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Direct and inverse fluid-structure interaction problems

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Riassunto: *Si sviluppa il metodo del potenziale per il problema tridimensionale dell'interazione fluido-corpo elastico. Si studiano i problemi di esistenza e unicità della soluzione e si riconosce la risolubilità del problema diretto per numeri d'onda arbitrari e per un'arbitraria onda incidente. Si rappresentano le soluzioni con integrali di tipo potenziale e se ne studiano le proprietà strutturali. Si dimostra che il campo scalare è determinato univocamente nel dominio esterno, mentre il campo vettoriale elastico è determinato nel dominio interno a meno dei modi di Jones. Sulla base di questi risultati si riconosce l'unicità della soluzione del problema inverso.*

Abstract: *The potential method is developed for the fluid-structure interaction three-dimensional problems. The uniqueness and existence questions are investigated and the solvability of the direct problem is shown for arbitrary wave numbers and for arbitrary incident wave functions. The solutions are represented by potential type integrals and their structural properties are studied. It is shown that the scalar field in the exterior domain is defined uniquely, while the elastic vector field in the interior domain is defined modulo Jones modes. On the basis of these results the uniqueness theorem for a solution to the inverse fluid-structure interaction problem is proved.*

– Introduction

Direct and inverse problems connected with the interaction between vector fields of different dimension have received much attention in the

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mathematical and engineering scientific literature and have been intensively investigated for the past years. They arise in many physical and mechanical models describing the interaction of two different media where the whole process is characterised by a vector-function of dimension k in one medium and by a vector-function of dimension n in another one (for example, fluid-structure interaction where a streamlined body is an elastic obstacle, scattering of acoustic and electromagnetic waves by an elastic obstacle, interaction between an elastic body and seismic waves, etc.).

Quite many authors have considered and studied in detail the direct problems of the interaction between an elastic *isotropic* body, which occupies a bounded region $\overline{\Omega^+}$ and where a three-dimensional elastic vector field is to be defined, and some *isotropic* medium (fluid say), which occupies the unbounded exterior region, the complement of Ω^+ with respect to the whole space, where a scalar field is to be defined. The time-harmonic dependent unknown vector and scalar fields are coupled by some kinematic and dynamic conditions on the boundary $\partial\Omega^+$, which lead to various type of non-classical interface problems of steady oscillations for a piecewise homogeneous isotropic medium. An exhaustive information in this direction concerning theoretical and numerical results can be found in [1], [2], [3], [4], [5], [8], [9], [11], [12], [14], [15], [16], [22], [23],[26], [34], [36], [37], [38], [39], [40].

Some particular cases where the elastic body under concideration is *anisotropic* have been treated in [33], [18], [19].

In the present paper we generalise the results of the above cited works into two directions: first, in the direct problems the both interacting media are assumed to be *anisotropic* and, second, the corresponding inverse problems of the wave scattering theory are considered. To the authors' best knowlege such kind of inverse problems have not been treated in the scientific literature even for the isotropic case.

When studying these problems there arise difficulties due to the above-mentioned anisotropy property of media in question, since for the both fields we need non-trivial analogues of the classical Rellich's lemma [41], [30]. Moreover, in the resonance case, i.e., when there exist Jones modes for exceptional values of the oscillation parameter, we analyse the cokernels of the corresponding pseudodifferential operators and establish efficient conditions of solvability for the non-homogeneous transmission problems. In particular, we have shown that the direct scattering problems are solvable for arbitrary values of the frequency parameter and for arbitrary incident wave functions.

On the basis of these results and applying the approach developed in [7] and [24], we have proved the uniqueness of solution to the inverse fluid-structure interaction (scattering) problem.

1 – Preliminary material. Formulation of direct and inverse problems

1.1 – Elastic field

Let $\Omega^+ \subset \mathbb{R}^3$ be a bounded domain ($\operatorname{diam}\Omega^+ < +\infty$) with a smooth, connected, nonselfintersecting boundary $S = \partial\Omega^+$ and $\Omega^- = \mathbb{R}^3 \backslash \overline{\Omega^+}$, $\overline{\Omega^+} = \Omega^+ \cup S$.

The region $\overline{\Omega^+}$ is supposed to be filled up by a homogeneous anisotropic medium with the elastic coefficients $c_{kjpq} = c_{pqkj} = c_{jkpq}$, $k, j, p, q = 1, 2, 3$, and the density $\rho_1 = \text{const} > 0$.

The homogeneous system of steady state oscillation equations of the linear elasticity reads as follows

$$C(D,\omega)u(x) := C(D)u(x) + \rho_1\omega^2 u(x) = 0\,, \tag{1.1}$$

where $u = (u_1, u_2, u_3)^\top$ is the complex-valued displacement vector (amplitude), $\omega > 0$ is the oscillation (frequency) parameter,

$$\begin{aligned} C(D,\omega) &: = C(D) + \rho_1\omega^2 I_3, \\ C(D) &: = [c_{kjpq}D_jD_q]_{3\times 3}, \quad D = (D_1, D_2, D_3), \quad D_j = \frac{\partial}{\partial x_j}\,; \end{aligned}$$

here and in what follows I_3 stands for the unit 3×3 matrix, the superscript $^\top$ denotes transposition, and the summation over repeated indices is meant from 1 to 3, unless otherwise stated.

The stress tensor $\{\sigma_{kj}\}$ and the strain tensor $\{\varepsilon_{kj}\}$ are related by Hook's law

$$\sigma_{kj} = c_{kjpq}\varepsilon_{pq}, \quad \varepsilon_{kj} = 2^{-1}(D_k u_j + D_j u_k)\,.$$

As usual, the quadratic form corresponding to the potential energy is assumed to be positive definite in the symmetric real variables $\varepsilon_{kj} = \varepsilon_{jk}$ (see, e.g., [13])

$$E(u,u) = \sigma_{kj}\varepsilon_{kj} = c_{kjpq}\varepsilon_{kj}\varepsilon_{pq} \geq \delta_1 \varepsilon_{kj}\varepsilon_{kj}, \quad \delta_1 = \text{const} > 0\,. \tag{1.2}$$

Due to the symmetry relations of the elastic constants we also have

$$E(u,\overline{u}) = \sigma_{kj}\overline{\varepsilon_{kj}} = c_{kjpq}\varepsilon_{kj}\overline{\varepsilon_{pq}} \geq \delta_1[\varepsilon'_{kj}\varepsilon'_{kj} + \varepsilon''_{kj}\varepsilon''_{kj}]\,, \tag{1.3}$$

where an overbar denotes complex conjugation, and where $\varepsilon_{kj} = \varepsilon'_{kj} + i\,\varepsilon''_{kj}$ are complex variables (complex strain tensor) corresponding to the complex vector $u = u' + i\,u''$, $i = \sqrt{-1}$.

The inequality (1.2) implies the positive definiteness of the matrix $C(\xi)$ for $\xi \in \mathbb{R}^3 \backslash \{0\}$:

$$C(\xi)\zeta \cdot \zeta = C_{kp}(\xi)\zeta_p\overline{\zeta}_k \geq \delta_2|\xi|^2|\zeta|^2, \quad \delta_2 = \text{const} > 0\,,$$

where ζ is an arbitrary three-dimensional complex vector $\zeta \in \mathbb{C}^3$. Throughout the paper $a \cdot b = \sum_{k=1}^m a_k\overline{b}_k$ denotes the scalar product of two vectors in $\mathbb{C}^m$.

Further we introduce the stress operator

$$T(D,n) = [T_{kp}(D,n)]_{3\times 3}\,, \quad T_{kp}(D,n) = c_{kjpq}n_jD_q\,,$$

where $n = (n_1, n_2, n_3)$ is a unit vector.

The k-th component of the stress vector acting on a surface element with the unit normal vector n is calculated by the formula

$$\sigma_{kj}n_j = c_{kjpq}n_jD_pu_q = [T(D,n)u]_k\,. \tag{1.4}$$

We note that throughout the paper we will use the following notations (when this causes no confusion):

a) if all elements of a vector $v = (v_1, \cdots, v_m)^\top$ (matrix $N = [N_{kj}]_{m\times p}$) belong to one and the same space X, we will write $v \in X$ $(N \in X)$ instead of $v \in [X]^m$ $(N \in [X]_{m\times n})$;

b) if $K : X_1 \times \cdots \times X_m \to Y_1 \times \cdots \times Y_p$ and $X_1 = \cdots = X_m$, $Y_1 = \cdots = Y_p$, we will write $K : X \to Y$ instead of $K : [X]^m \to [Y]^p$;

c) by $||u||_X$ we denote the norm of the element u in the space X.

1.2 – Scalar field

We assume that the exterior domain Ω^- is connected and it is filled up by a homogeneous anisotropic (fluid) medium with the constant density $\rho_2 > 0$. Further, let some physical process (the propagation of acoustic waves say) in Ω^- be described by a complex-valued scalar function (scalar field) $w(x)$ being a solution of the homogeneous "wave equation" (generalized Helmholtz equation)

$$a(D,\omega)w(x) := a(D)w(x) + \rho_2\omega^2 w(x) = 0, \quad x \in \Omega^- , \tag{1.5}$$

where $a(D) = a_{kj}D_kD_j$, the real constants $a_{kj} = a_{jk}$ define a positive definite matrix $\widetilde{a} = [a_{kj}]_{3\times 3}$, i.e.,

$$\widetilde{a}\zeta \cdot \zeta = a_{kj}\zeta_j\overline{\zeta}_k \geq \delta_3|\zeta|^2 , \quad \delta_3 = \text{const} > 0 ,$$

for arbitrary $\zeta \in \mathbb{C}^3$.

Denote by S_ω the characteristic surface (ellipsoid) given by the equation

$$\Phi_a(\xi,\omega) := \widetilde{a}\xi \cdot \xi - \rho_2\omega^2 = 0, \quad \xi \in \mathbb{R}^3 .$$

For an arbitrary vector $\eta \in \mathbb{R}^3$ with $|\eta| = 1$ there exists only one point $\xi(\eta) \in S_\omega$ such that the outward unit normal vector $n(\xi(\eta))$ to S_ω at the point $\xi(\eta)$ has the same direction as η, i.e., $n(\xi(\eta)) = \eta$. Note that $\xi(-\eta) = -\xi(\eta) \in S_\omega$ and $n(-\xi(\eta)) = -\eta$.

It can be easily verified that

$$\xi(\eta) = \omega\sqrt{\rho_2}\,(\widetilde{a}^{-1}\eta \cdot \eta)^{-1/2}\,\widetilde{a}^{-1}\eta , \tag{1.6}$$

where $\widetilde{a}^{-1}$ is the matrix inverse to $\widetilde{a}$.

Now we are in the position to define the class $\mathrm{Som}(\Omega^-)$ of complex-valued functions satisfying the generalized Sommerfeld type radiation conditions (see [41], [18], [19]).

A function w belongs to $\mathrm{Som}(\Omega^-)$ if $w \in C^1(\Omega^-)$ and for sufficiently large $|x|$

$$w(x) = O(|x|^{-1}) , \;\; D_kw(x) - i\xi_k(\eta)w(x) = O(|x|^{-2}) , \;\; k = 1,2,3 , \tag{1.7}$$

where $\xi(\eta) \in S_\omega$ corresponds to the vector $\eta = x/|x|$ (i.e., $\xi(\eta)$ is given by (1.6) with $\eta = x/|x|$).

The conditions (1.7) are equivalent to the classical Sommerfeld radiation conditions for the Helmholtz equation if the $a(D)$ is the Laplace operator (see, for example, [42], [6]). In the sequel elements of the class $\mathrm{Som}(\Omega^-)$ will also be referred to as *radiating functions*.

We have the following analogue of the classical Rellich's lemma (for details see [18])

LEMMA 1.1. *Let $w \in \mathrm{Som}(\Omega^-)$ be a solution of (1.5) in Ω^- and let*

$$\lim_{R\to+\infty} \mathrm{Im}\,\Big\{\int_{\Sigma_R} \overline{w(x)}\,\partial_n w(x)\,d\Sigma_R\Big\} = 0\,,$$

where Σ_R is the sphere centered at the origin and radius R, and ∂_n denotes the co-normal differentiation

$$\partial_n = a_{kj} n_k D_j\,.$$

Then $w = 0$ in Ω^-.

Note that, if w is a solution of the homogeneous equation (1.5), then w is an analytic function of the real variable x in the domain Ω^-. Moreover, if, in addition, $w \in C^1(\overline{\Omega^-}) \cap \mathrm{Som}(\Omega^-)$, then the following integral representation formula holds (cf. [41], [19])

$$\begin{aligned}(1.8)\qquad &\int_S \gamma(x-y,\omega)[\partial_n w(y)]^- dS_y - \int_S [\partial_{n(y)}\gamma(y-x,\omega)]\,[w(y)]^- dS_y = \\ &= \begin{cases} w(x) & \text{for } \ x \in \Omega^-\,, \\ 0 & \text{for } \ x \in \Omega^+\,, \end{cases}\end{aligned}$$

where

$$(1.9)\qquad \gamma(x,\omega) = -\frac{\exp\{i\omega\sqrt{\rho_2}(\widetilde{a}^{-1}x\cdot x)^{1/2}\}}{4\pi|\widetilde{a}|^{1/2}(\widetilde{a}^{-1}x\cdot x)^{1/2}}, \quad |\widetilde{a}| = \det\widetilde{a}\,,$$

is a radiating fundamental function (solution) to the equation (1.5) (see, e.g., Lemma 1.1 in [19], [28]), the symbols $[\,\cdot\,]^{\pm}$ denote limits on S from $\Omega^{\pm}$ and $n(y)$ is the outward unit normal vector to S at the point $y \in S$.

For sufficiently large $|x|$ we have the following asymptotic representation

$$
\begin{aligned}
\gamma(x-y,\omega) &= c(\xi)\frac{\exp\{i\xi\cdot(x-y)\}}{|x|} + O(|x|^{-2})\,,\\
c(\xi) &= -\frac{|\widetilde{a}\xi|}{4\pi\omega\,(\rho_2|\widetilde{a}|)^{1/2}}\,,
\end{aligned}
\tag{1.10}
$$

where y varies in a bounded subset of $\mathbb{R}^3$ and $\xi = \xi(\eta) \in S_\omega$ corresponds to the direction $\eta = x/|x|$; the asymptotic formula (1.10) can be differentiated any times with respect to x and y.

From formula (1.8) with the help of (1.10) we get the asymptotic representation (for sufficiently large $|x|$) of a radiating solution to the equation (1.5)

$$
w(x) = w_\infty(\xi)\frac{\exp\{i\xi\cdot x\}}{|x|} + O(|x|^{-2})\,, \tag{1.11}
$$

where

$$
w_\infty(\xi) = c(\xi)\int_S \exp\{-i\xi\cdot y\}\{[\partial_n w(y)]^- + i\,(\widetilde{a}\xi\cdot n(y))[w(y)]^-\}\,dS_y
$$

with ξ and $c(\xi)$ as in (1.10); $w_\infty(\xi)$ is the so-called *far field pattern* of the radiating solution $w(x)$ (cf. [7]).

1.3 – Formulation of direct and inverse interaction problems

First we set the direct fluid-structure interaction problem.

Let a total wave field in Ω^- is represented as a sum of incident and scattered fields

$$
w^{tot}(x) = w^{inc}(x) + w^{sc}(x)\,,
$$

where the incident field w^{inc} is taken in the form of a plane wave

$$
w^{inc}(x) = w^{inc}(x;d) = e^{i\,x\cdot d},\ \ x\in\mathbb{R}^3,\ \ d\in S_\omega\,, \tag{1.12}
$$

while the scattered field (scattered acoustic pressure) $w^{sc}(x) = w^{sc}(x;d)$ is a radiating solution of equation (1.15); here $d = (d_1, d_2, d_3)$ denotes the direction of propagation of the plane wave (cf. [17]).

Problem $P^{(\mathrm{dir})}$. Find a regular vector of displacements $u \in C^2(\Omega^+) \cap C^1(\overline{\Omega^+})$ and a regular radiating function $w^{sc} \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-}) \cap \mathrm{Som}(\Omega^-)$ which are solutions of equation (1.1) and (1.5), respectively, and satisfy the following (kinematic and dynamic) coupling conditions on $S = \partial\Omega^\pm$:

$$[u(x) \cdot n(x)]^+ = b_1\,[\partial_n w^{tot}(x)]^- = b_1\,[\partial_n w^{sc}(x)]^- + f_0(x)\,, \tag{1.13}$$

$$[T(D,n)u(x)]^+ = b_2\,[w^{tot}(x)]^- n(x) = b_2\,[w^{sc}(x)]^- n(x) + f(x)\,, \tag{1.14}$$

where $T(D,n)u$ is the stress vector given by formula (1.4), $\partial_n w = a_{pq} n_p D_q w$ is the co-normal derivative, throughout this paper $n(x)$ denotes the unit outward normal vector to S at the point $x \in S$, and

$$b_1 = [\rho_2 \omega^2]^{-1}\,, \quad b_2 = -1\,. \tag{1.15}$$

Here the boundary scalar function f_0 and the vector-valued function $f(x)$ are defined as follows:

$$f_0(x) = f_0(x;d) = b_1\,\partial_n w^{inc}(x;d)\,, \tag{1.16}$$

$$f(x) = (f_1(x), f_2(x), f_3(x))^\top = f(x;d) = b_2\,w^{inc}(x;d)\,n(x)\,. \tag{1.17}$$

As it follows from the above statement, in the direct problem the domains Ω^+ and Ω^- are fixed and we look for the displacement vector u and the radiating scalar function w^{sc} (scattered field).

The inverse fluid-structure acoustic interaction problem consists in finding the surface S (i.e., the scatterer Ω^+) if the corresponding far field pattern $w^{sc}_\infty(\cdot;d)$ is known for several or all direction vectors $d \in S_\omega$. More precise mathematical formulation of the inverse problem reads as follows:

Problem $P^{(\mathrm{inv})}$. Find an elastic scatterer Ω^+ with a compact, connected, nonselfintersecting, smooth boundary surface S provided that the conditions of Problem $P^{(\mathrm{dir})}$ are satisfied on S and the far field pattern $w^{sc}_\infty(\cdot;d)$ is a known function

$$w^{sc}_\infty(\xi;d) = G(\xi;d)\,, \quad \xi \in S_\omega\,,$$

for several (or all) direction vectors $d \in S_\omega$; here $G(\cdot;d)$ is a given function of ξ on S_ω and ξ corresponds to the vector $\eta = x/|x|$ (see (1.6)).

In the both problems the oscillation parameter ω is an arbitrarily fixed positive number. The investigation of the inverse problem becomes complicated due to the fact that, in general, the direct interaction problem for arbitrary scatterer Ω^+ is not unconditionally solvable for all ω. For exceptional values of the parameter ω, i.e., for those values of ω for which the corresponding homogeneous direct problem possesses non-trivial solutions (Jones modes), the boundary data f_0 and f, involved in the equations (1.13) and (1.14), have to satisfy special compatibility (necessary) conditions ([19]). However, as we shall show below for the functions given by (1.16) and (1.17) these necessary conditions are fulfilled and Problem $P^{(\mathrm{dir})}$ is always solvable. Moreover, the scalar field w^{sc} is defined uniquely in Ω^- for arbitrary ω, while the elastic field u, in general, is defined modulo Jones modes (see Section 2).

We will study the above problems by the potential (boundary integral equations) method. The properties of the corresponding potential type operators partly can be found in [29], [30], [31], [18], [19], [33], but for the readers convenient and self-containedness of the paper we bring needed material in the forthcoming subsections.

1.4 – Scalar potentials. Steklov-Poincaré type relations

Let us introduce the single- and double-layer scalar potentials related to the operator $a(D,\omega)$:

$$V_a(g)(x) = \int_S \gamma(x-y,\omega)\, g(y)\, dS_y\,, \qquad x \in \mathbb{R}^3 \backslash S\,,$$
$$W_a(g)(x) = \int_S [\partial_{n(y)}\gamma(y-x,\omega)]\, g(y)\, dS_y\,, \quad x \in \mathbb{R}^3 \backslash S\,,$$

where g is a scalar density function.

In what follows we essentially will use the following properties of these potentials (for details see [18], [19]).

Lemma 1.2. *Let $S \in C^{k+1+\alpha'}$ with integer $k \geq 0$ and $0<\alpha<\alpha' \leq 1$, $0 \leq m \leq k$.*

Then

i) *the operators* V_a *and* W_a *have the mapping properties:*

$$
\begin{aligned}
V_a &: C^{m+\alpha}(S) \to C^{m+1+\alpha}(\overline{\Omega^+})\,,\\
&: C^{m+\alpha}(S) \to C^{m+1+\alpha}(\overline{\Omega^-}) \cap \mathrm{Som}(\Omega^-)\,,\\
W_a &: C^{m+\alpha}(S) \to C^{m+\alpha}(\overline{\Omega^+})\,,\\
&: C^{m+\alpha}(S) \to C^{m+\alpha}(\overline{\Omega^-}) \cap \mathrm{Som}(\Omega^-)\,;
\end{aligned}
$$

ii) *for arbitrary* $g \in C^{m+\alpha}(S)$ *and* $z \in S$ *the following jump relations hold on S:*

$$
\begin{aligned}
[V_a(g)(z)]^{\pm} &= \int_S \gamma(z-y,\omega)\, g(y)\, dS_y =: \mathcal{H}_a\, g(z), \quad m \geq 0\,,\\
[\partial_{n(z)} V_a(g)(z)]^{\pm} &= \mp 2^{-1} g(z) + \int_S [\partial_{n(z)} \gamma(z-y,\omega)]\, g(y)\, dS_y =\\
&=: [\mp 2^{-1} I + \mathcal{K}_a^{(1)}]\, g(z), \quad m \geq 0\,,\\
[W_a(g)(z)]^{\pm} &= \pm 2^{-1} g(z) + \int_S [\partial_{n(y)} \gamma(y-z,\omega)]\, g(y)\, dS_y =\\
&=: [\pm 2^{-1} I + \mathcal{K}_a^{(2)}]\, g(z), \quad m \geq 0\,,\\
[\partial_{n(z)} W_a(g)(z)]^{+} &= [\partial_{n(z)} W_a(g)(z)]^{-} =: \mathcal{L}_a\, g(z), \quad m \geq 1\,,
\end{aligned}
$$

where I *stands for the identical operator;*

iii) *the operators*

$$
\begin{aligned}
\mathcal{H}_a &:\ C^{m+\alpha}(S) \to C^{m+1+\alpha}(S)\,,\\
\mathcal{K}_a^{(1)},\ \mathcal{K}_a^{(2)} &:\ C^{m+\alpha}(S) \to C^{m+\alpha}(S)\,,\\
\mathcal{L}_a &:\ C^{m+1+\alpha}(S) \to C^{m+\alpha}(S)\,,
\end{aligned}
$$

are bounded; *moreover,* $\mathcal{H}_a$ *has a weakly singular kernel-function of type* $O(|x-y|^{-1})$, *while* $\mathcal{K}_a^{(1)}$ *and* $\mathcal{K}_a^{(2)}$ *have weakly singular kernel-functions of type* $O(|x-y|^{-2+\alpha'})$ *on* S *for* $k = 0$ *and* $O(|x-y|^{-1})$ *for* $k \geq 1$, *and* $\mathcal{L}_a$ *is a singular integro-differential operator;*

iv) *the operators* $\mathcal{H}_a$, $\mp 2^{-1} I + \mathcal{K}_a^{(1)}$, $\pm 2^{-1} I + \mathcal{K}_a^{(2)}$ *and* $\mathcal{L}_a$ *are elliptic pseudodifferential operators of order* -1, 0, 0, *and* 1, *respectively, with the index zero;*

v) *the principal homogeneous symbols of the above operators read:*

$$\begin{aligned}
&\sigma(\mathcal{H}_a)(x,\widetilde{\xi}) = -(2\pi)^{-1}\int_{-\infty}^{+\infty}\frac{d\,\xi_3}{a_{pq}\,(B\xi)_p\,(B\xi)_q} < 0\,,\\
&\sigma(\pm 2^{-1}I + \mathcal{K}_a^{(1)})(x,\widetilde{\xi}) = \sigma(\pm 2^{-1}I + \mathcal{K}_a^{(2)})(x,\widetilde{\xi}) = \pm 2^{-1}\,,\\
&\sigma(\mathcal{L}_a)(x,\widetilde{\xi}) = -[4\sigma(\mathcal{H}_a)(x,\widetilde{\xi})]^{-1} > 0\,,\\
&\widetilde{\xi} = (\xi_1,\xi_2)\in \mathbb{R}^2\backslash\{0\},\ \ x\in S,\ \ \xi = (\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3\,;
\end{aligned}$$

here

$$B(x) = \begin{bmatrix} l_1(x) & m_1(x) & n_1(x)\\ l_2(x) & m_2(x) & n_2(x)\\ l_3(x) & m_3(x) & n_3(x)\end{bmatrix}$$

with det $B(x) = 1$, *where* $n(x)$ *is the exterior* (*with respect to* Ω^+) *unit normal vector, while* $l(x) = (l_1(x), l_2(x), l_3(x))$ *and* $m(x) = (m_1(x), m_2(x), m_3(x))$ *are orthogonal unit vectors in the tangent plane at the point* $x\in S$*;*

vi) *the following operator equations*

$$\begin{aligned}
&\mathcal{H}_a\mathcal{K}_a^{(1)} = \mathcal{K}_a^{(2)}\mathcal{H}_a\,, \quad \mathcal{L}_a\mathcal{K}_a^{(2)} = \mathcal{K}_a^{(1)}\mathcal{L}_a\,,\\
&\mathcal{H}_a\mathcal{L}_a = -4^{-1}I + [\mathcal{K}_a^{(2)}]^2\,, \quad \mathcal{L}_a\mathcal{H}_a = -4^{-1}I + [\mathcal{K}_a^{(1)}]^2\,,
\end{aligned}$$

hold in appropriate functional spaces.

LEMMA 1.3. *Let* $g\in C^{1+\alpha}(S)$ *with* $S\in C^{2+\alpha'}$, $0<\alpha<\alpha'\le 1$, *and*

$$w(x) = W_a(g)(x) - i\,V_a(g)(x), \quad x\in\Omega^-\,. \tag{1.18}$$

If w *vanishes in* Ω^-, *then* $g = 0$ *on* S.

Further, let

$$\mathcal{N} \,:=\, (-2^{-1}\,I + \mathcal{K}_a^{(2)}) \,-\, i\,\mathcal{H}_a\,, \tag{1.19}$$

$$\mathcal{M} \,:=\, \mathcal{L}_a \,-\, i\,(2^{-1}\,I + \mathcal{K}_a^{(1)})\,. \tag{1.20}$$

These operators are generated by the limiting values on S (from Ω^-) of the potential (1.18) and its co-normal derivative.

LEMMA 1.4. *Let S, k, α, α', and m be as in Lemma 1.2. Then*

i) *the operators*

$$\mathcal{N} : C^{m+1+\alpha}(S) \to C^{m+1+\alpha}(S)$$

and

$$\mathcal{M} : C^{m+1+\alpha}(S) \to C^{m+\alpha}(S)$$

are isomorphisms;

ii) *the exterior Dirichlet boundary value problem (BVP)*

$$\begin{aligned} &a(D,\omega)w(x) = 0 \quad in \quad \Omega^-, \quad w \in C(\overline{\Omega^-}) \cap \mathrm{Som}(\Omega^-)\,, \\ &[w(z)]^- = \varphi(z) \quad on \quad S\,, \quad \varphi \in C(S)\,, \end{aligned}$$

is uniquely solvable and the solution is representable in the form

$$w(x) = \Big(W_a - i\,V_a\Big)(\mathcal{N}^{-1}\varphi)(x), \;\; x \in \Omega^-\,.$$

If $\varphi \in C^{m+\alpha}(S)$, then $w \in C^{m+\alpha}(\overline{\Omega^-}) \cap \mathrm{Som}(\Omega^-)$.

iii) *the exterior Neumann BVP*

$$\begin{aligned} &a(D,\omega)w(x) = 0 \quad in \quad \Omega^-, \quad w \in C^1(\overline{\Omega^-}) \cap \mathrm{Som}(\Omega^-)\,, \\ &[\partial_n w(z)]^- = \psi(z) \quad on \quad S\,, \quad \psi \in C^\alpha(S)\,, \end{aligned}$$

is uniquely solvable and the solution is representable in the form

$$w(x) = \Big(W_a - i\,V_a\Big)(\mathcal{M}^{-1}\psi)(x)\,, \;\; x \in \Omega^-\,.$$

If $\psi \in C^{m+\alpha}(S)$, then $w \in C^{m+1+\alpha}(\overline{\Omega^-}) \cap \mathrm{Som}(\Omega^-)$.

iv) *the Dirichlet and the Neumann data for an arbitrary radiating solution w of the equation (1.5) are related on S by the following Steklov-Poincaré type equations*

$$[\partial_n w]^- = \mathcal{M}\mathcal{N}^{-1}[w]^-, \quad [w]^- = \mathcal{N}\mathcal{M}^{-1}[\partial_n w]^-\,.$$

1.5 – Robin type problem. Properties of plane waves

Let us consider the interior Robin type BVP

$$a(D,\omega)w(x)=0 \quad \text{in} \quad \Omega^+, \quad w\in C^1(\overline{\Omega^+})\,, \tag{1.21}$$

$$[\partial_n w(z)-i\,w(z)]^+=\mu(z) \quad \text{on} \quad S, \quad \mu\in C^\alpha(S)\,. \tag{1.22}$$

If we look for a solution to this problem in the form of a single-layer potential $w(x)=V_a(g)(x)$, we arrive at the Fredholm integral equation on S

$$\mathcal{P}_a g := (-2^{-1}I+\mathcal{K}_a^{(1)}-i\,\mathcal{H}_a)\,g=\mu\,,$$

where $\mathcal{P}_a$ is an invertible integral operator with a weakly singular kernel.

By standard arguments we can easily prove the following assertion (cf. [24]; see also the proof of Lemma 2.3 below).

LEMMA 1.5. i) *The BVP* (1.21)-(1.22) *is uniquely solvable.*

ii) *If* $S\in C^{k+1+\alpha'}$ *and* $\mu\in C^{k+\alpha}$, *then the solution* w *of the BVP* (1.21)-(1.22) *belongs to the space* $C^{k+1+\alpha}(\overline{\Omega^+})$. *Moreover, for* $\overline{\Omega_0^+}\in\Omega^+$ *there holds the uniform estimate*

$$|w(x)|\le c\,||[\partial_n w-iw]^+||_X\,, \quad x\in\overline{\Omega_0^+}\,,$$

where X *stands for one of the spaces* $C(S)$, $C^{k+\alpha}(S)$, *and* $L_2(S)$, *and the constant* c *is independent of* w.

iii) *An arbitrary solution* $w\in C^{1+\alpha}(\overline{\Omega^+})$ *of the equation* (1.21) *is uniquely representable in the form*

$$w(x)=V_a(\mathcal{P}_a^{-1}[\partial_n w-i\,w]^+)(x)\,, \quad x\in\overline{\Omega^+}\,.$$

From Lemma 1.5 it follows that the plane wave $\exp\{i\,d\cdot x\}$, where $d\in S_\omega$, can be uniquely represented in the form

$$e^{i\,d\cdot x}=V_a(\mathcal{P}_a^{-1}[(\partial_{n(\zeta)}-i)e^{i\,d\cdot\zeta}]_S^+)(x)\,, \quad x\in\Omega^+\,.$$

Note that $\exp\{i\,d\cdot x\}$ with $d\in S_\omega$ is a non-radiating solution to the homogeneous equation (1.5) in $\mathbb{R}^3$.

Let

$$P(S) := \{(\partial_{n(x)} - i)\, e^{i\, d\cdot x},\ x \in S,\ d \in S_\omega\}\,,$$

$$P_{sp}(S) := \Big\{\sum_{q=1}^{m} c_q\, p(x; d^{(q)}) : p(x; d^{(q)}) \in P(S),\ c_q \in \mathbb{C},\ d^{(q)} \in S_\omega,\ m \in \mathbb{N}\Big\},$$

$$P_{sp}(\mathbb{R}^3) := \Big\{\sum_{q=1}^{m} c_q\, e^{i\, d^{(q)}\cdot x},\ x \in \mathbb{R}^3,\ c_q \in \mathbb{C},\ d^{(q)} \in S_\omega,\ m \in \mathbb{N}\Big\}\,;$$

here $\mathbb{N}$ and $\mathbb{C}$ are the sets of all natural and complex numbers, respectively.

Lemma 1.6. *The set $P(S)$ is complete in $L_2(S)$.*

Proof. Let $f \in L_2(S)$ and

$$\int_S [(\partial_{n(y)} - i)e^{i\, d\cdot y}]\, f(y)\, dS_y = 0 \tag{1.23}$$

for all $d \in S_\omega$.

Let us consider the function

$$w(x) = (W_a - i\, V_a)(f)(x), \quad x \in \mathbb{R}^3 \backslash S\,.$$

Clearly, we have

$$w(x) = c(\xi)\, \frac{\exp\{i\, \xi \cdot x\}}{|x|} \int_S [(\partial_{n(y)} - i)e^{-i\, \xi\cdot y}]\, f(y)\, dS_y + O(|x|^{-2})$$

as $|x| \to +\infty$, where $\xi \in S_\omega$ corresponds to x and $c(\xi)$ is defined by (1.10).

By (1.23) we then conclude

$$w(x) = O(|x|^{-2})\,,$$

which implies $w(x) = 0$ in Ω^- due to Lemma 1.1. Therefore, we obtain

$$[w(x)]^- = \mathcal{N} f = 0 \quad \text{on} \quad S\,.$$

By ellipticity of the operator $\mathcal{N}$ we have the inclusion $f \in C^{1+\alpha}(S)$, whence by Lemma 1.4 we arrive at the equation $f = 0$ on S. This completes the proof. $\square$

Lemma 1.7. *Let Ω^+ be a bounded domain with C^2 boundary such that Ω^- be connected and let $w \in C^1(\overline{\Omega^+}) \cap C^2(\Omega^+)$ be a solution to the equation* (1.21) *in Ω^+.*

Then there exists a sequence $v_m \in P_{sp}(\mathbb{R}^3)$ such that $v_m \to w$ and $D^\beta v_m \to D^\beta w$ as $m \to \infty$ uniformly on compact subsets of Ω^+ ($\beta = (\beta_1, \beta_2, \beta_3)$ is an arbitrary multiindex).

Proof. From Lemma 1.6 it follows that there exists in $P_{sp}(S)$ a sequence of type

$$\sum_{q=1}^{m} c_q \, (\partial_{n(x)} - i) \exp\{i \, d^{(q)} \cdot x\} \,, \quad x \in S \,,$$

which converges (in the L_2-sense) to the function $[(\partial_{n(x)} - i)w]^+ \in C(S) \subset L_2(S)$.

We set

$$v_m(x) = \sum_{q=1}^{m} c_q \, e^{i \, d^{(q)} \cdot x} \,, \quad x \in \overline{\Omega^+} \,.$$

Hence,

$$(\partial_{n(x)} - i) v_m(x) \to [(\partial_{n(x)} - i) w(x)]^+ \ \text{ in } \ L_2(S) \,.$$

By Lemma 1.5 the functions v_m and w can be represented in the form

$$\begin{aligned} v_m(x) &= V_a(\mathcal{P}_a^{-1}[(\partial_n - i)v_m]^+)(x) \,, \quad x \in \Omega^+ \,, \\ w(x) &= V_a(\mathcal{P}_a^{-1}[(\partial_n - i)w]^+)(x) \,, \quad x \in \Omega^+ \,. \end{aligned}$$

Now, let $\overline{\Omega_0^+} \subset \Omega^+$ and $x \in \overline{\Omega_0^+}$. Denote by δ the distance between $\overline{\Omega_0^+}$ and $S = \partial\Omega^+$. The above representations of v_m and w together with Lemma 1.5 then imply

$$\begin{aligned} &|D^\beta w(x) - D^\beta v_m(x)| \leq \\ &\leq c_1(\delta) \, ||\mathcal{P}_a^{-1}[(\partial_n - i)v_m]^+ - \mathcal{P}_a^{-1}[(\partial_n - i)w]^+||_{L_2(S)} \leq \\ &\leq c_2(\delta) \, ||[(\partial_n - i)v_m]^+ - [(\partial_n - i)w]^+||_{L_2(S)} \to 0 \end{aligned}$$

as $m \to +\infty$ (uniformly in $\overline{\Omega_0^+}$) for arbitrary multiindex β. □

Corollary 1.8. *Let* $x_0 \notin \overline{\Omega^+}$. *Then there exists a sequence* $v_m \in P_{sp}(\mathbb{R}^3)$ *such that (for arbitrary multiindex* β*)*

$$D^\beta v_m(x) \to D^\beta \gamma(x - x_0, \omega)$$

uniformly in $\overline{\Omega^+}$, *i.e., for arbitrary* $k \in \mathbb{N} \cup \{0\}$ *and* $\alpha \in (0;\, 1)$

$$||v_m(x) - \gamma(x - x_0, \omega)||_{C^{k+\alpha}(\overline{\Omega^+})} \to 0$$

as $m \to \infty$.

1.6 – Vector-valued potential operators of the theory of steady state elastic oscillations

Let

$$\Phi_C(\xi,\omega) := \det\, C(i\xi,\omega) = \det\left[\rho_1\omega^2 I_3 - C(\xi)\right], \quad \xi \in \mathbb{R}^3\,, \tag{1.24}$$

be a characteristic polynomial of the operator $C(D,\omega)$.

We assume that the following two conditions are fullfiled (cf. [41], [29], [30]):

I^0. $\nabla_\xi \Phi_C(\xi,\omega) \neq 0$ at real zeros of the polynomial (1.24);

II^0. The Gaussian curvature of the surface, defined by the real zeros of the polynomial $\Phi_C(\xi,\omega)$, does not vanish anywhere.

From the above conditions I^0 and II^0 it follows that the real zeros of the polynomial $\Phi_C(\xi,\omega)$ form nonselfintersecting, closed, convex two-dimensional surfaces $S_{\omega,1}$, $S_{\omega,2}$, $S_{\omega,3}$, enveloping the origin of co-ordinates. For an arbitrary unit vector $\eta = x/|x|$ with $x \in \mathbb{R}^3\backslash\{0\}$, there exists only one point on each $S_{\omega,j}$, namely $\xi^j = (\xi_1^j, \xi_2^j, \xi_3^j) \in S_{\omega,j}$ such that the outward unit normal vector $n(\xi^j)$ to $S_{\omega,j}$ at the point ξ^j has the same direction as η, i.e., $n(\xi^j) = \eta$. In this case we say that the points ξ^j $(j = 1,2,3)$ correspond to the vector η.

A function (vector, matrix) u is said to belong to the class $\mathrm{SK}(\Omega^-)$ if $u \in C^1(\Omega^-)$ and for sufficiently large $|x|$ the following relations hold (no summation over j in the last equation):

$$\begin{gathered} u(x) = \sum_{j=1}^{3} u^j(x), \\ u^j(x) = O(|x|^{-1}),\ D_k u^j(x) - i\,\xi_k^j u^j(x) = O(|x|^{-2}),\ k = 1,2,3\,, \end{gathered} \tag{1.25}$$

where the point $\xi^j \in S_{\omega,j}$ corresponds to the vector $\eta = x/|x|$. These conditions are *the generalized Sommerfeld-Kupradze type radiation conditions* in the anisotropic elasticity. It is easy to verify that in the isotropic case the conditions (1.25) coincide with the well-known Sommerfeld-Kupradze radiation conditions (for details see [25], [29], [30]).

Denote by $\Gamma(\cdot\,,\omega) \in \mathrm{SK}(\mathbb{R}^3\backslash\{0\})$ the fundamental matrix of the operator $C(D,\omega)$. By means of the Fourier transform method and the limiting absorption principle this matrix has been constructed and investigated in [29] (see also [41], [30]). The matrix reads as

$$\Gamma(x,\omega) = \lim_{\varepsilon\to 0+} \mathcal{F}^{-1}_{\xi\to x}[C^{-1}(-i\,\xi,\omega+i\,\varepsilon)]\,,$$

where $\mathcal{F}^{-1}$ stands for the generalized Fourier inverse transform. Note that for the isotropic case this matrix can be written explicitely in terms of elementary functions (see [25]).

Further, we construct single- and double-layer vector potentials:

$$\begin{aligned} V_C(g)(x) &= \int_S \Gamma(x-y,\omega)\,g(y)\,dS_y\,,\\ W_C(g)(x) &= \int_S [T(D_y,n(y))\Gamma(y-x,\omega)]^\top\,g(y)\,dS_y\,. \end{aligned}$$

Properties of these potentials and boundary operators generated by them are studied in [29], [30].

For a regular solution u to the equation (1.1) in Ω^+ we have the following integral representation

$$u(x) = W_C([u]^+)(x) - V_C([Tu]^+)(x)\,,\quad x\in\Omega^+\,. \tag{1.26}$$

For $x\in\Omega^-$ the right-hand side is zero.

The similar representation holds also for a radiating regular solution to the equation (1.1) in Ω^-:

$$u(x) = V_C([Tu]^-)(x) - W_C([u]^-)(x)\,,\quad x\in\Omega^-\,.$$

For $x\in\Omega^+$ the right-hand side is zero.

The analogue of Rellich's lemma in this case reads as follows.

LEMMA 1.8. *Let $u \in SK(\Omega^-)$ be a solution of* (1.1) *in Ω^- and let*

$$\lim_{R\to+\infty} \operatorname{Im} \Big\{ \int_{\Sigma_R} (Tu)_k \overline{(u)_k}\, d\Sigma_R \Big\} = 0\,,$$

where Σ_R is the same as in Lemma 1.1.

Then $u = 0$ in Ω^-.

This lemma implies that the exterior homogeneous BVPs (with given zero displacements or zero stresses on the boundary) have only the trival solution (see [29], [30]).

Further, we set

$$\mathcal{H}_C\, g(z) := \int_S \Gamma(z-y,\omega)\, g(y)\, dS_y, \tag{1.27}$$

$$\begin{aligned} \mathcal{K}_C^{(1)}\, g(z) &:= \int_S [T(D_z, n(z))\Gamma(z-y,\omega)]\, g(y)\, dS_y\,, \\ \mathcal{K}_C^{(2)}\, g(z) &:= \int_S [T(D_y, n(y))\Gamma(y-z,\omega)]^\top\, g(y)\, dS_y\,, \\ \mathcal{L}_C\, g(z) &:= [T(D_z, n(z)) W_C(g)(z)]^\pm, \quad z \in S\,. \end{aligned} \tag{1.28}$$

Note that the potential and boundary operators V_C, W_C, $\mathcal{H}_C$, $\mathcal{K}_C^{(1)}$, $\mathcal{K}_C^{(2)}$, and $\mathcal{L}_C$ have quite the same jump and mapping properties as the corresponding scalar operators considered in Subsection 1.4 (see Lemma 1.2). Moreover, the matrix operators $\mathcal{H}_C$, $\pm 2^{-1} I_3 + \mathcal{K}_C^{(1)}$, $\pm 2^{-1} I_3 + \mathcal{K}_C^{(2)}$, and $\mathcal{L}_C$ are elliptic pseudodifferential operators of index zero. In particular, the principal symbol matrices of the operators $-\mathcal{H}_C$ and $\mathcal{L}_C$ are positive definite (for details see [29], [30], [33], [20], [21]).

The $\mathcal{H}_C$ is an integral operator on S with a weakly singular kernel of type $O(|x-y|^{-1})$, $\mathcal{K}_C^{(1)}$ and $\mathcal{K}_C^{(2)}$ are singular integral operators on S, while $\mathcal{L}_C$ is a singular integro-differential operator on S.

2 – The direct fluid-structure interaction problem

2.1 – Uniqueness theorem. Jones modes and Jones eigenfrequencies

We denote by $J(\Omega^+)$ the set of values of the frequency parameter $\omega > 0$ for which the following boundary value problem

$$\begin{aligned} &C(D,\omega)\, u(x) = 0, \quad x \in \Omega^+, \\ &[T(D,n)u(x)]^+ = 0, \quad [u(x)\cdot n(x)]^+ = 0, \quad x \in S, \end{aligned}$$

admits a nontrivial solution. Such solutions (vectors) are called *Jones modes*, while the corresponding values of ω are called *Jones eigenfrequencies* (cf. [26], [18]). The space of Jones modes corresponding to ω we denote by $X_\omega(\Omega^+)$. Note that $J(\Omega^+)$ is at most enumerable, and for each $\omega \in J(\Omega^+)$ the space of associated Jones modes is of finite dimension (see [29]). Clearly, if $u \in X_\omega(\Omega^+)$, then $\overline{u} \in X_\omega(\Omega^+)$.

Let us consider the homogeneous version of the direct problem ($f_0 = 0$ and $f = 0$ in (1.13) and (1.14)).

THEOREM 2.1. *Let a pair* (u, w^{sc}) *be a solution of the homogeneous direct problem* (1.1), (1.5), (1.13) *and* (1.14), *with* b_1 *and* b_2 *given by* (1.15).

Then $w^{sc} = 0$ *in* Ω^- *and* $u \in X_\omega(\Omega^+)$.

PROOF. It is verbatim the proof of Theorem 4.1 in [33]. ∎

COROLLARY 2.2. *Let* $\omega \notin J(\Omega^+)$. *Then the homogeneous direct problem possesses only the trivial solution.*

2.2 – Existence results

In what follows we assume that $S \in C^{k+1+\alpha'}$ with integer $k \geq 0$ and $0 < \alpha < \alpha' \leq 1$.

First we prove the following assertion.

LEMMA 2.3. *Let* $\partial\Omega^+ = S \in C^{1+\alpha'}$. *Then an arbitrary solution* $u \in C^{1+\alpha}(\overline{\Omega^+})$ *of equation* (1.1) *is representable in the form of a single-layer potential.*

PROOF. Let $u \in C^{1+\alpha}(\overline{\Omega^+})$ be a solution of equation (1.1). The vector function

$$[T(D,n)u(x)]^+ - i\,[u(x)]^+ =: F(x) \tag{2.1}$$

then belongs to the space $C^\alpha(S)$.

Let us consider the following boundary value problem

$$C(D,\omega)u(x) = 0 \quad \text{in} \quad \Omega^+\,, \tag{2.2}$$

$$[T(D,n)u(x)]^+ - i\,[u(x)]^+ = \Phi(x), \quad x \in S\,, \tag{2.3}$$

where $\Phi = (\Phi_1, \Phi_2, \Phi_3) \in C^\alpha(S)$ is an arbitrary vector-function.

We look for a solution to the BVP (2.2)-(2.3) in the form of a single-layer potential

$$u(x) = V_C(g)(x)\,, \quad x \in \Omega^+\,,$$

where $g = (g_1, g_2, g_3)^\top$ is a sought for density.

The boundary condition (2.3) then leads to the system of singular integral equations of normal type with index equal to zero

$$[\,-2^{-1}I_3 + \mathcal{K}_C^{(1)} - i\,\mathcal{H}_C]\,g = \Phi\,, \tag{2.4}$$

where $\mathcal{K}_C^{(1)}$ and $\mathcal{H}_C$ are given by (1.28) and (1.27), respectively.

Further we show that the operator

$$\mathcal{P}_C := -2^{-1}I_3 + \mathcal{K}_C^{(1)} - i\,\mathcal{H}_C \;:\; C^\alpha(S) \to C^\alpha(S) \tag{2.5}$$

is invertible.

To this end we first prove that the homogeneous BVP ($\Phi = 0$) has only the trivial solution.

Applying Green's identity we arrive at the equation

$$\int_{\Omega^+} \{E(u,\overline{u}) - \omega^2\,|u|^2\}\,dx = \int_S [Tu]^+ \cdot [u]^+\,dS\,,$$

which due to (2.3) with $\Phi = 0$ implies

$$\int_{\Omega^+} \{E(u,\overline{u}) - \omega^2\,|u|^2\}\,dx = i\,\int_S |[u]^+|^2\,dS\,;$$

here $E(u,\overline{u})$ is a real non-negative function given by (1.3).

From this equation it follows: $[u(x)]^+ = 0$ on S. Therefore, $[Tu]^+ = 0$ on S, again due to (2.3). With the help of the general integral representation (1.3), we conclude that $u = 0$ in Ω^+, which shows that the homogeneous BVP in question has only the trivial solution.

Let $g_0 \in C^\alpha(S)$ be an arbitrary solution to the homogeneous system (2.4) ($\Phi = 0$). The potential $V_C(g_0) \in C^{1+\alpha}(\overline{\Omega^+})$ then solves the homogeneous BVP (2.2)-(2.3) and therefore

$$V_C(g_0)(x) = 0\,, \quad x \in \Omega^+\,.$$

Using the continuity property of single-layer potentials, we have

$$[V_C(g_0)(x)]^+ = [V_C(g_0)(x)]^- = 0\,.$$

Evidently, $V_C(g_0)(x) \in C^{1+\alpha}(\overline{\Omega^-}) \cap \mathrm{SK}(\Omega^-)$ and this potential solves the homogeneous Dirichlet type exterior BVP (with zero dicplacements on S). By Theorem 4.2 in [30], we then get $V_C(g_0)(x) = 0$ in Ω^- and taking into consideration the equation $[TV_C(g_0)]^- - [TV_C(g_0)]^+ = g_0$, we conclude that $g_0 = 0$ on S, i.e., $\ker \mathcal{P}_C$ is trivial.

Since $\mathcal{P}_C$ is an elliptic pseudodifferential operator of order 0 with zero index, it follows that the operator (2.5) is invertible. Therefore, from (2.4) we have

$$g = \mathcal{P}_C^{-1}\,\Phi = \mathcal{P}_C^{-1}\{[T(D,n)u]^+ - i\,[u]^+\}\,.$$

In turn, this proves that an arbitrary solution $u \in C^{1+\alpha}(\overline{\Omega^+})$ to equation (1.1) can be represented in the form of single-layer potential

$$u(x) = V_C(\mathcal{P}_C^{-1}\,F)(x)\,, \quad x \in \Omega^+\,,$$

with F given by (2.1).

This completes the proof. □

Further, we show that the nonhomogeneous problem $P^{(\mathrm{dir})}$ is solvable for arbitrary incident wave and for arbitrary value of the oscillation parameter ω.

Let us look for a solution to the problem in the following form:

$$u(x) = V_C(g)(x)\,, \quad x \in \Omega^+\,, \tag{2.6}$$

$$w^{sc}(x) = W_a(g_4)(x) - i\,V_a(g_4)(x), \quad x \in \Omega^-\,, \tag{2.7}$$

where $g = (g_1, g_2, g_3)^\top \in C^\alpha(S)$ and $g_4 \in C^{1+\alpha}(S)$ are sought for densities.

The boundary conditions (1.13) and (1.14) lead to the system of equations

$$(-2^{-1}I_3 + \mathcal{K}_C^{(1)})g(x) - b_2\,n(x)\mathcal{N}\,g_4(x) = f(x), \tag{2.8}$$

$$(\mathcal{H}_C\,g(x)) \cdot n(x) - b_1\,\mathcal{M}\,g_4(x) = f_0(x), \qquad x \in S, \tag{2.9}$$

where $\mathcal{K}_C^{(1)}$, $\mathcal{H}_C$, $\mathcal{N}$, $\mathcal{M}$, f, and f_0 are given by (1.28), (1.27), (1.19), (1.20), (1.17), and (1.16), respectively. The constants b_1 and b_2 are defined by (1.15).

It can easily be shown that this system is equivalent to the following one

$$g_4 = b_1^{-1}\mathcal{M}^{-1}\{(\mathcal{H}_C\, g)\cdot n\} - b_1^{-1}\mathcal{M}^{-1} f_0, \tag{2.10}$$

$$\begin{aligned} g + \mathcal{P}_C^{-1}\{i\,\mathcal{H}_C\, g - b_2\, b_1^{-1} n \mathcal{N}\,\mathcal{M}^{-1}[(\mathcal{H}_C\, g)\cdot n]\} &= \\ = \mathcal{P}_C^{-1}(f - b_2\, b_1^{-1} n \mathcal{N}\,\mathcal{M}^{-1} f_0). & \end{aligned} \tag{2.11}$$

Note that, in the latter equation, the second term in the left-hand side represents an integral operator with a weakly singular kernel function of type $O(|x-y|^{-1})$.

The matrix operator generated by the left-hand sides of (2.8) and (2.9) reads as

$$\mathcal{K} := \begin{bmatrix} \left[-2^{-1} I_3 + \mathcal{K}_C^{(1)}\right]_{3\times 3} & [\,-b_2 n(x)\,\mathcal{N}]_{3\times 1} \\ \left[n_j(\mathcal{H}_C)_{jk}\right]_{1\times 3} & -b_1\mathcal{M} \end{bmatrix}_{4\times 4}.$$

Therefore, the system (2.8) and (2.9) can be rewritten in the matrix form

$$\mathcal{K}\, G = F\,, \tag{2.12}$$

where $G = (g_1, g_2, g_3, g_4)^\top$ and $F = (f_1, f_2, f_3, f_0)^\top$.

From the results outlined in Subsection 1.4 and 1.6 it follows that $\mathcal{K}$ is an elliptic pseudodifferential operator on S (in the sense of Douglis-Nirenberg) with zero index and has the mapping property

$$\mathcal{K}\ :\ [C^{k+\alpha}(S)]^3 \times [C^{k+1+\alpha}(S)] \to [C^{k+\alpha}(S)]^4\,.$$

Let us introduce the scalar integral operator

$$\mathcal{R}\,\varphi(x) := \int_S \frac{1}{|x-y|}\,\varphi(y)\, dS_y\,.$$

The operator

$$\mathcal{R}\ :\ C^{k+\alpha}(S) \to C^{k+1+\alpha}(S)$$

is an isomorphism (see [18], Lemma 3.3; [32]). Moreover, it is a formally self-adjoint, elliptic, equivalent lifting pseudodifferential operator of order -1.

Further, let

$$\mathcal{Q} := \begin{bmatrix} [I_3]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{R} \end{bmatrix}_{4\times 4} \quad \text{and} \quad \widetilde{\mathcal{K}} := \mathcal{Q}\,\mathcal{K}\,.$$

Clearly, $\widetilde{\mathcal{K}}$ is a matrix singular integral operator of normal type with zero index ([25], [10], [27]). It is evident that the equation (2.12) is equivalent to the system of singular integral equations

$$\widetilde{\mathcal{K}}\,G = \widetilde{F}, \quad \widetilde{F} = \mathcal{Q}F = (f_1, f_2, f_3, \mathcal{R}\,f_0)^\top\,, \tag{2.13}$$

which is obtained from (2.12) by applying the operator $\mathcal{Q}$ to the both sides.

Note that

$$\widetilde{\mathcal{K}}\ :\ C^{k+\alpha}(S) \to C^{k+\alpha}(S) \quad [L_2(S) \to L_2(S)]\,.$$

The formally adjoint operator to $\widetilde{\mathcal{K}}$ with respect to the complex L_2-scalar product is defined by the relation

$$\langle \widetilde{\mathcal{K}}\,G, H\rangle_S = \langle G, \widetilde{\mathcal{K}}^*\,H\rangle_S\,,$$

where $G = (g_1, g_2, g_3, g_4)^\top$ and $H = (h_1, h_2, h_3, h_4)^\top$ are complex-valued vector-functions, and

$$\langle G, H\rangle_S := \int_S \sum_{k=1}^{4} G_k\,\overline{H_k}\,dS\,.$$

With the help of the equalities $[\mathcal{K}_a^{(1)}]^* = \overline{\mathcal{K}_a^{(2)}}$, $\mathcal{H}_a^* = \overline{\mathcal{H}_a}$, $\mathcal{L}_a^* = \overline{\mathcal{L}_a}$, $[\mathcal{K}_C^{(1)}]^* = \overline{\mathcal{K}_C^{(2)}}$, $\mathcal{R}^* = \mathcal{R}$, it can be shown that the homogeneous adjoint equation to (2.13)

$$\widetilde{\mathcal{K}}^*\,H = 0\,, \quad H = (h, h_4)^\top\,, \quad h = (h_1, h_2, h_3)^\top\,, \tag{2.14}$$

is equivalent to the system of singular integral equations

$$(2.15)\qquad (-2^{-1}I_3 + \mathcal{K}_C^{(2)})\overline{h} + \mathcal{H}_C\, n\, \mathcal{R}\, \overline{h_4} = 0\,,$$

$$(2.16)\qquad -b_2[-2^{-1}I+\mathcal{K}_a^{(1)}-i\,\mathcal{H}_a]\,(\overline{h}\cdot n)-b_1[\mathcal{L}_a-i(2^{-1}I+\mathcal{K}_a^{(2)})]\mathcal{R}\,\overline{h_4}=0.$$

In fact, this system coincides with the equation $\overline{\widetilde{\mathcal{K}}^*\, H} = 0$.

Lemma 2.4. *Let* $H = (h_1, h_2, h_3, h_4)^\top \in C^{1+\alpha}(S)$ *be a solution of equation* (2.14) (*i.e., of the system* (2.15)-(2.16)).

Then

$$(h\cdot n) = 0\,, \quad h_4 = 0\,,$$

where $h = (h_1, h_2, h_3)^\top$.

If h is nontrivial, then it represents the restriction on S of some Jones mode and vice versa: if $u \in X_\omega(\Omega^+)$, then $([u]_S^+, 0)^\top \in \ker \widetilde{\mathcal{K}}^*$.

Moreover, $\dim\ker\widetilde{\mathcal{K}}^* = \dim\ker\widetilde{\mathcal{K}} = \dim\ker\mathcal{K} = \dim\ker X_\omega(\Omega^+)$.

Proof. Let us construct the potential type functions

$$\begin{aligned} u(x) &= W_C(\overline{h})(x) + V_C(n\,\mathcal{R}\,\overline{h_4})(x), \ \ x\in\Omega^-\,,\\ w(x) &= -b_2\, V_a(\overline{h}\cdot n)(x) - b_1\, W_a(\mathcal{R}\,\overline{h_4})(x)\,, \ \ x\in\Omega^+\,. \end{aligned}$$

It is evident, that $u_j, w \in C^1(\overline{\Omega^\pm})$, and $[u(x)]^- = 0$ and $[(\partial_n - i)w(x)]^+ = 0$ on S, due to (2.15) and (2.16).

By Theorem 4.2 in [30] and Lemma 1.5 we conclude that $u(x) = 0$ in Ω^- and $w(x) = 0$ in Ω^+. Therefore,

$$(2.17)\qquad \begin{aligned} &[u]^+ - [u]^- = [u]^+ = \overline{h},\\ &[Tu]^+ - [Tu]^- = [Tu]^+ = -n\,\mathcal{R}\,\overline{h_4}\,,\\ &[w]^+ - [w]^- = -[w]^- = -b_1\,\mathcal{R}\,\overline{h_4},\\ &[\partial_n w]^+ - [\partial_n w]^- = -[\partial_n w]^- = b_2\,\overline{h}\cdot n\,, \end{aligned}$$

which imply

$$(2.18)\qquad [Tu]^+ = -b_1^{-1}[w]^-\, n\,, \quad [u\cdot n]^+ = -b_2^{-1}[\partial_n w]^-\,.$$

This is a similar problem to Problem $P^{(\mathrm{dir})}$. One can easily show that the conditions (2.18) yield: $u \in X_\omega(\Omega^+)$ and $w = 0$ in Ω^- (cf. Theorem 2.1).

From these results together with the invertibility of the operator $\mathcal{R}$ and equation (2.17) the conditions (2.13) follow immediately.

Thus we have shown that $\overline{h}$ is the restiction of some Jones mode on S. The inverse assertion easily follows from the general representation formula (1.26).

The last statement of the lemma is a consequence of the equivalence of (2.12) and (2.13), Lemma 2.3 and the fact that the index of $\mathcal{K}$ equals to zero. This completes the proof. ∎

COROLLARY 2.5. i) *If $\omega \notin J(\Omega^+)$, then the system* (2.8)-(2.9) *is uniquely solvable for arbitrary f and f_0.*

ii) *Let $\omega \in J(\Omega^+)$. The equation*

$$\int_S f(y) \cdot h(y)\, dS = \int_S \sum_{j=1}^{3} f_j(y)\, \overline{h_j(y)}\, dS = 0\,, \tag{2.19}$$

where $h = (h_1, h_2, h_3)^\top$ is the restriction on S of an arbitrary Jones mode, represents a necessary and sufficient condition for the system (2.8)-(2.9) (*i.e., for the equation* (2.12)) *to be solvable in the space $[C^\alpha(S)]^3 \times C^{1+\alpha}(S)$, provided that $f_j, f_0 \in C^\alpha(S)$, $S \in C^{2+\alpha'}$.*

Note that, if $f(x) = n(x)\,\varphi(x)$, where φ is some scalar function and, as above, n is the unit normal vector to S, then the condition (2.19) is automatically satisfied.

THEOREM 2.6. *The direct scattering problem $P^{(\mathrm{dir})}$ is solvable for arbitrary incident wave w^{inc} and for arbitrary value of the oscillation parameter ω.*

Moreover, a solution is representable in the form of (2.6) *and* (2.7), *where g_4 and w^{sc} are defined uniquely, while g and u are defined uniquely if $\omega \notin J(\Omega^+)$ and, if ω is exceptional ($\omega \in J(\Omega^+)$), then g is defined modulo vector-functions of $\ker \mathcal{K}$ and u is defined modulo Jones modes ($X_\omega(\Omega^+)$).*

When ω is an exceptional value, then the vector $(-\frac{1}{2} I_3 + \mathcal{K}_C^{(1)})\, g$ and the normal component of the vector $\mathcal{H}_C\, g$ are determined uniquely.

PROOF. For arbitrary incident wave and for arbitrary value of the oscillation parameter the solvability of the nonhomogeneous problem $P^{(\mathrm{dir})}$ with interface conditions (1.13)-(1.17) follows from Lemmata 2.3, 2.4, and corollary 2.5, since the problem is equivalently reduced to the system (2.8)-(2.9).

It is evident that, if $\omega \notin J(\Omega^+)$, then $\ker\mathcal{K}$ is trivial and due to Theorem 2.1 and corollary 2.2 the problem $P^{(\mathrm{dir})}$ is uniquely solvable. In this case the solution is representable in the form of (2.6) and (2.7), where $(g, g_4)^\top$ is a unique solution of the system (2.8)-(2.9).

Let $\omega \in J(\Omega^+)$, i.e., the problem $P^{(\mathrm{dir})}$ is solvable but not uniquely. An arbitrary solution of the problem is again representable in the form of (2.6)-(2.7), where $(g, g_4)^\top$ solves the system (2.8)-(2.9).

To prove the last assertion of the theorem one needs to consider the homogeneous version of the above mentioned system and show that, if $G = (g, g_4)^\top \in \ker\mathcal{K} = \ker\widetilde{\mathcal{K}}$, then

$$g_4 = 0, \quad (-2^{-1}I_3 + \mathcal{K}_C^{(1)})\, g = 0\,, \quad \mathcal{H}_C\, g \cdot n = 0\,.$$

In turn , this follows from Theorem 2.1, Lemma 1.3 and the structure of the system (2.8)-(2.9). $\square$

It is evident that the far field pattern $w_\infty^{\mathrm{sc}}(\xi)$ corresponding to the scalar field (2.7), is defined uniquely as well and

$$w_\infty^{\mathrm{sc}}(\xi) = c(\xi) \int_S [(\partial_{n(y)} - i)e^{-i\xi\cdot y}]\, g_4(y)\, dS_y\,, \tag{2.20}$$

where $\xi \in S_\omega$ corresponds to the vector $\eta = x/|x|$ and $c(\xi)$ is given by (1.10).

COROLLARY 2.7. *Let* $G = (g_1, g_2, g_3, g_4)^\top$ *be a solution to the equation* (2.13)*, i.e., to the system* (2.8)-(2.9).

Then

$$\begin{aligned}
&\| g_4 \|_{L_2(S)} \le C_1 \| \tilde{F} \|_{L_2(S)} \le C_2\{\| f \|_{L_2(S)} + \| f_0 \|_{L_2(S)}\}\,,\\
&\| g_4 \|_{C^{k+1+\alpha}(S)} \le C_3 \| \tilde{F} \|_{C^{k+1+\alpha}(S)} \le C_4\{\| f \|_{C^{k+1+\alpha}(S)} + \| f_0 \|_{C^{k+\alpha}}\}\,,
\end{aligned}$$

where C_j $(j = \overline{1,4})$ *do not depend on* g_4, f, f_0.

The proof is a consequence of the following – more general assertion.

LEMMA 2.8. *Let a Banach space X be the direct product of two Banach spaces X_1 and X_2, i.e., $X = X_1 \times X_2$ with the norm $\| x \|_X = \| x_1 \|_{X_1} + \| x_2 \|_{X_2}$, where $x = (x_1, x_2) \in X$, $x_k \in X_k$, $k = 1, 2$.*

Let $T : X \to Y$ be a linear continuous operator from X into Banach space Y (with the norm $\| \cdot \|_Y$) and assume that the linear equation

$$Tx = y\,, \tag{2.21}$$

where $y \in Y$ is a given element and $x \in X$ is an unknown, is normally solvable, i.e., the range $R(T)$ is closed in Y.

Moreover, let $\ker T \subset X_1 \times \{\theta_2\}$ *where θ_k, $k = 1, 2$, are zero elements of X_k.*

If $x = (x_1, x_2) \in X$ is a solution of equation (2.21) *then there exists a constant $c > 0$, independent of y, such that*

$$\| x_2 \|_{X_2} \leq c \| y \|_Y = c \| Tx \|_Y \ .$$

PROOF. Denote by T_1 and T_2 the restrictions of the operator T onto the spaces $X_1 \times \{\theta_2\}$ and $\{\theta_1\} \times X_2$. In fact, the operators T_1 and T_2 uniquely induce linear continuous operators from X_k into Y for which we keep the same notation, i.e., $T_1 x_1 := T_1(x_1, \theta_2)$ and $T_2 x_2 := T_2(\theta_1, x_2)$. Evidently, $Tx = T_1 x_1 + T_2 x_2$ for $x = (x_1, x_2)$.

For simplicity we also use the notation x_1 and x_2 for the elements (x_1,θ_2) and (θ_1,x_2) and assume $T(x_1,\theta_2) = T_1(x_1,\theta_2) = T_1 x_1$ and $T(\theta_1,x_2) = T_2(\theta_1, x_2) = T_2 x_2$.

Further, we introduce the quotient space $Z = X_{1/\ker T} = \{[x_1]\}$ endowed with the norm $\| [x_1] \|_Z = \inf_{\xi \in \ker T} \| x_1 + \xi \|_{X_1}$. Since $\ker T$ is closed, Z is a Banach space.

Due to our notation $\ker T = \ker T_1$.

By standard approach the linear continuous operator $T_1 : X_1 \to Y$ in the natural way induces the corresponding linear operator $\widetilde{T}_1 : Z = X_{1/\ker T_1} \to Y$ which is also continuous with respect to the corresponding norms.

Note that $Y_0 = R(T)$ is a Banach space with the norm induced by the Y-norm.

Now, let us construct the operator $T_0 : Z \times X_2 \to Y_0 = R(T)$, where

$$T_0(z, x_2) = \widetilde{T}_1 z + T_2 x_2 .$$

It is evident that $Z \times X_2$ is a Banach space and T_0 is a one-to-one operator. Therefore, due to the Banach theorem for arbitrary $(z, x_2) \in Z \times X_2$ there holds the inequality

$$\| (z, x_2) \|_{Z\times X_2} = \| z \|_Z + \| x_2 \|_{X_2} \le c_1 \| T_0(z, x_2) \|_Y \tag{2.22}$$

with a constant $c_1 > 0$ independent of (z, x_2).

Let $x = (x_1, x_2)$ be an arbitrary element of X and let $z \in Z$ corresponds to the element $x_1 \in X_1$. By applying (2.22) and the equality $T_0(z, x_2) = T_1 x_1 + T_2 x_2 = T(x_1, x_2) = Tx$ we get

$$\| x_2 \|_{X_2} \le \| z \|_Z + \| x_2 \|_{X_2} \le c_1 \| T_0(z, x_2) \|_Y = c_1 \| Tx \|_Y ,$$

which completes the proof. □

Corollary 2.9. *Let (u, w^{sc}) be a solution to Problem $P^{(\mathrm{dir})}$. Then*

$$\| w^{sc} \|_{C^{k+1+\alpha}(\Omega^-)} \le C_5 \{ \| f \|_{C^{k+1+\alpha}(S)} + \| f_0 \|_{C^{k+\alpha}(S)} \} ,$$

where $C_5 > 0$ does not depend on f, f_0, w^{sc}.

Proof. It follows from properties of single- and double-layer potentials and corollary 2.7. □

Corollary 2.10. *Let g be a solution to the equation* (2.11). *If $\{g^{(q)}\}_{q=1}^N$ is a complete system of linearly independent solutions to the homogeneous version of system* (2.11) *and $g^{(0)}$ is a particular solution to the nonhomogeneous system* (2.13), *orthogonal to all of $g^{(q)}$ $(q = \overline{1, n})$, then*

$$\| g^{(0)} \|_{L_2(S)} \le C_6 \{ \| f \|_{L_2(S)} + \| f_0 \|_{L_2(S)} \} ,$$
$$\| g^{(0)} \|_{C(S)} \le C_7 \{ \| f \|_{C(S)} + \| f_0 \|_{C(S)} \} ,$$
$$\| g^{(0)} \|_{C^{k+1+\alpha}(S)} \le C_8 \{ \| f \|_{C^{k+1+\alpha}(S)} + \| f_0 \|_{C^{k+\alpha}(S)} \} ,$$

where the constants C_6, C_7, and C_8 are independent of $f, f_0, g^{(0)}$.

The proof follows form the assertion.

Lemma 2.11. *Let X and Y be Banach spaces, $T : X \to Y$ be a linear continuous operator, and the equation*

$$Tx = y \tag{2.23}$$

be normally solvable.

Moreover, let $\dim \ker T = N < \infty$, $\{e_j\}_{j=1}^N$ *be a basis in* $\ker T$, *and* $\{f_j\}_{j=1}^N$ *be a corresponding biorthogonal system in the adjoint space* X^*, *i.e.,* $f_j \in X^*$ *and* $f_i(e_j) = \delta_{ij}$, *where* δ_{ij} *is the Kronecker's delta.*

If x is an arbitrary solution of the equation (2.23), *then* $\widetilde{x} = x - \sum_{i=1}^N f_i(x)e_i$ *is a particular solution of the same equation satisfying the inequality*

$$\| \widetilde{x} \|_X \leq C_9 \| y \|_Y , \tag{2.24}$$

where C_9 does not depend on x and y.

Proof. It is well-known that, if $X_0 = \ker T$ and $X_1 = \cap_{i=1}^N \ker f_i$, then X is a direct sum of the spaces X_0 and X_1: $X = X_0 \oplus X_1$ (see [35], Ch. 4). Denote by T_1 the restriction of the operator T onto the subspace $X_1 \subset X$ and let $Y_1 := R(T)$ be the range of the operator T, i.e., $Y_1 = T(X)$. Since equation (2.23) is normally solvable, Y_1 is a closed subset of Y and due to the Banach theorem the bijective operator $T_1 : X_1 \to Y_1$ possesses a continuous inverse operator $T_1^{-1} : Y_1 \to X_1$. Moreover, there holds an inequality

$$\| T_1^{-1} z \|_X \leq C_{10} \| z \|_Y \quad \text{for all} \quad z \in Y_1 = R(T) , \tag{2.25}$$

where C_{10} does not depend on z.

Further, if x is an arbitrary solution of equation (2.23), then $\widetilde{x} = x - \sum_{i=1}^N f_i(x)e_i$ is also a solution to the same equation, and, evidently, $\widetilde{x} \in X_1$ and $\widetilde{x} = T^{-1}y$ with $y \in Y_1$. If we put $z = y$ and $T_1^{-1}z = \widetilde{x}$ in (2.25) we get the inequality (2.24). ∎

3 – Inverse problem. Uniqueness theorem

This section deals with the uniqueness of solution to the inverse fluid-structure interaction problem (see Subsection 1.3).

THEOREM 3.1. *Let $\overline{\Omega_j^+}$, $j = 1, 2$, be two bounded elastic scatterers with $C^{2+\alpha'}$–smooth boundaries $\partial\Omega_j^+ = S_j$ and with simply connected complements $\Omega_j^- = \mathbb{R}^3 \setminus \overline{\Omega_j^+}$, and let for a fixed wave number ω the far field patterns $w_\infty^{(j)sc}(\cdot\,; d)$ for the both scatterers coincide for all incident directions $d \in S_\omega$.*

Then $\Omega_1^+ = \Omega_2^+$.

PROOF. STEP 1. We denote the elastic vector field in the domain Ω_j^+ by $u^{(j)}(x; d)$ and the scattered scalar field in the domain Ω_j^- by $w^{(j)sc}(x; d) =: w^{(j)}(x; d)$, $j = 1, 2$.

In the both cases the incident field is represented in the form of a plane wave (see (1.12)). Thus, the pair $(u^{(j)}, w^{(j)})$ is a solution to Problem $P^{(\mathrm{dir})}$ for the scatterer Ω_j^+ $(j = 1, 2)$ with a fixed oscillation parameter ω (see (1.13)-(1.17)).

Let $\Omega_1^+ \neq \Omega_2^+$ and $\Omega_{12}^- = \mathbb{R}^3 \setminus \{\overline{\Omega_1^+ \cup \Omega_2^+}\}$.

Since $w^{(1)}(x; d)$ and $w^{(2)}(x; d)$ are radiating solutions of the equation (1.5) in Ω_{12}^- and have the same far field patterns $w_\infty^{(1)}(\xi; d) = w_\infty^{(2)}(\xi; d)$ for all $d \in S_\omega$, we conclude that

$$w^{(1)}(x; d) = w^{(2)}(x; d) \quad \text{in} \quad \Omega_{12}^- \,, \tag{3.1}$$

due to the asymptotic relation (1.11) and Lemma 1.1.

STEP 2. Let us consider Problem $P^{(\mathrm{dir})}$ with the domains Ω_j^+, Ω_j^- $(j = 1, 2)$, where the incident field is taken in the form $w^{inc}(x) = v_m(x) \in P_{sp}(\mathbb{R}^3)$, i.e., $f_0(x) = b_1 \partial_{n(x)} v_m(x)$, $f(x) = b_2\, v_m(x)\, n(x)$. The corresponding elastic field in Ω_j^+ and the scattered field in Ω_j^- we denote by $u^{(j,m)}(x)$ and $w^{(j,m)}(x)$, respectively. From the conditions of Theorem 3.1 and the equality (3.1) it follows that

$$w_\infty^{(1,m)}(\xi) = w_\infty^{(2,m)}(\xi), \;\; \xi \in S_\omega, \;\; \text{and} \;\; w^{(1,m)}(x) = w^{(2,m)}(x), \;\; x \in \Omega_{12}^- \,.$$

STEP 3. Let x_0 be an arbitrary point in Ω_{12}^- and let us consider Problem $P^{(\mathrm{dir})}$ with the same domains Ω_j^+, Ω_j^- $(j = 1, 2)$, where the interface data are given as follows

$$f_0(x) = f_0(x; x_0) := b_1\, \partial_{n(x)}\gamma(x - x_0; \omega)\,, \tag{3.2}$$

$$f(x) = f(x; x_0) := b_2\, \gamma(x - x_0; \omega) n(x)\,; \tag{3.3}$$

here $\gamma(\cdot\,; \omega)$ is the fundamental function defined by (1.9). The corresponding elastic field (in Ω_j^+) and scalar scattered field (in Ω_j^-) we denote by $u^{(j)}(x; x_0)$ and $w^{(j)}(x; x_0)$.

Due to Lemma 1.8 there exists a sequence $v_m \in P_{sp}(\mathbb{R}^3)$ such that (for arbitrary multi-index β)

$$D^\beta v_m(x) \to D^\beta \gamma(x - x_0; \omega) \tag{3.4}$$

uniformly in $\overline{\Omega_1^+ \cup \Omega_2^+}$.

Applying the linearity of the direct problem, equation (2.20), corollary 2.7 and the results obtained in step 2, we get

$$\begin{aligned}
|w_\infty^{(1)}(\xi;x_0) - w_\infty^{(2)}(\xi;x_0)| &= |w_\infty^{(1)}(\xi;x_0) - w_\infty^{(1,m)}(\xi) + w_\infty^{(2,m)}(\xi) - w_\infty^{(2)}(\xi;x_0)| \le \\
&\le |w_\infty^{(1)}(\xi;x_0) - w_\infty^{(1,m)}(\xi)| + |w_\infty^{(2)}(\xi;x_0) - w_\infty^{(2,m)}(\xi)| \le \\
&\le C\, \{||\gamma(x - x_0; \omega) - v_m(x)||_{C^{1+\alpha}(S_1)} + \\
&\qquad + ||\gamma(x - x_0; \omega) - v_m(x)||_{C^{1+\alpha}(S_2)}\} \to 0
\end{aligned}$$

as $m \to \infty$; here $w_\infty^{(j,m)}(\xi)$ denotes the far field pattern of the scattered field $w^{(j,m)}(x)$ corresponding to the function $v_m \in P_{sp}(\mathbb{R}^3)$, involved in (3.4).

This implies $w_\infty^{(1)}(\xi; x_0) = w_\infty^{(2)}(\xi; x_0)$ and, consequently,

$$w^{(1)}(x; x_0) = w^{(2)}(x; x_0) \quad \text{in} \quad \Omega_{12}^-\,. \tag{3.5}$$

Step 4. Since $\Omega_1^+ \ne \Omega_2^+$, there exists a point $x^* \in \partial(\overline{\Omega_1^+ \cup \Omega_2^+})$ such that the closed ball $\underline{B(x^*, 2\delta)}$ centered at x^* and radius $2\delta > 0$ does not intersect either $\overline{\Omega_1^+}$ or $\overline{\Omega_2^+}$. Without restriction of generality, we assume that $\overline{B(x^*, 2\delta)} \cap \overline{\Omega_2^+} = \emptyset$. Evidently, $S_1^* := \partial\Omega_1^+ \cap \overline{B(x^*, 2\delta)} \subset S_1$ and $\text{dist}\{\overline{B(x^*, \delta)}\,, \overline{\Omega_2^+}\} \ge \delta$.

Further we choose a sequence $x^p \in B(x^*, \delta) \cap \Omega_{12}^-$ on the normal line to S_1 at the point $x^* \in S_1^*$ such that $|x^* - x^p| \to 0$ as $p \to \infty$.

Now let us consider the problem described in step 3 with the point x^p in the place of x_0.

For the domains Ω_1^+ and Ω_1^-, the interface conditions of type (1.13) on S_1 reads as follows:

$$[u^{(1)}(x;x^p)\cdot n(x)]^+_{S_1} = b_1[\partial_{n(x)}w^{(1)}(x;x^p)]^-_{S_1} + b_1\partial_{n(x)}\gamma(x-x^p;\omega)\,,\quad x\in S_1\,.$$

Taking into account the fact that $w^{(2)}(\cdot;x^p)$ is bounded in $\overline{B(x^*,\delta)}\subset\Omega_2^-$ together with its derivatives uniformly with respect to $x^p\in\overline{B(x^*,\delta)}$ (see corollary 2.9) and applying the equation (3.5) with x^p in the place of x_0, we arrive at the inequality

$$\begin{aligned}\big|[u^{(1)}(x;x^p)\cdot n(x)]^+_{S_1^*} - b_1[\partial_{n(x)}\gamma(x-x^p;\omega)]_{S_1^*}\big| \le |b_1[\partial_{n(x)}w^{(1)}(x;x^p)^-_{S_1^*}| =\\ = |b_1[\partial_{n(x)}w^{(2)}(x;x^p)^-_{S_1^*}|\le C,\end{aligned}$$

where C does not depend on $u^{(1)}$ and x^p.

In particular,

$$|[u^{(1)}(x^*;x^p)\cdot n(x^*)]^+ - b_1[\partial_{n(x^*)}\gamma(x^*-x^p;\omega)]|\le C,\quad p=1,2,3,\cdots\,. \tag{3.6}$$

Step 5. Here we prove that

$$|[u^{(1)}(x^*;x^p)\ \cdot n(x^*)]^+|\le C\ |\ \log|x^*-x^p|\ | \tag{3.7}$$

with a constant $C>0$ independent of x^p and $u^{(1)}$. Note that $u^{(1)}(x;x^p)$ and $w^{(1)}(x;x^p)$ can be represented in the form (2.6) and (2.7), where the densities g and g_4 are to be defined from the system (2.8)-(2.9), which is equivalent to the system (2.10)-(2.11), with f_0 and f given by (3.2) and (3.3). The right-hand side vector-function in (2.11) can be estimated as $O(|x-x^p|^{-1})$ $(x\in S_1)$ (see [31], Theorem 2.1).

Moreover,

$$g(x;x^p) = g^{(0)}(x;x^p) - \sum_{q=1}^{N} c_q g^{(q)}(x)\,,$$

where c_q $(q=\overline{1,N})$ are arbitrary constants, $\{g^{(q)}\}_{q=1}^N$ is a complete (orthonormal) system of linearly independent solutions of the corresponding homogeneous equations, and $g^{(0)}$ is a fixed particular solution of (2.11) orthogonal to this system. Remark, that $V(g^{(q)}) \in X_\omega(\Omega^+)$ $(q = \overline{1, N})$ if $\omega \in J(\Omega^+)$. Applying corollary 2.10 it can be shown that $g^{(0)}(x; x^p) = O(|x - x^p|^{-1})$. Therefore,

$$\begin{aligned} |[u^{(1)}(x; x^p) \ \cdot n(x)]^+| &= |[V_C(g(\cdot; x^p))(x) \ \cdot n(x)]^+| = \\ &= |[V_C(g^{(0)}(\cdot; x^p))(x) \ \cdot n(x)]^+| = \\ &= |[\mathcal{H}_C g^{(0)}(\cdot; x^p) \ \cdot n(x)]^+| \leq \\ &\leq |[\mathcal{H}_C g^{(0)}(\cdot; x^p)| \leq C \, | \ \log|x - x^p|| \,, \qquad x \in S_1, \end{aligned}$$

where C does not depend on x^p and $g^{(0)}$. In particular (3.7) holds.

Step 6. The inequality (3.7) contradicts to the inequality (3.6), since

$$\partial_{n(x^*)}\gamma(x^* - x^p; \omega) = \frac{1}{4\pi|\widetilde{a}^{1/2}| \ [\widetilde{a}^{-1}n(x^*) \cdot n(x^*)]^{3/2}} \frac{1}{|x^* - x^p|^2} + O(1) \,,$$

and the left-hand side in (3.6) is not bounded as x^p approaches x^*.

This completes the proof. □

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