alpha_2$-plane, whose equation is

$$c_\nu \alpha_1 + \gamma_\nu \alpha_2 = \sigma_\nu \,, \tag{3.15}$$

passes through the intersection of the two straight lines whose equations are

$$\begin{aligned} (c_1 - c_3)\alpha_1 + (\gamma_1 - \gamma_3)\alpha_2 &= \sigma_1 - \sigma_3 \,, \\ (c_2 - c_3)\alpha_1 + (\gamma_2 - \gamma_3)\alpha_2 &= \sigma_2 - \sigma_3 \,. \end{aligned} \tag{3.16}$$

Thus, all five of the straight lines (3.15) and (3.16) have a common intersection and therefore from any three of them a universal relation analogous to (3.10) can be obtained. For example, from the fact that the three straight lines (3.15) with $\nu = 4, 5, 6$ have a common intersection, we obtain the universal relation

$$\begin{vmatrix} \sigma_4 & \sigma_5 & \sigma_6 \\ c_4 & c_5 & c_6 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{vmatrix} = 0 \,. \tag{3.17}$$

It follows from (3.16) that

$$(c_1 - c_2)\alpha_1 + (\gamma_1 - \gamma_2)\alpha_2 = \sigma_1 - \sigma_2 \,. \tag{3.18}$$

Accordingly this straight line also passes through the common intersection of (3.15) and (3.16). Thus, a universal relation can be obtained by eliminating α_1 and α_2 from any three of the six linear equations (3.15), (3.16) and (3.18).

In Sections 4, 5 and 6, three special cases of the relations (3.12)-(3.14) are presented. In each case the deformation consists of a simple shear superposed on a pure homogeneous deformation with principal extension ratios λ_1, λ_2, λ_3.

4 – Isotropic material - special case I

We choose the axes of the rectangular cartesian reference system x to be parallel to the principal directions of the pure homogeneous deformation. We superpose on the pure homogeneous deformation a simple shear of amount K/λ_3; the direction of shear lies in the 12-plane and the plane of shear is parallel to the 3-axis. The deformation is described by

$$x_1 = \lambda_1 X_1 + K_1 X_3, \quad x_2 = \lambda_2 X_2 + K_2 X_3, \quad x_3 = \lambda_3 X_3\,, \tag{4.1}$$

where

$$K_1 = K\cos\theta, \quad K_2 = K\sin\theta \tag{4.2}$$

and θ is the angle between the direction of shear and the 1-direction. Then, from (3.2),

$$\begin{aligned} &c_1 = \lambda_1^2 + K_1^2, \quad c_2 = \lambda_2^2 + K_2^2, \quad c_3 = \lambda_3^2\,,\\ &c_4 = K_2\lambda_3, \qquad c_5 = K_1\lambda_3, \qquad c_6 = K_1K_2\,. \end{aligned} \tag{4.3}$$

By substituting from (4.3) in (3.12)-(3.14), we obtain

$$\begin{aligned} K_2\lambda_3\{(\sigma_1-\sigma_3)K_1^2\lambda_2^2 - (\sigma_2-\sigma_3)[(\lambda_2^2-\lambda_1^2)(\lambda_1^2-\lambda_3^2)+\\ +\lambda_1^2(K_2^2-K_1^2)+K_1^2\lambda_2^2]\} = \sigma_4 A\,, \end{aligned} \tag{4.4}$$

$$\begin{aligned} K_1\lambda_3\{(\sigma_1-\sigma_3)[(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)+\lambda_2^2(K_1^2-K_2^2)+K_2^2\lambda_1^2]+\\ -(\sigma_2-\sigma_3)K_2^2\lambda_1^2\} = \sigma_5 A\,, \end{aligned} \tag{4.5}$$

$$\begin{aligned} K_1K_2\{(\sigma_1-\sigma_3)[\lambda_1^2(\lambda_2^2-\lambda_3^2)+\lambda_1^2K_2^2+\lambda_2^2K_1^2]+\\ -(\sigma_2-\sigma_3)[\lambda_2^2(\lambda_1^2-\lambda_3^2)+\lambda_1^2K_2^2+\lambda_2^2K_1^2]\} = \sigma_6 A\,. \end{aligned} \tag{4.6}$$

where A, defined in (3.11), is given by

$$\begin{aligned} A = &-(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)+\\ &+K_1^2\lambda_2^2(2\lambda_1^2-\lambda_2^2)-K_2^2\lambda_1^2(2\lambda_2^2-\lambda_1^2)+\\ &-\lambda_3^2(\lambda_1^2K_1^2-\lambda_2^2K_2^2)+(K_1^2-K_2^2)(\lambda_1^2K_2^2+\lambda_2^2K_1^2)\,. \end{aligned} \tag{4.7}$$

If $\theta = 0$, so that $K_2 = 0$ and $K_1 = K$, the relation (4.5) becomes

$$\{(\sigma_1 - \sigma_3)K\lambda_3 - \sigma_5(\lambda_1^2 - \lambda_3^2 + K^2)\}\{(\lambda_2^2 - \lambda_3^2)(\lambda_1^2 - \lambda_2^2) + K^2\lambda_2^2\} = 0\,, \tag{4.8}$$

whence

$$(\sigma_1 - \sigma_3)K\lambda_3 = \sigma_5(\lambda_1^2 - \lambda_3^2 + K^2)\,, \tag{4.9}$$

or

$$(\lambda_2^2 - \lambda_3^2)(\lambda_1^2 - \lambda_2^2) + K^2\lambda_2^2 = 0\,. \tag{4.10}$$

Since σ_1, σ_3 and σ_5 depend continuously on λ_1, λ_2, λ_3 and K, the relation (4.9) must be valid even if (4.10) is satisfied.

The universal relation (4.9) was previously obtained by WINEMAN and GANDHI [3]. If $\lambda_1 = \lambda_2 = \lambda_3 = 1$, we obtain RIVLIN's [1] universal relation

$$\sigma_1 - \sigma_3 = K\sigma_5\,, \tag{4.11}$$

If $\theta = 0$, the relations (4.4) and (4.6) become

$$\sigma_4 A = 0, \quad \sigma_6 A = 0\,, \tag{4.12}$$

where

$$A = -(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2) + K^2\lambda_2^2(2\lambda_1^2 - \lambda_2^2) - \lambda_1^2\lambda_3^2K^2 + \lambda_2^2K^4. \tag{4.13}$$

It follows that $\sigma_4 = \sigma_6 = 0$. This also follows trivially from the constitutive equation for $\boldsymbol{\sigma}$ and the fact that if $\theta = 0$, then $c_4 = c_6 = \gamma_4 = \gamma_6 = 0$.

With (4.3), the universal relation (3.17) yields

$$\lambda_1^2 K_2\sigma_5 - \lambda_2^2 K_1\sigma_4 = (\lambda_1^2 - \lambda_2^2)\lambda_3\sigma_6\,. \tag{4.14}$$

This result could also be obtained, with greater difficulty, by eliminating $\sigma_1 - \sigma_3$ and $\sigma_2 - \sigma_3$ from the universal relations (4.4)-(4.6).

By substituting from (4.2) in (4.4) and (4.6) and differentiating with respect to θ we can obtain the two universal relations

$$K\lambda_3\{(\sigma_1-\sigma_3)K^2\lambda_2^2+(\sigma_2-\sigma_3)(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2+K^2)\}=\frac{d\sigma_4}{d\theta}\Big|_{\theta=0}A, \tag{4.15}$$

$$\begin{aligned} &K^2\{(\sigma_1-\sigma_3)[\lambda_1^2(\lambda_2^2-\lambda_3^2)+\lambda_2^2K^2]+\\ &\quad-(\sigma_2-\sigma_3)[\lambda_2^2(\lambda_1^2-\lambda_3^2)+\lambda_2^2K^2]\}=\frac{d\sigma_6}{d\theta}\Big|_{\theta=0}A\,, \end{aligned} \tag{4.16}$$

where A is given by (4.13).

5 – Isotropic material - special case II

We again choose the axes of the rectangular cartesian reference system x to be parallel to the principal directions of the pure homogeneous deformation. On the pure homogeneous deformation is superposed a simple shear of amount K, for which the plane of shear is the 12-plane of the system x and the direction of shear is inclined at an angle θ to the 1-direction. The deformation is described, in the reference system x, by

$$\begin{aligned} x_1&=(1-Kab)\lambda_1X_1+Ka^2\lambda_2X_2\,,\\ x_2&=-Kb^2\lambda_1X_1+(1+Kab)\lambda_2X_2\,,\\ x_3&=\lambda_3X_3\,, \end{aligned} \tag{5.1}$$

where

$$a=\cos\theta,\quad b=\sin\theta\,. \tag{5.2}$$

With (3.1), (3.2) and (3.6) we obtain

$$\begin{aligned} c_1&=(1-Kab)^2\lambda_1^2+K^2a^4\lambda_2^2,\\ c_2&=K^2b^4\lambda_1^2+(1+Kab)^2\lambda_2^2\,,\\ c_3&=\lambda_3^2,\\ c_4&=c_5=0,\\ c_6&=-Kb^2\lambda_1^2(1-Kab)+Ka^2\lambda_2^2(1+Kab)\,. \end{aligned} \tag{5.3}$$

By introducing $c_4 = c_5 = 0$ into the universal relation (3.14) and into (3.11) we obtain

$$(5.4) \qquad (\sigma_1 - \sigma_2)c_6 = (c_1 - c_2)\sigma_6 \,,$$

or

$$(5.5) \qquad (c_2 - c_3)(c_1 - c_3) - c_6^2 = 0 \,,$$

where the c's are given by (5.3). From the fact that $\boldsymbol{\sigma}$ depends continuously on $\mathbf{c}$, it follows that the universal relation (5.4) is satisfied even if (5.5) is satisfied.

A relation equivalent to the relation (4.9) of Wineman and Gandhi can (allowing for difference of notation) be recovered by taking $\theta = 0$ (i.e. $a = 1$, $b = 0$) in (5.3) and substituting the resulting expressions in (5.4).

With $c_4 = c_5 = 0$, the universal relations (3.12) and (3.13) yield $\sigma_4 = \sigma_5 = 0$.

6 – Isotropic material - special case III

In this section we again suppose that the isotropic material is subjected to a pure homogeneous deformation and adopt a rectangular cartesian reference system x whose axes are parallel to its principal directions. On this deformation is superposed a simple shear of amount K. The direction of shear is parallel to the 3-direction of the reference system x and the plane of shear is inclined at an angle θ to the 1-direction.

The deformation is described in the reference system x by

$$(6.1) \qquad x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3 + K(a\lambda_1 X_1 + b\lambda_2 X_2) \,,$$

where

$$(6.2) \qquad a = \cos\theta, \qquad b = \sin\theta \,.$$

From (6.1) and the definition of $\mathbf{c}$ in (3.2) we obtain

$$
\begin{aligned}
c_1 &= \lambda_1^2,\\
c_2 &= \lambda_2^2,\\
c_3 &= \lambda_3^2 + K^2(a^2\lambda_1^2 + b^2\lambda_2^2)\,,\\
c_4 &= bK\lambda_2^2,\\
c_5 &= aK\lambda_1^2,\\
c_6 &= 0\,.
\end{aligned}
\tag{6.3}
$$

By substituting from (6.3) in eqs (3.11)-(3.14) we obtain

$$
\begin{aligned}
bK\lambda_2^2\{(\sigma_3-\sigma_2)[(\lambda_1^2-\lambda_2^2)(\lambda_3^2-\lambda_1^2)+\lambda_1^2K^2(b^2\lambda_2^2+a^2(\lambda_1^2-\lambda_2^2))]+\\
+(\sigma_1-\sigma_3)a^2K^2\lambda_1^4\} = \sigma_4 A\,,
\end{aligned}
\tag{6.4}
$$

$$
\begin{aligned}
aK\lambda_1^2\{(\sigma_1-\sigma_3)[(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)+\lambda_2^2K^2(a^2\lambda_1^2+b^2(\lambda_2^2-\lambda_1^2))]+\\
+(\sigma_3-\sigma_2)b^2K^2\lambda_2^4\} = \sigma_5 A,
\end{aligned}
\tag{6.5}
$$

$$
\begin{aligned}
abK^2\lambda_1^2\lambda_2^2\{(\sigma_1-\sigma_3)[\lambda_2^2-\lambda_3^2-K^2(a^2\lambda_1^2+b^2\lambda_2^2)]+\\
+(\sigma_3-\sigma_2)[\lambda_1^2-\lambda_3^2-K^2(a^2\lambda_1^2+b^2\lambda_2^2)]\} = \sigma_6 A\,,
\end{aligned}
\tag{6.6}
$$

where

$$
\begin{aligned}
A = &-(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)+\\
&+K^2\{\lambda_1^2a^2[\lambda_2^2(\lambda_2^2-\lambda_3^2)+\lambda_3^2(\lambda_1^2-\lambda_2^2)]+\\
&+\lambda_2^2b^2[\lambda_1^2(\lambda_3^2-\lambda_1^2)+\lambda_3^2(\lambda_1^2-\lambda_2^2)]\}+\\
&+K^4\lambda_1^2\lambda_2^2(b^2-a^2)(a^2\lambda_1^2+b^2\lambda_2^2)\,.
\end{aligned}
\tag{6.7}
$$

If $\theta = 0$, so that $a = 1$ and $b = 0$, equation (6.5) becomes

$$
(\sigma_1-\sigma_3)K\lambda_1^2\{(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)+\lambda_1^2\lambda_2^2K^2\} = \sigma_5 A\,,
\tag{6.8}
$$

where, from equation (6.7),

$$
A = -\{(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)+\lambda_1^2\lambda_2^2K^2\}(\lambda_3^2-\lambda_1^2+K^2\lambda_1^2)\,.
\tag{6.9}
$$

It follows that

$$(\sigma_3-\sigma_1)K\lambda_1^2=\sigma_5(\lambda_3^2-\lambda_1^2+K^2\lambda_1^2)\,. \tag{6.10}$$

If in eqs. (6.4) and (6.6) we take $\theta=0$, we obtain $\sigma_4=\sigma_6=0$. These results also follow trivially from the constitutive equation (3.4). However, by differentiating the equations with respect to θ and taking $\theta=0$, we obtain the universal relations

$$\begin{aligned}K\lambda_2^2\{(\sigma_3-\sigma_2)[(\lambda_1^2-\lambda_2^2)(\lambda_3^2-\lambda_1^2)+\lambda_1^2K^2(\lambda_1^2-\lambda_2^2)]+\\+(\sigma_1-\sigma_3)K^2\lambda_1^4\}=\frac{d\sigma_4}{d\theta}\Big|_{\theta=0}A\,,\end{aligned} \tag{6.11}$$

$$\begin{aligned}K^2\lambda_1^2\lambda_2^2\{(\sigma_1-\sigma_3)(\lambda_2^2-\lambda_3^2-K^2\lambda_1^2)+\\+(\sigma_3-\sigma_2)(\lambda_1^2-\lambda_3^2-K^2\lambda_1^2)\}=\frac{d\sigma_6}{d\theta}\Big|_{\theta-0}A\,,\end{aligned} \tag{6.12}$$

where A is given by (6.9).

Allowing for differences of notation, the relation (6.10) is equivalent to the relation (4.9). This is not surprising since both relations apply to deformations in which the direction of shear is a principal direction of the pure homogeneous deformation and the plane of shear is a principal plane. We note, however, that the relations (4.15) and (4.16) are not equivalent to (6.11) and (6.12). This reflects the fact that the approach to the deformation is along different paths.

7 – Transversely isotropic elastic materials

We now suppose that the elastic material has transverse isotropy. We choose the reference system x to have its 3-axis normal to the plane of isotropy. Then the strain energy W, per unit initial volume, is a function of the five strain invariants I_μ ($\mu=1,\dots,5$), where I_1, I_2, I_3 are given [6] by eqs. (3.3) and

$$I_4=C_{33},\quad I_5=C_{3A}C_{A3}\,, \tag{7.1}$$

and $\mathbf{C}=\|C_{AB}\|$ is defined in (3.2).

The Cauchy stress $\boldsymbol{\sigma} = \|\sigma_{ij}\|$ is now given by

$$\boldsymbol{\sigma} = \alpha_1 \mathbf{c} + \alpha_2 \boldsymbol{\gamma} + \alpha_4 \boldsymbol{\phi} + \alpha_5 \boldsymbol{\psi} + \alpha_3 \boldsymbol{\delta}\,, \tag{7.2}$$

where α_1, α_2, α_3 are given by (3.5),

$$\alpha_4 = 2 I_3^{-1/2} W_4, \qquad \alpha_5 = 2 I_3^{-1/2} W_5\,, \tag{7.3}$$

and the notation $W_\mu = \partial W / \partial I_\mu$ is used. $\boldsymbol{\gamma} = \mathbf{c}^2$ and $\mathbf{c}$ is given by (3.2). $\boldsymbol{\phi} = \|\phi_{ij}\|$, $\boldsymbol{\psi} = \|\psi_{ij}\|$ are defined by

$$\phi_{ij} = \frac{\partial x_i}{\partial x_3}\frac{\partial x_j}{\partial x_3}, \quad \psi_{ij} = \frac{\partial x_i}{\partial x_3}\frac{\partial x_k}{\partial x_3} c_{kj} + \frac{\partial x_j}{\partial x_3}\frac{\partial x_k}{\partial x_3} c_{ki}\,. \tag{7.4}$$

We adopt the notation (3.6) with analogous meanings for γ_ν, δ_ν, σ_ν, ϕ_ν and ψ_ν.

The matrix equation (7.2) may be regarded as six linear simultaneous equations in the five quantities $\alpha_1, \ldots, \alpha_5$. With the assumption that the expressions for the six stress components are linearly independent, we can eliminate $\alpha_1, \ldots, \alpha_5$ to obtain the universal relation

$$\Delta = 0\,, \tag{7.5}$$

where

$$\Delta = -\begin{vmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 \\ \psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{vmatrix}\,. \tag{7.6}$$

This can be rewritten as

$$\Delta = \begin{vmatrix} \sigma_1 - \sigma_3 & \sigma_2 - \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 \\ c_1 - c_3 & c_2 - c_3 & c_4 & c_5 & c_6 \\ \gamma_1 - \gamma_3 & \gamma_2 - \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ \phi_1 - \phi_3 & \phi_2 - \phi_3 & \phi_4 & \phi_5 & \phi_6 \\ \psi_1 - \psi_3 & \psi_2 - \psi_3 & \psi_4 & \psi_5 & \psi_6 \end{vmatrix}\,. \tag{7.7}$$

If the material has a direction of inextensibility parallel to the 3-axis of the reference system, then

$$I_4 = 1 \tag{7.8}$$

and, in the constitutive equation (7.2), α_4 is arbitrary if the deformation is specified. The relation (7.5), with (7.7), remains valid, in general.

In the next three sections we apply the universal relation (7.5) to the three deformations discussed, for an isotropic material, in Sections 4, 5 and 6. Each of these consists of a simple shear superposed on a pure homogeneous deformation. We adopt a rectangular cartesian reference system x whose axes are parallel to the principal directions of the pure homogeneous deformation and whose 3-axis is parallel to the axis of rotational symmetry of the material.

8 – Transverse isotropy - special case I

In this section we suppose that the transversely isotropic material undergoes the deformation considered in Section 4. The deformation is described, in the rectangular cartesian coordinate system x, by eqs. (4.1). It follows (cf. eqs. (4.3)) that

$$\begin{gathered} c_1 - c_3 = \lambda_1^2 + K_1^2 - \lambda_3^2, \quad c_2 - c_3 = \lambda_2^2 + K_2^2 - \lambda_3^2\,, \\ c_4 = K_2\lambda_3, \quad c_5 = K_1\lambda_3, \quad c_6 = K_1K_2\,, \end{gathered} \tag{8.1}$$

$$\begin{aligned} \gamma_1 - \gamma_3 &= (\lambda_1^2 + K_1^2)^2 + K_1^2K_2^2 - \lambda_3^2(\lambda_3^2 + K_2^2)\,, \\ \gamma_2 - \gamma_3 &= (\lambda_2^2 + K_2^2)^2 + K_1^2K_2^2 - \lambda_3^2(\lambda_3^2 + K_1^2)\,, \\ \gamma_4 &= K_2\lambda_3(\lambda_2^2 + \lambda_3^2 + K^2)\,, \\ \gamma_5 &= K_1\lambda_3(\lambda_1^2 + \lambda_3^2 + K^2)\,, \\ \gamma_6 &= K_1K_2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + K^2)\,, \end{aligned} \tag{8.2}$$

$$\begin{gathered} \phi_1 - \phi_3 = K_1^2 - \lambda_3^2, \quad \phi_2 - \phi_3 = K_2^2 - \lambda_3^2\,, \\ \phi_4 = K_2\lambda_3, \quad \phi_5 = K_1\lambda_3, \quad \phi_6 = K_1K_2\,, \end{gathered} \tag{8.3}$$

$$
(8.4)\qquad
\begin{aligned}
\psi_1-\psi_3 &= 2K_1^2(\lambda_1^2+K^2)-2\lambda_3^2(\lambda_3^2+K_2^2)\,,\\
\psi_2-\psi_3 &= 2K_2^2(\lambda_2^2+K^2)-2\lambda_3^2(\lambda_3^2+K_1^2)\,,\\
\psi_4 &= K_2\lambda_3[2(\lambda_3^2+K^2)+\lambda_2^2]\,,\\
\psi_5 &= K_1\lambda_3[2(\lambda_3^2+K^2)+\lambda_1^2]\,,\\
\psi_6 &= K_1K_2[2(\lambda_3^2+K^2)+\lambda_1^2+\lambda_2^2]\,.
\end{aligned}
$$

With these expressions we find (see Appendix I, eq. (12.8)) that Δ, defined in (7.7), is given by

$$
(8.5)\quad \Delta = K_1K_2\lambda_3\lambda_1^2\lambda_2^2(\lambda_1^2-\lambda_2^2)\{\lambda_1^2K_2\sigma_5-\lambda_2^2K_1\sigma_4-(\lambda_1^2-\lambda_2^2)\lambda_3\sigma_6\}\,.
$$

With (7.5), we obtain the single universal relation (cf. eq. (4.14))

$$
(8.6)\qquad \lambda_1^2K_2\sigma_5-\lambda_2^2K_1\sigma_4=(\lambda_1^2-\lambda_2^2)\lambda_3\sigma_6\,.
$$

If $\lambda_1 = \lambda_2$, so that the 12-plane remains isotropic after the pure homogeneous deformation, equation (8.6) becomes

$$
(8.7)\qquad \sigma_5/K_1=\sigma_4/K_2\,;
$$

the shearing force is parallel to the direction of shear, as is also evident on physical grounds.

9 – Transverse isotropy - special case II

We now suppose that the deformation is described by eqs. (5.1). Then **c** is given by eqs. (5.3). From eqs. (5.1), (5.3) and (7.4), we obtain

$$
(9.1)\qquad
\begin{gathered}
c_3=\phi_3=\lambda_3^2,\quad 2\gamma_3=\psi_3=2\lambda_3^4,\quad \phi_\nu=\psi_\nu=0\quad(\nu\neq 3)\,,\\
c_4=c_5=\gamma_4=\gamma_5=0\,.
\end{gathered}
$$

We see that the relation (7.5), with (7.7), is now satisfied identically. Accordingly we return to the constitutive equation (7.2). This yields, with (9.1),

$$
(9.2)\qquad
\begin{gathered}
\sigma_1=\alpha_1c_1+\alpha_2\gamma_1+\alpha_3,\quad \sigma_2=\alpha_1c_2+\alpha_2\gamma_2+\alpha_3\,,\\
\sigma_3=\alpha_3+(\alpha_1+\alpha_4)\lambda_3^2+(\alpha_2+2\alpha_5)\lambda_3^4\,,\\
\sigma_6=\alpha_1c_6+\alpha_2\gamma_6,\quad \sigma_4=\sigma_5=0\,.
\end{gathered}
$$

We note that

$$\gamma_1 = c_1^2 + c_6^2, \quad \gamma_2 = c_2^2 + c_6^2, \quad \gamma_6 = c_6(c_1 + c_2)\,. \tag{9.3}$$

We now substitute for γ_1, γ_2, γ_6 in the expressions (9.2) for σ_1, σ_2, σ_6 and eliminate α_1, α_2, α_3 from the resulting equations to obtain the universal relation (cf. eq. (5.4))

$$(\sigma_1 - \sigma_2)c_6 = (c_1 - c_2)\sigma_6\,. \tag{9.4}$$

10 – Transverse isotropy - special case III

We now suppose that the deformation is described by eqs. (6.1). Then **c** is given by eqs. (6.3). From eqs. (6.1), (6.3) and (7.4) we obtain

$$\begin{gathered} c_1 = \lambda_1^2, \quad c_2 = \lambda_2^2, \quad c_3 = \lambda_3^2 + K^2(a^2\lambda_1^2 + b^2\lambda_2^2)\,, \\ c_4 = bK\lambda_2^2, \quad c_5 = aK\lambda_1^2, \quad c_6 = 0\,, \end{gathered} \tag{10.1}$$

$$\begin{aligned} \gamma_1 &= \lambda_1^4(1 + a^2K^2), \quad \gamma_2 = \lambda_2^4(1 + b^2K^2)\,, \\ \gamma_3 &= \{\lambda_3^2 + K^2(a^2\lambda_1^2 + b^2\lambda_2^2)\}^2 + K^2(a^2\lambda_1^4 + b^2\lambda_2^4)\,, \\ \gamma_4 &= \lambda_2^2 bK\{\lambda_2^2 + \lambda_3^2 + K^2(a^2\lambda_1^2 + b^2\lambda_2^2)\}\,, \\ \gamma_5 &= \lambda_1^2 aK\{\lambda_1^2 + \lambda_3^2 + K^2(a^2\lambda_1^2 + b^2\lambda_2^2)\}\,, \\ \gamma_6 &= \lambda_1^2\lambda_2^2 abK^2\,, \end{aligned} \tag{10.2}$$

$$\phi_3 = \lambda_3^2, \quad \phi_\nu = 0 \quad (\nu \neq 3)\,, \tag{10.3}$$

$$\begin{aligned} \psi_3 &= 2\lambda_3^2\{\lambda_3^2 + K^2(a^2\lambda_1^2 + b^2\lambda_2^2)\}\,, \\ \psi_4 &= \lambda_2^2\lambda_3^2 bK, \quad \psi_5 = \lambda_1^2\lambda_3^2 aK\,, \\ \psi_1 &= \psi_2 = \psi_6 = 0\,. \end{aligned} \tag{10.4}$$

With these expressions we obtain from (7.7) (see Appendix II, eqs. (13.1) and (13.3))

$$\Delta = K^2 ab\lambda_1^2\lambda_2^2\lambda_3^4(\lambda_2^2 - \lambda_1^2)\{(\lambda_1^2 - \lambda_2^2)\sigma_6 + K(a\lambda_1^2\sigma_4 - b\lambda_2^2\sigma_5)\}\,. \tag{10.5}$$

With (10.5) we obtain from (7.5) the universal relation

$$(\lambda_1^2 - \lambda_2^2)\sigma_6 = K(b\lambda_2^2\sigma_5 - a\lambda_1^2\sigma_4)\,. \tag{10.6}$$

If $\theta = 0$, so that the plane of shear is a principal plane for the pure homogeneous deformation, equation (10.6) becomes

$$(\lambda_2^2 - \lambda_1^2)\sigma_6 = K\lambda_1^2\sigma_4\,. \tag{10.7}$$

11 – Beatty's procedure

Beatty [4] obtained universal relations for an isotropic material using a different procedure from that adopted here. From the constitutive equation (3.4) it follows that

$$\boldsymbol{\sigma}\mathbf{c} = \mathbf{c}\boldsymbol{\sigma}\,. \tag{11.1}$$

This yields, in our notation, the three universal relations

$$\begin{aligned} c_6(\sigma_1 - \sigma_2) &= (c_1 - c_2)\sigma_6 + c_5\sigma_4 - c_4\sigma_5\,,\\ c_4(\sigma_2 - \sigma_3) &= (c_2 - c_3)\sigma_4 + c_6\sigma_5 - c_5\sigma_6\,,\\ c_5(\sigma_3 - \sigma_1) &= (c_3 - c_1)\sigma_5 + c_4\sigma_6 - c_6\sigma_4\,. \end{aligned} \tag{11.2}$$

Beatty concludes that any other universal relation can be obtained from these.

We note that each of eqs. (11.2) involves four terms in the stress components, while the relations we have derived involve only three. We can, however, recover our relations (3.12)-(3.14) by eliminating a pair of the stress components σ_4, σ_5, σ_6 from (11.2). For example, we can obtain the relation (3.12) by eliminating σ_5 and σ_6 from (11.2) and making the substitution

$$\sigma_1 - \sigma_2 = (\sigma_1 - \sigma_3) - (\sigma_2 - \sigma_3)\,. \tag{11.3}$$

Again, we can obtain the relation (3.17) by multiplying eqs. (11.2) by c_4c_5, c_5c_6 and c_6c_4 respectively and adding the resulting equations.

The relation (11.1), which provides the three universal relations (11.2), is valid only when the material is isotropic. Beatty discusses a material in which the possible deformations are constrained by a single direction of inextensibility. If the 3-axis of the rectangular cartesian reference system is taken as the direction of inextensibility in the undeformed material, Beatty's constitutive equation has the form

$$\boldsymbol{\sigma} = \mathbf{t} + q\boldsymbol{\phi}\,, \tag{11.4}$$

where q is undetermined if the deformation is specified and $\mathbf{t}$, called by Beatty the *extra stress*, is given by (cf. (3.4))

$$\mathbf{t} = \alpha_1 \mathbf{c} + \alpha_2 \boldsymbol{\gamma} + \alpha_3 \boldsymbol{\delta}\,, \tag{11.5}$$

where the α's are given in terms of a strain-energy function W by eqs. (3.5) and W is a function of the three strain invariants I_1, I_2, I_3 defined in eqs. (3.3). The extra stress $\mathbf{t}$ evidently satisfies a relation analogous to (11.1) and therefore yields three independent relations, which can be obtained from (11.2) by replacing σ_ν by t_ν $(\nu = 1, \ldots, 6)$.

The constitutive equation (11.4) is a particular case of the constitutive equation (7.2) with $\alpha_4 = q$ undetermined and $\alpha_5 = 0$. By eliminating α_3, we obtain (cf. eqs. (3.15) and (3.16))

$$\begin{aligned} c_\nu \alpha_1 + \gamma_\nu \alpha_2 + \phi_\nu q &= \sigma_\nu \qquad (\nu = 4, 5, 6)\,, \\ (c_1 - c_3)\alpha_1 + (\gamma_1 - \gamma_3)\alpha_2 + (\phi_1 - \phi_3) q &= \sigma_1 - \sigma_3\,, \\ (c_2 - c_3)\alpha_1 + (\gamma_2 - \gamma_3)\alpha_2 + (\phi_2 - \phi_3) q &= \sigma_2 - \sigma_3\,. \end{aligned} \tag{11.6}$$

We can eliminate the α's and q from any four of these five equations to obtain a universal relation. In this way five universal relations can be obtained and it is easily shown, by an argument similar to that used in Section 3, that any two of these imply the others. Two such universal relation are

$$\begin{vmatrix} \sigma_1 - \sigma_3 & \sigma_2 - \sigma_3 & \sigma_\mu & \sigma_\nu \\ c_1 - c_3 & c_2 - c_3 & c_\mu & c_\nu \\ \gamma_1 - \gamma_3 & \gamma_2 - \gamma_3 & \gamma_\mu & \gamma_\nu \\ \phi_1 - \phi_3 & \phi_2 - \phi_3 & \phi_\mu & \phi_\nu \end{vmatrix} = 0 \qquad (\mu, \nu = 4, 5; 4, 6)\,. \tag{11.7}$$

We note that these universal relations involve the actual stress rather than the extra stress.

12 – Appendix I

In this section we show how the result in equation (8.5) can be obtained from eqs. (7.7) and(8.1)-(8.4). This is done by making repeated use of the fact that addition or subtraction of two rows or two columns in a determinant does not alter its value. Also, multiplying the elements of a row or of a column by the same constant multiplies the value of the determinant by that constant.

Accordingly, in the determinant Δ defined in equation (7.7), we

(i) multiply row 3 by 2 and subtract from it row 5,
(ii) divide columns 3, 4 and 5 by $\lambda_3 K_2$, $\lambda_3 K_1$ and $K_1 K_2$ respectively,
(ii) subtract column 3 from columns 4 and 5,
(iv) divide columns 4 and 5 by $\lambda_1^2 - \lambda_2^2$ and λ_1^2 respectively,
(v) subtract column 4 from column 5.

In this way, we obtain, with the expressions (8.1)-(8.4),

$$\Delta = \frac{1}{2} K_1^2 K_2^2 \lambda_3^2 \lambda_1^2 (\lambda_1^2 - \lambda_2^2) \bar{\Delta} \,, \tag{12.1}$$

where

$$\bar{\Delta} = \begin{vmatrix} \overline{\sigma}_1 & \overline{\sigma}_2 & \overline{\sigma}_4 & \overline{\sigma}_5 & \overline{\sigma}_6 \\ \overline{c}_1 & \overline{c}_2 & 1 & 0 & 0 \\ \widehat{\gamma}_1 & \widehat{\gamma}_2 & \lambda_2^2 & 1 & 0 \\ \overline{\phi}_1 & \overline{\phi}_2 & 1 & 0 & 0 \\ \overline{\psi}_1 & \overline{\psi}_2 & \overline{\psi}_4 & 1 & 0 \end{vmatrix} , \tag{12.2}$$

with the definitions

$$\overline{\sigma}_1 = \sigma_1 - \sigma_3, \quad \overline{\sigma}_2 = \sigma_2 - \sigma_3 \tag{12.3}$$

and analogous definitions for $\overline{c}_1$, $\overline{c}_2$, $\overline{\gamma}_1$, $\overline{\gamma}_2$, $\overline{\phi}_1$, $\overline{\phi}_2$, $\overline{\psi}_1$, $\overline{\psi}_2$, and

$$\widehat{\gamma}_1 = 2\overline{\gamma}_1 - \overline{\psi}_1, \quad \widehat{\gamma}_2 = 2\overline{\gamma}_2 - \overline{\psi}_2 \,. \tag{12.4}$$

Also,

$$
\begin{aligned}
&\overline{\psi}_4 = 2(\lambda_3^2 + K^2) + \lambda_2^2, \\
&\overline{\sigma}_4 = \frac{\sigma_4}{K_2\lambda_3}, \\
&\overline{\sigma}_5 = \frac{1}{\lambda_1^2 - \lambda_2^2}\Big(\frac{\sigma_5}{K_1\lambda_3} - \frac{\sigma_4}{K_2\lambda_3}\Big), \\
&\overline{\sigma}_6 = \frac{1}{\lambda_1^2}\Big(\frac{\sigma_6}{K_1K_2} - \frac{\sigma_4}{K_2\lambda_3}\Big) - \frac{1}{\lambda_1^2 - \lambda_2^2}\Big(\frac{\sigma_5}{K_1\lambda_3} - \frac{\sigma_4}{K_2\lambda_3}\Big).
\end{aligned}
\tag{12.5}
$$

In the determinant $\bar{\Delta}$, defined in (12.2), we

(i) subtract row 4 from row 2,
(ii) subtract row 5 from row 3.

We thus obtain

$$
\bar{\Delta} = \overline{\sigma}_6 \begin{vmatrix} \overline{c}_1 - \overline{\phi}_1 & \overline{c}_2 - \overline{\phi}_2 & 0 \\ \widehat{\gamma}_1 - \overline{\psi}_1 & \widehat{\gamma}_2 - \overline{\psi}_2 & \lambda_2^2 - \overline{\psi}_4 \\ \overline{\phi}_1 & \overline{\phi}_2 & 1 \end{vmatrix}. \tag{12.6}
$$

By expanding the determinant and substituting from eqs. (12.4), (12.5) and (8.1)-(8.4), we obtain

$$
\bar{\Delta} = 2\lambda_1^2\lambda_2^2(\lambda_2^2 - \lambda_1^2)\overline{\sigma}_6. \tag{12.7}
$$

It follows from eqs. (12.1), (12.7) and the expression for $\overline{\sigma}_6$ in eqs. (12.5) that

$$
\Delta = K_1K_2\lambda_3\lambda_1^2\lambda_2^2(\lambda_1^2 - \lambda_2^2)\{\lambda_1^2K_2\sigma_5 - \lambda_2^2K_1\sigma_4 - (\lambda_1^2 - \lambda_2^2)\lambda_3\sigma_6\}. \tag{12.8}
$$

13 – Appendix II

In this section we show how the result in equation (10.5) can be obtained from equation (7.7).

In the determinant Δ, defined in (7.7), we

(i) divide columns 4, 5 and 6 by $\lambda_2^2 bK$, $\lambda_1^2 aK$ and $\lambda_1^2\lambda_2^2 abK^2$ respectively,
(ii) subtract column 4 from column 5,

(iii) divide column 5 by $\lambda_1^2 - \lambda_2^2$,
(iv) subtract column 5 from column 6.

Using the expressions in (10.1)-(10.4) for c_ν, γ_ν, ϕ_ν and ψ_ν ($\nu = 4, 5, 6$), we obtain

$$\Delta = \lambda_1^2\lambda_2^2 abK^2\{(\lambda_1^2 - \lambda_2^2)\sigma_6 + K(a\lambda_1^2\sigma_4 - b\lambda_2^2\sigma_5)\}\bar{\Delta}\,, \tag{13.1}$$

where

$$\bar{\Delta} = \begin{vmatrix} c_1 - c_3 & c_2 - c_3 & 1 \\ \phi_1 - \phi_3 & \phi_2 - \phi_3 & 0 \\ \psi_1 - \psi_3 & \psi_2 - \psi_3 & \lambda_3^2 \end{vmatrix}\,. \tag{13.2}$$

With the expressions in (10.1), (10.3), (10.4) for c_ν, ϕ_ν, ψ_ν ($\nu = 1, 2, 3$), we obtain

$$\bar{\Delta} = \lambda_3^4(\lambda_2^2 - \lambda_1^2)\,. \tag{13.3}$$

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