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Newtonian capacity and quasi-balayage

W. K. HAYMAN – L. KARP – H. S. SHAPIRO

RIASSUNTO: *Per varie applicazioni (analisi di regolarità di frontiere libere, approssimazioni in media delle funzioni armoniche su domini illimitati) sono utili le stime, con funzioni regolari ϕ a supporto compatto in $\mathbb{R}^n$, che limitino $|\phi|$ sulle sfere unitarie B in termini dell'estremo superiore di $|\Delta\phi|$ in $\mathbb{R}^n$. Stime di questo tipo non esistono senza assunzioni aggiuntive su ϕ; in un precedente lavoro è stato però riconosciuto che se ϕ è nulla su un sottoinsieme $E \subset B$ di volume non nullo, sussiste una limitazione di questo tipo ed in modo uniforme rispetto ai sottoinsiemi E di volume maggiore di un valore assegnato. In questo lavoro si ottiene una stima analoga per le funzioni ϕ con $\phi(0) \leq 1$ quando* $\operatorname{grad}(\phi)$ *sia nullo su un sottinsieme $E \subset B$ la cui capacità newtoniana superi un valore assegnato. Si indicano varie applicazioni di questo risultato. La derivazione delle stime di base coinvolge delle idee che hanno un certo interesse in generale; in particolare le idee di "quasi-balayage" cioè l'allargamento di una misura dal suo supporto ad un altro compatto, in modo tale che i potenziali delle due misure, se pure non coincidono su di un intorno dell'infinito (come richiesto dal "balayage"), siano asintoticamente uguali con un ordine prescritto.*

ABSTRACT: *For various applications (including: regularity study of free boundaries, and approximation in the mean of harmonic functions on unbounded domains by rapidly decreasing ones) it is desirable to have an estimate, valid for an arbitrary smooth function ϕ of compact support in $\mathbb{R}^n$, bounding* $\sup|\phi|$ *in the unit ball B in terms of* $\sup|\Delta\phi|$ *on $\mathbb{R}^n$. Such an estimate cannot exist with no further assumptions on ϕ, but in the predecessor of this paper it was shown that if ϕ vanishes on a subset E of B with volume $|E| > 0$, such a bound holds , and uniformly with respect to all sets E with $|E|$ not less than any prescribed positive constant. In the present paper an analogous estimate is obtained for ϕ which $|\phi(0)| \leq 1$, and* $\operatorname{grad}(\phi)$ *vanishes on a subset E of B whose Newtonian capacity exceeds a positive constant. Various applications are given. The derivation of the basic estimate involves ideas which may have some independent interest, in particular "quasi-balayage", the sweeping-out of a measure from its support to some other compact set such that the potentials of the two measures, while not equal on a neighborhood of infinity as required by (true) balayage, are asymptotically equal to some prescribed degree.*

1 – Introduction

This paper complements and extends some of the results in [14]. Here is a brief description of the contents. Section 2 presents in a systematic way a scheme for proving, under suitable hypotheses, that functions of certain classes can be bounded on large sets in terms of a bound on a "small" set of suitable kind. Estimates of this kind are the basis of the method of quasi-balayage, but have independent interest. While Section 2 probably contains nothing that is new in principle, we could not find references where this material is developed in the form we require.

Section 3 treats the problem of approximating a harmonic function on an unbounded domain, in L^1 metric, by potentials of compactly supported signed measures which are $O(|x|^{-m})$ at infinity, for some (large) m. We show that such approximation is possible when the complement of the domain is "sufficiently large". The technique employed can equally well be used to study approximation of harmonic functions on bounded domains, by ones which have a zero of high order at a given boundary point. Later, in Section 5, we continue the discussion of approximating harmonic functions by rapidly decreasing ones, from a more general point of view, whereby the approximating functions are not required a priori to be potentials of compactly supported measures.

Sections 2 and 3 do not use the notion of quasi-balayage, and can be read independently of the earlier paper [14]. In Section 4, quasi-balayage is described, and used to obtain an estimate for a C^∞ function of compact support on $\mathbb{R}^n$ with a sufficiently rich set of critical points, in terms of a bound on its Laplacian. We prove also a closely related estimate for functions of polynomial growth at infinity, of a type that has been found useful in studying the regularity of free boundaries in obstacle problems, Hele-Shaw flows and related problems, as well as the approximation problem in Section 5.

Characteristic for our main results is a condition that the complement of some domain in $\mathbb{R}^n$ be sufficiently "large". In [14] this was achieved usually by imposing a condition (there called "rich complement") of positive density, defined in terms of *Lebesgue volume measure*. In the present

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paper we replace this kind of density by one computed in terms of Newtonian capacity, which leads to an essential strengthening of corresponding results of [14].

2 – Preliminary results

2.1 – The Λ-operation

Let F be a closed bounded set in Euclidean n-dimensional space $\mathbb{R}^n$ (in practice, it shall usually be a ball, sphere, or cube), and $C(F)$ the Banach space of real-valued continuous functions on F, endowed with the usual norm

$$\|f\| = \|f\|_\infty = \|f\|_{C(F)} := \max_{x\in F} |f(x)|\,, \qquad f \in C(F)\,. \tag{2.1}$$

Occasionally we will want to adapt the considerations about to be presented to analogous situations where F is in $\mathbb{C}^n$, and/or the functions comprising $C(F)$ are complex-valued, or vector-valued with values in $\mathbb{R}^m$ or $\mathbb{C}^m$; insofar as only trivial changes are needed to achieve these modifications we shall feel free to use such adaptations without further discussion.

Let $\{E_m\}_{m=1}^\infty$ be a sequence of non-empty subsets of F. From this sequence we construct a set $E^\#$, also denoted by $\Lambda\{E_m\}$ that plays an important role in this paper:

$$\Lambda\{E_m\} = E^\# := \bigcap_{M=1}^\infty \operatorname{clos}\Big[\bigcup_{m=M}^\infty E_m\Big]\,. \tag{2.2}$$

Here $\operatorname{clos}[\,\cdot\,]$ denotes closure. Observe that Λ is very similar to the usual "lim inf" operation in set theory, except for the insertion of the closure operation before taking intersections. Since F is compact, $E^\#$ is a nonempty closed subset of F. We take note of some elementary properties of the Λ-operator.

Proposition 2.3. *$E^\#$ is identical with the set E defined as follows: $x \in E$ if and only if there is a sequence $1 \le m_1 < m_2 < \dots$, and points $x_i \in E_{m_i}$ such that $x_{m_i} \to x$.*

Proof. Suppose first that $x \in E$. Then, since for each M, all but a finite number of the x_i belong to $\bigcup_{m=M}^{\infty} E_m =: T_M$, x belongs to clos$[T_M]$ and hence to $\bigcap_{M=1}^{\infty} \operatorname{clos}[T_M] = E^{\#}$.

Conversely, if $x \in E^{\#}$, then for each M there is a point x_M in T_M at distance less than $1/M$ from x, and x_M is in some E_m with $m \geq M$. Thus, since F is compact, we may recursively pick a sequence $\{x_M\}$ and $x_M \to x$ such that x_M belongs to one of the E_m. Moreover, for fixed M we may choose m as large as we please, hence without loss of generality can arrange that the rank m of the chosen set E_m containing x_M, is at each stage larger than those for the previously considered points x_j $(j < M)$. This shows that $x \in E$. ∎

Proposition 2.4. *If for two sequences $\{E_m\}_1^{\infty}$ and $\{E'_m\}_1^{\infty}$ we have $E_m \subset E'_m$ for every m, then $\Lambda\{E_m\} \subset \Lambda\{E'_m\}$.*

Proof. Obvious. ∎

Proposition 2.5. *$\Lambda\{E_m\}$ depends only on the sets E_m, not upon their ordering.*

Proof. Let $\{F_m\}$ denote a permutation of the sequence $\{E_m\}$. For each M, $\bigcup_{m=M}^{\infty} F_m \supset \bigcup_{j=J}^{\infty} E_j$ where J is chosen as the least integer such that all the sets $F_1, F_2, \ldots, F_{M-1}$, in their capacity as elements of the sequence $\{E_m\}$, have subscripts less than J.

Hence,

$$\operatorname{clos}\Big[\bigcup_{m=M}^{\infty} F_m\Big] \supset \operatorname{clos}\Big[\bigcup_{j=J}^{\infty} E_j\Big] \supset E^{\#}$$

and consequently $F^{\#}$, the intersection of the sets on the left-hand side, satisfies $F^{\#} \supset E^{\#}$. By symmetry the reverse inclusion also holds. ∎

2.2 – Some families stable with respect to the Λ-operation

It is important for us to identify certain families $\mathcal{E}$ of subsets of F which *are Λ-stable*, by which we mean: *if $E_j \in \mathcal{E}$ for $j = 1, 2, \ldots,$ then $\Lambda\{E_j\} \in \mathcal{E}$.*

Proposition 2.6. *Let μ denote a positive finite Borel measure on (the Borel subsets of) F, and suppose that $c > 0$. The family of all Borel* *subsets E of F such that*

$$\mu(E) \geq c \tag{2.7}$$

is Λ-stable.

Proof. Observe that, to verify that a family $\mathcal{E}$ of subsets of F is Λ-stable, one has only to check the following three conditions:

(i) E is stable under formation of countable unions.
(ii) $E \in \mathcal{E} \Longrightarrow \text{clos}[E] \in \mathcal{E}$.
(iii) The intersection of a countable, monotone decreasing family of closed nonempty subsets of $\mathcal{E}$ is again in $\mathcal{E}$.

It is clear that the family defined by (2.7) has these properties. □

Similar results hold for other set functions in place of μ; of special importance in the present paper is the *Newtonian capacity.* For each Borel set E of $\mathbb{R}^n$, we denote by $\text{cap}\,(E)$ the Newtonian capacity for $n \geq 3$ and the logarithmic capacity for $n = 2$. Thus $\text{cap}\,(E)$ is a non-negative real number, or $+\infty$ (see [10]). We have

Proposition 2.8. *Suppose that $c > 0$. The family of all Borel subsets E of some given compact set F satisfying*

$$\text{cap}(E) \geq c \tag{2.9}$$

is Λ-stable (or empty).

The proof follows the same pattern as the preceding one. We require only two well-known properties of the Newtonian capacity $\text{cap}\,(\cdot)$: monotonicity, that is, $\text{cap}(E_1) \leq \text{cap}(E_2)$ whenever $E_1 \subset E_2$ (this is evident); and upper semi continuity, which for our purpose means

$$\text{cap}\,\Big(\bigcap_{j=1}^{\infty} K_j\Big) = \lim_{j\to\infty} \text{cap}\,(K_j)$$

holds for any decreasing sequence $\{K_j\}$ of compact sets (see e.g. [5; Theorem 5.5]).

One can also construct Λ-stable families based on geometric properties:

Proposition 2.10. *For a fixed positive integer k, and $c > 0$ let $\mathcal{M}(k, c)$ denote the collection of all subsets E of F such that E contains k points no two of which have mutual distance less than c. (We assume there is at least one such set). Then, $\mathcal{M}(k, c)$ is Λ-stable.*

Proof. We need only check that, if $\{E_j\}$ are closed sets in $\mathcal{M}(k, c)$ such that $E_1 \supset E_2 \supset \dots$, then $E := \cap E_j$ is in $\mathcal{M}(k, c)$. Let $\{x_{j,1}, \dots, x_{j,k}\}$ denote points of E_j with mutual distance at least c. Since there is clearly no loss of generality if we replace $\{E_j\}$ by any infinite subsequence, we may, and do, assume that $x_i := \lim_{j\to\infty} x_{j,i}$ exists for $i = 1, 2, \dots, k$. Then $|x_i - x_m| \geq c$ for $i \neq m$; moreover for fixed i, and fixed r, all $x_{j,i}$ belongs to E_r as soon as $j \geq r$, hence $x_i \in E_r$. Thus, $x_i \in \bigcap E_r = E$ $(i = 1, 2, \dots, m)$. ∎

Proposition 2.11. *Let G denote any nonempty closed subset of F. The collection of all subsets of F which contain a "copy" of G (i.e. σG, for some rigid motion σ of $\mathbb{R}^m$) is Λ-stable.*

Proof. Again, we have only to check that if $\{E_j\}$ are closed subsets of F such that $E_1 \supset E_2 \supset \dots$, and each of which contains a copy of G (say, $\sigma_j G \subset E_j$ where σ_j belongs to the group R of rigid motions on $\mathbb{R}^n$), then $\sigma G \subset E := \bigcap E_j$ for some σ in R. Since the group R (with the usual topology) is locally compact and obviously the "translation" component of σ_j remains bounded, we may assume w.l.o.g. that σ_j converges to σ in R. When $\varepsilon > 0$, let E^ε denote the set of points in $\mathbb{R}^n$ at distance less than ε from E. For all sufficiently large j, $E_j \subset E^\varepsilon$ and hence $\sigma_j G \subset E^\varepsilon$ for all large j, so $\sigma G \subset \operatorname{clos}[E^\varepsilon]$. Thus,

$$\sigma G \subset \bigcap_{\varepsilon > 0} \operatorname{clos}[E^\varepsilon] = E\,. \qquad \blacksquare$$

One can prove any number of similar results. Here is one example in the direction of generalizing the last proposition. We leave its proof to the reader:

PROPOSITION 2.12. *For some $\alpha > 0$ and $a > 0$, let $T(a, \alpha)$ denotes the set of all triangles in $\mathbb{R}^n$ (here, by "triangle", we mean the closed convex hull of a set consisting of 3 points) each side of which has length at least a, and each angle of which is at least α. The family of all sets which contain an element of $T(a, \alpha)$ is Λ-stable.*

REMARK. If, in place of $\{T(a, \alpha)\}$ we consider the family $\{V(a, \alpha)\}$, where each element is the set of vertices of one of the allowed triangles, the corresponding proposition remains true.

2.3 – "Sets of uniqueness" sometimes entail estimates

Let P denote a closed subspace of $C(F)$. A closed set K of F is a *set of uniqueness* for P, if $p(x) = 0$ for all x in K implies $p \equiv 0$, for all p in P.

Our main concern here is when the stronger property

$$\|p\| := \max_{x \in F} |p(x)| \leq C \max_{x \in K} |p(x)| \tag{2.13}$$

holds for all p in P, where C is a constant only depending on K (and, in interesting cases, as we shall see, even uniform for all sets K in some specified class). Of course, without further hypotheses, K being a set of uniqueness for P by no means implies an estimate of type (2.13) (for example, if F is the closed unit ball of $\mathbb{R}^n$, $n \geq 2$ and P denotes the set of p in $C(F)$ which are harmonic in the open ball, then $K := \{x : |x| \leq 1/2\}$ is a set of uniqueness but no estimate (2.13) holds).

The main result of the present section is:

PROPOSITION 2.14. *Suppose that P is finite-dimensional, and that $\mathcal{E}$ is a Λ-stable family of subsets of F, each of which is a set of uniqueness for P. Then, there is a constant $C_{\mathcal{E}}$ (depending only on $\mathcal{E}$) such that for each K in $\mathcal{E}$*

$$\|p\| := \max_{x \in F} |p(x)| \leq C_{\mathcal{E}} \max_{x \in K} |p(x)| \,. \tag{2.15}$$

PROOF. Suppose the contrary; then, there exists a sequence $\{K_j\}$ in $\mathcal{E}$ and corresponding sequence $\{p_j\}$ in P such that

$$\|p_j\| > j \max_{x \in K_j} |p_j(x)| .$$

Clearly we may normalize so that $\|p_j\| = 1$. Since the unit sphere of P is compact, we can find p in P with $\|p\| = 1$ and a subsequence of $\{p_j\}$ converging (in $C(F)$) to p. We may, w.l.o.g. assume this is true for the whole sequence $\{p_j\}$. Thus, $\|p_j - p\| \to 0$ and

$$\max_{x \in K_j} |p_j(x)| < 1/j .$$

Hence, given that $\varepsilon > 0$, we have

$$\max_{x \in K_j} |p(x)| < (1/j) + \|p - p_j\| < \varepsilon$$

as soon as j is sufficiently large. Hence for some J

$$|p(x)| < \varepsilon \qquad \text{for} \qquad x \in \bigcup_{j \geq J} K_j =: T_J$$

and so

$$|p(x)| \leq \varepsilon \qquad \text{on} \qquad \operatorname{clos}[T_J] ,$$

so *a fortiori*

$$|p(x)| \leq \varepsilon \qquad \text{on} \qquad \bigcap_{J \geq 1} \operatorname{clos}[T_J] = \Lambda\{K_j\} .$$

Thus, on the set $K^{\#} := \Lambda\{K_j\}$, we have $|p(x)| \leq \varepsilon$, and ε being arbitrary, so $p = 0$ on $K^{\#}$. Since by hypothesis $K^{\#}$ is a set of uniqueness for P, $p \equiv 0$, which contradicts the assumption $\|p\| = 1$. $\square$

By combining (2.14) with earlier propositions, we now can deduce any number of concrete estimates. We start with some which are well known, just for illustrative purposes. In what follows, $B(y, r)$ denotes $\{x \in \mathbb{R}^n : |x - y| < r\}$ and $B := B(0, 1)$, $\overline{B} = \mathrm{clos}[B]$. By Σ_n we denote the unit sphere of $\mathbb{R}^n$, and by Q the cube $\{0 \le x_1 \le 1, \ldots, 0 \le x_n \le 1\}$.

$P_r^{[n]} = P_r$ will denote the set of polynomials in $x_1, \ldots, x_n$ of degree at most r, and $HP_r^{[n]} = HP_r$ the subset of P_r consisting of harmonic polynomials.

Proposition 2.16. ([3], [7]) *If $c > 0$ and n, r are given, then there is a constant $C = C(n, r, c)$ such that*

$$\max_{x \in Q} |p(x)| \le C \max_{x \in K} |p(x)| \tag{2.17}$$

holds for every p in $P_r^{[n]}$ and every compact K, where $K \subset Q$ and K has Lebesgue n-dimensional measure at least c.

Proof. Since sets of positive measure are sets of uniqueness for polynomials (indeed, for real analytic functions), (2.16) follows from Propositions 2.14 and 2.6. ∎

Remark. It is an interesting problem, and by no means completely understood, to find the sharp quantitative dependence of C (n, r, c) on its parameters (moreover, although we have placed the action in Q we could as well have done so in $\overline{B}$ or other compact set, and then the dependence of C also on this set is a matter of interest). The same applies to all the estimates we shall obtain in this paper; thus, our estimates are really of a "zeroth order", or rather superficial nature which is to be expected since they are based only on such general considerations as uniqueness, and compactness. Still, they suffice for some nice applications. It is our hope to pursue more precise estimates in future works.

Proposition 2.18. *If $c > 0$ and n, r are given, then there is a constant $C = C(n, r, c)$ (we use C, $C(n, r, c)$ etc. as generic constants defined "locally", there is no implied relation e.g. with the $C(n, r, c)$ of the preceding proposition) such that*

$$\max_{x \in \Sigma_n} |p(x)| \le C \max_{x \in K} |p(x)| \tag{2.19}$$

holds for every p in $HP_r^{[n]}$ and every compact subset K of Σ_n whose "hypersurface measure" ($= (n-1)$-dimensional Hausdorff measure) is at least c.

PROOF. The proof is analogous to that of (2.17), the only difference being that we now require the knowledge that a subset K of Σ_n of positive "hypersurface measure" is a set of uniqueness for harmonic polynomials. This is so because, roughly speaking, if we introduce local real-analytic coordinates on Σ_n, the harmonic polynomial p is real-analytic in those $n-1$ coordinates, and hence vanishes on a relatively open portion of Σ_n. From this, and the irreducibility over $\mathbb{C}$ of the polynomial $x_1^2 + \cdots + x_n^2 - 1$ one deduces that p vanishes on all Σ_n, and hence (being harmonic) identically. $\square$

Before moving on to the main results of the paper, we illustrate the use of (2.14) on a few other examples. The first of these is rather trivial and included for pedagogic purposes.

PROPOSITION 2.20. *Let $p(z) = c_0 + c_1 z + \cdots + c_r z^r$ denote a polynomial in the complex variable z with complex coefficients and $z_0, z_1, z_2, \ldots, z_r$ points of the closed unit disk $\overline{\mathbb{D}}$ such that $|z_i - z_j| \geq c > 0$ for $i \neq j$. Then, there is a constant $C = C(r, c)$ such that*

$$\max_{|z|\leq 1} |p(z)| \leq C \max\{|p(z_i)|; i = 0, 1, 2, \ldots, r\} \tag{2.21}$$

PROOF. Combine (2.14) and (2.10). $\square$

REMARK. In this case, a simple direct argument gives a better result: writing $p(z) = \sum_{j=0}^{r} p(z_j) Q_j(z)$, where Q_j are the "Lagrange fundamental polynomials" for the sequence $z_0, \ldots, z_r$, one gets (2.21) with $C = (2/c)^r$. Yet, the abstract technique is useful, we could e.g. obtain analogous estimates to (2.21) for multivariable polynomials where one lacks a complete theory of interpolating polynomials.

In our next example, which is a special case of an important general result to be proved later, we broaden the context to vector-valued functions. Here our basic space is $C(\overline{B}, \mathbb{R}^3)$, the continuous functions from the closed unit ball $\overline{B}$ of $\mathbb{R}^3$, into $\mathbb{R}^3$ or, what is equivalent the space of triples $f = (f_1, f_2, f_3)$ where $f_i \in C(\overline{B}, \mathbb{R})$ and $|f(x)| := \big(f_1(x)^2 + f_2(x)^2 + f_3(x)^2\big)^{1/2}$. As our finite-dimensional subspace P, we take the set of *gradients of harmonic polynomials h in (x_1, x_2, x_3) of degree at most r*, normalized so that $h(0) = 0$.

PROPOSITION 2.22. *Let T denote a nondegenerate planar triangle. There is a constant C depending only on r and T such that for every harmonic polynomial h in (x_1, x_2, x_3) of degree at most r, with $h(0) = 0$*

$$\max_{x\in\overline{B}} |h(x)| \le C \cdot \max_{x\in K} \Big[\Big(\partial_1 h(x)\Big)^2 + \Big(\partial_2 h(x)\Big)^2 + \Big(\partial_3 h(x)\Big)^2\Big]^{1/2}$$

holds, whenever $K \subset \overline{B}$, and K contains a triangle congruent to T.

PROOF. By combining (2.14) and (2.11), all we have to check is that if $\operatorname{grad} h$ vanishes on a triangle (congruently embedded in $\mathbb{R}^3$), h is constant (and hence zero, since $h(0) = 0$). And this is clear: for, h is then constant on the triangle and, being harmonic, is constant on a neighborhood of it in $\mathbb{R}^3$ (e.g. by the Cauchy-Kovalewskii theorem, see e.g. [6]) and hence everywhere. ∎

The next result shows that (2.22) extends to sets K *with Newtonian capacity bounded from below*; putting a triangle into K was just one way of achieving this, that permitted an elementary proof. In order to do that we need to show that compact sets with positive capacity are uniqueness sets for harmonic functions; that is, if u is harmonic in a neighborhood of K and $\operatorname{grad} u$ vanishes on K, then u is a constant.

The set $\{x \in K : \ |\operatorname{grad} u(x)| = 0\}$ ($|\operatorname{grad} u(x)| = \big((\partial_1 u(x))^2 + \cdots + (\partial_n u(x))^2\big)^{1/2}$) is called the critical set, where in general, u is a solution to an elliptic partial differential operator. There are many papers dealing with the "size" of the critical sets. We did not trace who first proved lemma 2.23 below. We refer to a very elegant proof of HARDT and SIMON [4; lemma 1.9].

LEMMA 2.23. [4] *Suppose that u is a non-constant harmonic function in a neighborhood of a compact set K. Then the critical set $\{x \in K : \ |\operatorname{grad} u(x)| = 0\}$ decomposes into a countable union of subsets each with a finite $(n-2)$-Hausdorff measure.*

We recall that cap$(\cdot)$ denotes the logarithmic capacity in $\mathbb{R}^2$ and the Newtonian capacity in $\mathbb{R}^n, n \geq 3$.

Lemma 2.24. *Let u be harmonic in a domain Ω and let K be a compact subset of Ω with* $\operatorname{cap}(K) > 0$. *If* $|\operatorname{grad} u(x)| = 0$ *for all x in K, then u is a constant in Ω.*

Proof. If u is not a constant, then by lemma 2.23 $K = \cup_{j=1}^{\infty} K_j$, where K_j has a finite $(n-2)$-Hausdorff measure for each j. So by Frostman's comparison theorem (see e.g. [5; theorem 5.14]), $\operatorname{cap}(K_j) = 0$ for all j. Since the capacity is a subadditive set-function, $\operatorname{cap}(K) = 0$ which contradicts the assumptions of the lemma. ∎

Remark. If $n = 2$, then $f(z) := \partial_1 u(z) - i\partial_2 u(z), z = x_1 + ix_2$ is an analytic function in Ω. Hence, in that case, (2.24) deals with the zero sets of analytic functions.

Corollary 2.25. *If $c > 0$ and n, r are given, then there is a constant $C = C(n, r, c)$ such that ($\overline{B}$ denoting the closed unit ball in $\mathbb{R}^n$)*

$$\max_{x \in \overline{B}} |p(x)| \leq C \max_{x \in K} \Big| \operatorname{grad} p(x) \Big| \tag{2.26}$$

holds for every harmonic polynomial p of degree at most r, satisfying

$$p(0) = 0 \tag{2.27}$$

and every compact subset K of $\overline{B}$ with $\operatorname{cap}(K) \geq c$.

Proof. Combine (2.15) and lemma 2.24 (same pattern as in earlier proofs). ∎

3 – Approximating harmonic functions by rapidly decreasing ones

In this and the following sections, we present applications of the estimates in Section 2, especially (2.26). They are the same kind of applications as in [14] but, owing to the use of "capacitary" estimate (2.26) in place of the measure-theoretic (2.17) and (2.19) we get sharper results all along the line. We begin with an approximation problem.

Let Ω denote an open, *unbounded* set $\Omega \subset \mathbb{R}^n$. We denote by $L^p(\Omega)$ the usual Lebesgue space of measurable functions f on Ω such that $|f|^p$ is integrable, and by $HL^p(\Omega)$ its subspace of harmonic functions. We address the question: *Can every f in $HL^1(\Omega)$ be approximated arbitrarily well* (in $L^1(\Omega)$) *by elements of $HL^1(\Omega)$ which, moreover are $O(|x|^{-k})$ at ∞, for some given (large) k?*

In general, the answer is no. Here is a simple and instructive example. (It can be presented in any number of dimensions, we illustrate it in the three dimensional space). Take for Ω the set $\{|x| > 1\}$ in $\mathbb{R}^3$, and $h(x) = H_3(x)/|x|^7$ where H_3 denotes an arbitrary nontrivial homogenous harmonic polynomial of degree 3 (then, as is well known, h is harmonic in $\mathbb{R}^3 \setminus \{0\}$). Thus, $h \in HL^1(\Omega)$. We assert that h *cannot be approximated arbitrarily closely in $L^1(\Omega)$ by harmonic functions that are $O(|x|^{-5})$ at ∞*. For, every such harmonic function u has an expansion

$$u(x) = \sum_{j=4}^{\infty} a_j \frac{H_j(x)}{|x|^{2j+1}} \tag{3.1}$$

converging on $\{x > 1\}$, where H_j denotes a homogenous harmonic polynomial of degree j. Indeed, by Kelvin transformation $u(x) = |x|^{-1}\, U(x/|x|^2)$ where $U(y)$ is harmonic on $\{|y| < 1\}$, and it is easy to see that (3.1) is nothing else than the Taylor expansion of $U(y)$ about $y = 0$, transformed by the Kelvin transformation. If a sequence of functions of the form (3.1) converged to h in $L^1(\Omega)$, then their restrictions to the sphere $\{|x| = 2\}$ would converge to the restriction of h to that sphere, in the norm of $L^2(\{|x| = 2\}, d\sigma)$ where $d\sigma$ denotes surface measure on that sphere. But, w.r.t. this norm all summands in (3.1) are orthogonal to h, so this is impossible. We recall that H_j and H_k are orthogonal for $j \neq k$, on every $\{|x| = r\}$ w.r.t. the $(n-1)$ dimensional Lebesgue measure.

There is another, "dual" way to present this counterexample which is, in some ways, more useful to us. We'll only work it through in $\mathbb{R}^2$. Again, Ω is $\{|x| > 1\}$. We'll show that

$$h(x) = r^{-3} \cos 3\theta, \qquad x_1 + ix_2 = re^{i\theta} \tag{3.2}$$

cannot be approximated arbitrarily closely in $L^1(\Omega)$ by linear combinations of the functions

$$\{r^{-m}\cos m\theta, r^{-m}\sin m\theta\}, \qquad m \geq 4 \tag{3.3}$$

(or, what comes to the same, by harmonic functions on Ω that are $O(r^{-4})$, or even $o(r^{-3})$, at ∞).

The dual space of $L^1(\Omega)$ is $L^\infty(\Omega)$, the bounded measurable functions on Ω. The function

$$f(x) = f(r,\theta) := \cos 3\theta, \qquad r > 1 \tag{3.4}$$

is in $L^\infty(\Omega)$ and annihilates the functions (3.3) but not h, so these functions do not span h.

Define $\tilde{f} \in L^\infty(\mathbb{R}^2)$ by

$$\tilde{f}(x) = \begin{cases} f(x), & x \in \Omega \\ 0, & |x| \leq 1 \end{cases} \tag{3.5}$$

and let us solve the equation (understood distributionally)

$$\Delta\tilde{v} = \tilde{f}. \tag{3.6}$$

We can obtain a solution $\tilde{v}$ *which vanishes for* $|x| \leq 1$ if we can solve

$$\Delta v = f \qquad \text{on } \Omega \tag{3.7}$$

with the boundary conditions

$$v = 0, \qquad \operatorname{grad} v = 0 \qquad \text{on } \{|x| = 1\} \tag{3.8}$$

and then set

$$\tilde{v}(x) = \begin{cases} v(x), & x \in \Omega \\ 0, & |x| \leq 1. \end{cases} \tag{3.9}$$

Indeed, the "Cauchy data zero" condition (3.8) implies that it is correct to compute $\Delta\tilde{v}$ "sectionally" so that $\Delta\tilde{v} = \tilde{f}$ (mere "Dirichlet data zero", i.e. $v = 0$ on $\{|x| = 1\}$ would not suffice for this, $\Delta\tilde{v}$ would pick up an unwanted measure living on the unit circle, in addition to $\tilde{f}$).

The problem (3.7) and (3.8), although in principle "overdetermined", has a unique solution. To find it, make the Ansatz

$$v = v(r, \theta) = F(r) \cos 3\theta$$

where F is a smooth function on $[1, \infty)$ such that

$$F(1) = F'(1) = 0 \tag{3.10}$$

so that (3.8) will be satisfied. Then

$$\begin{aligned}\Delta v &= \frac{\partial^2 v}{\partial r^2} + (1/r)\,\frac{\partial v}{\partial r} + (1/r^2)\,\frac{\partial^2 v}{\partial \theta^2} = \\ &= \left[F''(r) + (1/r)F'(r) - (9/r^2)F(r)\right] \cos 3\theta\,,\end{aligned}$$

so (3.7) will hold with $f = \cos 3\theta$ if

$$F''(r) + (1/r)F'(r) - (9/r^2)F(r) = 1\,. \tag{3.11}$$

Since the solutions to the corresponding homogeneous equation are r^3 and r^{-3}, it is easy to check that

$$F(r) = \frac{1}{6}r^3 - \frac{1}{5}r^2 + \frac{1}{30}r^{-3} \tag{3.12}$$

solves (3.11) with the initial conditions (3.10).

Forgetting for a moment the motivation that led us to this result, the upshot is that we have constructed v on Ω satisfying

$$\Delta v = f \qquad \text{on } \Omega \tag{3.13}$$

$$v(x) = O(|x|^3), \quad |\operatorname{grad} v(x)| = O(|x|^2) \text{ as } x \to \infty \tag{3.14}$$

and

$$v(x) = 0,\ \operatorname{grad} v(x) = 0, \text{ on } \{|x| = 1\}\,. \tag{3.15}$$

Now, from (3.14) and (3.15) alone it follows that

$$\int_{\Omega} (\Delta v) \cdot u \, dx = 0$$

for every harmonic function u on a neighborhood of the closure of Ω such that

$$u(x) = o(|x|^{-3}) \qquad \text{as } x \to \infty \tag{3.16}$$

and

$$|\operatorname{grad} u(x)| = o(|x|^{-4}) \qquad \text{as } x \to \infty\,, \tag{3.17}$$

and in particular, for all functions (3.3). Note that the estimates (3.14), (3.16), (3.17) and the application of Green's identity on $\Omega_R := \{1 < |x| < R\}$ enable us, by letting $R \to \infty$, to deduce that $\int_\Omega (\Delta v) \cdot u \, dx = \int_\Omega v(\Delta u) dx = 0$. Thus, *an "obstacle" to spanning $L^1(\Omega)$ by rapidly decreasing harmonic functions is the existence of a solution to a certain overdetermined boundary value problem* (3.7), (3.8) *with growth restrictions at* ∞ (namely (3.14)). This is a general fact, and keeping it in mind should simplify the understanding of the more general situation to be considered shortly.

What "goes wrong" in the preceding counter-example is that the complement of Ω is "too small"—indeed it was bounded. But, using the "Cauchy problem" reformulation we can easily give a counter-example in which ∞ is a boundary point. Namely, look at the domain $G := \Omega \times \mathbb{R}^1$ where Ω is, as above, $\{x \in \mathbb{R}^2 : |x| > 1\}$. Thus, G is the exterior of a circular cylinder. We introduce coordinates so that

$$G = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 > 1\}\,.$$

We shall show that, *in $L^1(G)$, the set of harmonic functions u on G satisfying*

$$u(x) = o(|x|^{-4}), \quad |\operatorname{grad} u(x)| = o(|x|^{-5}) \qquad \textit{as } x \to \infty, \ \textit{in } G\,, \tag{3.18}$$

do not span $HL^1(G)$. In fact, they fail to span the function

$$h(x_1, x_2, x_3) := (x_1 - ix_2)^3 / |x|^7\,. \tag{3.19}$$

(It is convenient here to work with complex-valued functions; but one can also present the example adapted to real-valued functions, by just taking real parts throughout.)

To see this,observe first that

$$\varphi(x_1, x_2, x_3) := (x_1 + ix_2)^3/(x_1^2 + x_2^2)^{3/2} \tag{3.20}$$

is bounded (indeed, of constant modulus 1) on G and $\int_G h\varphi dx \neq 0$, since $h\varphi > 0$ on G. Hence, our assertion will follow if we verify

$$\int_G h\, u\, dx = 0 \tag{3.21}$$

for all $u \in HL^1(G)$ satisfying (3.18). Now, by a slight modification of the calculation just presented,

$$v(x_1, x_2) := \Big(\frac{1}{6}r^3 - \frac{1}{5}r^2 + \frac{1}{30}r^{-3}\Big)e^{3i\theta}, \text{ where } x_1 + ix_2 = re^{i\theta} \tag{3.22}$$

satisfies $\Delta v = e^{3i\theta}$ as well as (3.14) and (3.15). Hence

$$V(x_1, x_2, x_3) := v(x_1, x_2)$$

satisfies in G,

$$\Delta V = \varphi \tag{3.23}$$

and

$$V(x) = O(|x|^3), \quad |\operatorname{grad} V(x)| = O(|x|^2) \qquad \text{as } x \to \infty, \text{ in } G \tag{3.24}$$

as well as

$$V = 0, \quad \operatorname{grad} V = 0 \qquad \text{on } \partial G\,. \tag{3.25}$$

These estimates, together with a "Green's identity" argument as before imply

$$0 = \int_G V \Delta\, u\, dx = \int_G u\, \Delta V\, dx\,,$$

that is, in view of (3.23) we have proved (3.21) and therewith our assertion.

We turn next to sufficient conditions for the desired approximation, but need first some preliminary notions and notations.

For a (Schwartz) distribution μ of compact support in $\mathbb{R}^n$, its *Newtonian potential* U^μ can be defined either as

$$U^\mu = E_n * \mu \tag{3.26}$$

where E_n is the fundamental solution to the Laplace operator:

$$E_n = \begin{cases} c_2 \log |x|\,, & n = 2 \\ c_n |x|^{2-n}\,, & n \geq 3\,, \end{cases} \tag{3.27}$$

c_n being suitable constants so that $\Delta E_n = -\delta$ in the distributional sense, *or* as the unique solution to

$$\Delta U = \mu \tag{3.28}$$

satisfying

$$U(x) \sim \text{const} \log |x|, \qquad \text{as } x \to \infty \qquad (n = 2) \tag{3.29}$$

or

$$U(x) \to 0, \qquad \text{as } x \to \infty \qquad (n \geq 3)\,. \tag{3.30}$$

In [14; Proposition 4.2] it was proved that if μ is a measure with compact support, then for each nonnegative integer m, the conditions

$$U^\mu(x) = O(|x|^{1-m-n})\,, \qquad x \to \infty \tag{3.31}$$

and

$$\mu \textit{ annihilates } HP_m, \textit{ in the sense } \int f\, d\mu = 0, \textit{ for all } f \in HP_m \tag{3.32}$$

are equivalent. Thus, with *no* extra assumptions U^μ is $O(|x|^{2-n})$, whereas if $\int d\mu = 0$ then U^μ is $O(|x|^{1-n})$, *and if* μ *annihilates* HP_2, *then* U^μ *is* $O(|x|^{-1-n})$, *and hence is integrable over* $\mathbb{R}^n$.

From a practical, as well as theoretical point of view, it is very convenient to work with *potentials* as harmonic approximants. The simplest potentials, those of point masses (in other words $x \mapsto E_n(x-y)$) are too slowly decreasing at ∞ and we must take linear combinations to generate harmonic functions of rapid decrease at ∞. In this paper we work with the following special classes of potentials.

Definition 3.33. For an open set Ω, $R_m(\Omega)$ denotes the class of potentials of measures compactly supported in $\mathbb{R}^n \setminus \Omega$ and which annihilate HP_m. By $R_m^\#(\Omega)$ we denote the subset of $R_m(\Omega)$ containing the potentials of measures with compact support in $\mathbb{R}^n \setminus \overline{\Omega}$. (Observe that the symbol # is used here in a completely different sense than in (2.2).)

Observe that $R_m(\Omega) \subset HL^1(\Omega)$ if $m \geq 2$, by (3.31). Of course, for $R_m(\Omega)$ to be nonempty, $\mathbb{R}^n \setminus \Omega$ must contain at least $1 + \dim(HP_m)$ points. To avoid trivialities, we shall always assume

Condition 3.34. Every boundary point of Ω is the limit of a sequence of interior points of $(\mathbb{R}^n \setminus \overline{\Omega})$.

This condition implies that if f belongs to $L^\infty(\Omega)$ and it annihilates $R_m^\#(\Omega)$, then it annihilates $R_m(\Omega)$ as well. Hence, under condition 3.34, $R_m^\#(\Omega)$ is dense in $R_m(\Omega)$ (in the $L^1(\Omega)$ metric), for all $m \geq 2$. The functions in $R_m^\#(\Omega)$ are nicer, of course, being harmonic on a neighborhood of $\overline{\Omega}$.

One can also consider instead of $R_m(\Omega)$ the broader class of potentials of *distributions* $\mathcal{D}_m$ compactly supported in $\mathbb{R}^n \setminus \Omega$ which annihilate HP_m. Thus, Karp and others used Newton kernels $E_n(x-y)$ from which a number of terms of the Taylor expansion (w.r.t. y) have been subtracted off, to achieve rapid decay w.r.t. x for large $|x|$. This amounts to choosing elements of $\mathcal{D}_m$ based on distributions supported at a single point y. Since $R_m^\#(\Omega)$ is $L^1(\Omega)$-dense in $\mathcal{D}_m(\Omega)$ for $m \geq 2$ and Ω satisfying (3.34), it is largely a matter of indifference which of these classes we work with. It is a result due to Karp [8; Corollary 2.4] that $R_2^\#(\Omega)$ is dense in $HL^1(\Omega)$ when (3.34) holds. One can also prove the corresponding result for $R_2^\#(\Omega)$ directly, by methods used in the present paper, but we shall not do so here.

The analogous theorem for approximation by holomorphic functions (in an open set of the complex plane) was proved in [1]: Rational functions of the form

$$f(z) = \int \frac{d\mu(w)}{z-w} \qquad (z \in \Omega) \tag{3.35}$$

where μ is a complex measure compactly supported on $\mathbb{R}^2 \setminus \Omega$ and satisfying

$$\int d\mu(w) = \int w \, d\mu(w) = 0 \,, \tag{3.36}$$

(which implies $f(z) = O(|z|^{-3})$ and hence is integrable) are dense in $AL^1(\Omega)$, the space of holomorphic functions in $L^1(\Omega)$. Moreover, it is enough to allow only discrete measures μ, supported at 3 points. Similar refinements can be obtained for harmonic approximation. But we leave this aside now, and formulate one of our main results.

THEOREM 3.37. *Suppose that* Ω *satisfies the following conditions:*

$$\limsup_{R\to\infty} \frac{\text{cap } (\overline{B(0,R)} \setminus \Omega)}{\text{cap } \overline{B(0,R)}} > 0 \tag{3.38}$$

and (3.34). *Then, whenever* $m \geq 3$, $R_m^\#(\Omega)$ *is dense in* $R_2^\#(\Omega)$ *in the metric of* $L^1(\Omega)$, *and hence in* $HL^1(\Omega)$.

REMARKS. A condition like (3.38), but with lim inf on the left, has appeared in works of several authors on potential theory, as a measure of the "massiveness" of a set (in this case, the complement of Ω), and various terms have been used for it ("uniformly perfect"—Pommerenke, "uniformly fat"—J. Lewis: see [11], [12]). It is a stronger requirement than ∞ being a regular point for Dirichlet's problem. We shall have need of this "lim inf" condition later in this paper.

In the condition (3.38) it is easy to see that the origin plays no special role; centering the balls at some other point would give an equivalent condition.

For the proof we shall require a lemma, of independent interest.

LEMMA 3.39. *Let* p *denote a harmonic polynomial in* $x = (x_1, \ldots, x_n)$. *Suppose that there is a closed set* $K \subset \mathbb{R}^n$ *such that*

$$\limsup_{R\to\infty} \frac{\text{cap}\,(\overline{B(0,R)} \cap K)}{\text{cap}\,(\overline{B(0,R)})} > 0 \tag{3.40}$$

and

$$(3.41) \qquad |\operatorname{grad} p(x)| \leq C(1+|x|)^c \qquad \textit{for } x \in K\,,$$

where C, c are positive constants. Then the degree of p is at most $c+1$.

Proof. The hypotheses imply there is a sequence $\{R_j\}$ and $R_j \to \infty$, and there is a positive constant a such that

$$(3.42) \qquad \frac{\operatorname{cap}\,(\overline{B(0,R_j)} \cap K)}{\operatorname{cap}\,(\overline{B(0,R_j)})} \geq a \qquad j=1,2,\dots$$

Since our hypotheses remain unchanged if we add a constant to p, we may assume w.l.o.g. that $p(0)=0$. Now, consider the sequence $\{f_j\}$ where $f_j(y) := p(R_j y)$. This is a sequence of harmonic polynomials of fixed degree and $f_j(0)=0$. We write $K_j = \{y \in \overline{B} : R_j y \in K\}$. Then the left member in (3.42) equals

$$(3.43) \qquad \frac{\operatorname{cap}\,(K_j)}{\operatorname{cap}\,(\overline{B})}$$

since the ratio of capacities is unaffected by scale change (see e.g [10; theorem 2.9, p. 158]), so the quantity in (3.43) is at least a.

Since $\operatorname{grad} f_j(y) = R_j \operatorname{grad} p(R_j y)$, we have by (3.41) that

$$(3.44) \qquad |\operatorname{grad} f_j(y)| \leq C\, R_j (1+R_j)^c \qquad \text{for } y \in K_j\,.$$

We now invoke (2.25), which is applicable since $\operatorname{cap}(K_j) \geq a$ for all j, and obtain

$$|f_j(y)| \leq C_1\, R_j (1+R_j)^c \qquad \text{for } |y| \leq 1$$

where C_1 is some new constant (independent of j). Thus,

$$|p(R_j y)| \leq C_1\, R_j (1+R_j)^c \qquad \text{for } |y| \leq 1$$

so

$$|p(x)| \leq C_1\, R_j (1+R_j)^c \qquad \text{for } |x| \leq R_j\,.$$

Hence, $M(R) := \max |p(x)| : |x| \leq R$ satisfies

$$\liminf_{R\to\infty} \frac{M(R)}{R^{c+1}} < \infty \tag{3.45}$$

and it is well known that this implies that the harmonic polynomial p has degree at most $c+1$. ∎

Remark. Just for completeness, here is a very simple proof that (3.45) implies $\deg p \leq c+1$ (indeed, for any polynomial p, not necessarily harmonic). Let $p = p_0 + p_1 + \cdots + p_r$ be the decomposition of p into homogeneous polynomials ($r = \deg p$), so that each p_k is either 0 or homogeneous of degree k. Write $x = t\xi$ where $|\xi| = 1$ and $t = |x|$, then

$$p(x) = \sum_{k=0}^{r} p_k(\xi) t^k\,. \tag{3.46}$$

For fixed ξ, $p(t\xi)$ is a polynomial in t, and since

$$\liminf_{t\to\infty} \frac{p(t\xi)}{t^{c+1}} < \infty$$

(from (3.45)), this polynomial has degree at most $c+1$. Consequently, from (3.46), each p_k with $k > c+1$ must vanish on the unit sphere, and hence (being homogeneous) identically.

Proof of Theorem 3.37. By well-known functional analysis (F. Riesz representation theorem, Hahn-Banach theorem), we have to show that if $f \in L^\infty(\Omega)$ and

$$\int_\Omega f\, h\, dx = 0 \qquad \text{when } h \in R_m^\#(\Omega)\,, \tag{3.47}$$

then

$$\int_\Omega f\, h\, dx = 0 \qquad \text{when } h \in R_2^\#(\Omega)\,. \tag{3.48}$$

Define $\tilde{f} \in L^\infty(\mathbb{R}^n)$ by

$$\tilde{f}(x) = \begin{cases} f(x)\,, & x \in \Omega \\ 0\,, & x \in \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{3.49}$$

Then by Theorem 4.1 in [8], there exists a solution v to

$$\Delta v = \tilde{f} \qquad \text{on } \mathbb{R}^n , \tag{3.50}$$

in the sense of distributions (and, continuously differentiable on $\mathbb{R}^n$) satisfying

$$|v(x)| \le C(1+|x|)^2 \log(2+|x|) , \tag{3.51}$$

$$|\operatorname{grad} v(x)| \le C(1+|x|) \log(2+|x|) . \tag{3.52}$$

Now, (3.47) can be written

$$\int_{\mathbb{R}^n} (\Delta v) U^\mu dx = 0 , \tag{3.53}$$

for all measures μ which annihilate HP_m and which are supported in $\mathbb{R}^n \setminus \overline{\Omega}$. It remains to deduce the corresponding result for μ that annihilates HP_2.

Now, for μ which annihilates HP_2 we have as $x \to \infty$ (see [14; Section 4])

$$U^\mu(x) = O(|x|^{-n-1}) \tag{3.54}$$

$$|\operatorname{grad} U^\mu(x)| = O(|x|^{-n-2}) . \tag{3.55}$$

In view of Green's formula (still valid in this "distributional" setting)

$$\int_{B(0,R)} (\Delta v) U^\mu dx - \int_{B(0,R)} (\Delta U^\mu) v \, dx \tag{3.56}$$

is equal to a sum of integrals over the sphere of radius R, which are easily seen to be estimated by

$$C\, R^{n-1} (|\operatorname{grad} v| \cdot |U^\mu| + |v| \cdot |\operatorname{grad} U^\mu|) \tag{3.57}$$

(the functions being evaluated at points of the sphere), and by virtue of (3.51), (3.52), (3.54), (3.55) we see that (3.57) is $O(R^{-1/2})$ so, (3.56) tends to 0 as $R \to \infty$ and we have

$$\int_{\mathbb{R}^n} (\Delta v) U^\mu dx = \int_{\mathbb{R}^n} v (\Delta U^\mu) dx = - \int v \, d\mu . \tag{3.58}$$

So by (3.53) and (3.58), every compactly supported measure μ in $\mathbb{R}^n \setminus \overline{\Omega}$ which annihilates HP_m, annihilates also v. Now, since HP_m is a finite dimensional subspace of $C(\mathbb{R}^n \setminus \overline{\Omega})$ (the space of continuous functions in $\mathbb{R}^n \setminus \overline{\Omega}$), $C(\mathbb{R}^n \setminus \overline{\Omega})$ is a direct sum, with HP_m as one of its summands (see e.g. [13; theorem 4.21]). Therefore, *there is a harmonic polynomial p of degree at most m such that*

$$v(x) = p(x) \qquad \text{for } x \in \mathbb{R}^n \setminus \overline{\Omega}\,. \tag{3.59}$$

We now set $K = \mathbb{R}^n \setminus \Omega$. Then K satisfies (3.40) and, by (3.59), we have, if $x \in K$,

$$|\operatorname{grad} p(x)| = |\operatorname{grad} v(x)| = O\left(|x| \log |x|\right)$$

for large $|x|$, because of (3.52). Hence, by lemma (3.39) p has degree at most 2. That is, v coincides on $\mathbb{R}^n \setminus \Omega$ with an element of HP_2. Hence, (3.58) implies that (3.53) holds for all measures μ compactly supported in $\mathbb{R}^n \setminus \overline{\Omega}$ which annihilate HP_2. Thus (3.48) holds and the proof is now completed. ∎

Remark. The condition (3.40), in terms of Newtonian capacity, while yielding useful results in lemma 3.39 and theorem 3.37 is by no means necessary, and the search for better conditions leads to interesting problems concerning harmonic polynomials to which we hope to return. For example, there even exist *countable* sets K which, in (3.41) lead to the same conclusion; indeed, it is easy to see that there are *countable* sets K, tending to ∞ arbitrarily fast, such that any polynomial (harmonic or not) that is $O(|x|^m)$ on this set has degree at most m: just take a dense subset $\{\xi_j\}$ of the unit sphere, and construct K so that it meets each of the half-rays from 0 passing through one of the ξ_j at a sequence of points tending to ∞.

We shall return in Section 5 to the approximation problem discussed above, and present a different approach.

4 – Quasi-balayage, a priori estimates, and Phragmén-Lindelöf theorems

What is quasi-balayage, and what is it good for? The "philosophy" of quasi-balayage was explained in [14] and need not be repeated here in detail. We shall instead illustrate it with an example: Consider functions φ in $C_c^\infty(\Omega)$ (the class of infinitely differentiable functions on some open set Ω of $\mathbb{R}^n$, with compact support). An important problem in p.d.e's is to obtain "a priori estimates" which means typically, to estimate some norm of φ or of its partial derivatives in terms of some, in general different, norm of $\mathcal{M}\varphi$ where $\mathcal{M}$ is some differential operator. Such estimates are needed for a broad range of applications to questions of existence, uniqueness, regularity, unique continuation, etc.

Quasi-balayage is a technique specifically designed to permit sup norm estimation of φ (and its derivatives) from sup norm estimates of $\Delta\varphi$. This is of course a rather narrow problem, but important in various aspects of potential theory. Quite probably some of the techniques could be adapted for other elliptic operators and L^p norms, although that is for the future. Especially, it is useful in exploiting the vanishing of φ on parts of the space to improve the estimates.

For $y \in \Omega$ we have, when $\varphi \in C_c^\infty(\Omega)$,

$$\varphi(y) = -\int_{\mathbb{R}^n} (\Delta\varphi)(x)E(x-y)dx = -\int_\Omega (\Delta\varphi)(x)E(x-y)dx$$

where E denotes the fundamental solution, so

$$|\varphi(y)| \leq \|\Delta\varphi\|_\infty \cdot \int_\Omega |E(x-y)|dx\,.$$

If the last integral is finite, this gives an estimate controlling the growth of $\varphi(y)$. However, for sharper estimates (see (4.11) below) or for "large" domains, e.g. Ω a half-space, the integral diverges for every y in Ω and this approach gives nothing, even though (as we shall now show) nontrivial estimates are possible.

THEOREM 4.1. *Suppose that K is a compact subset of the closed unit ball $\overline{B}$ such that* $\operatorname{cap}(K) \geq a > 0$. *Then for any φ in $C_c^\infty(\mathbb{R}^n)$ such that* $\operatorname{grad}\varphi$ *vanishes on K, we have*

$$|\varphi(y)| \leq |\varphi(0)| + C(n,a)\|\Delta\varphi\|_\infty \qquad \text{when } y \in \overline{B}. \tag{4.2}$$

REMARK. It is important that $\|\Delta\varphi\|_\infty$ on the right side refers to the sup norm of $\Delta\varphi$ *over all of* $\mathbb{R}^n$; it is remarkable that the more ambitious inequality where $\|\cdot\|_\infty$ is taken only over B is *false*. So, in order to appreciate (4.2) let us straight away give such an example, adapted from [9].

We take $n = 2$. Let θ be any function in $C_c^\infty(\mathbb{R}^2)$ such that $\theta = 1$ on $B \cap \{x_1 \geq 0\}$, and $\theta = 0$ on $B \cap \{x_1 \leq -1/2\}$, and define

$$\varphi_\lambda(x) = \theta(x)e^{\lambda x_1} \sin \lambda x_2\,, \qquad \lambda > 0\,.$$

Then

$$\Delta\varphi_\lambda = (\Delta\theta)(e^{\lambda x_1} \sin \lambda x_2) + 2\lambda e^{\lambda x_1}[\theta_1 \sin \lambda x_2 + \theta_2 \cos \lambda x_2] \tag{4.3}$$

where θ_i denotes $\partial\theta/\partial x_i$. If $x \in B \cap \{x_1 \geq 0\}$, then the right side of (4.3) is 0, so $|\Delta\varphi_\lambda(x)|$ is bounded in B by $C\lambda$ with C independent of λ. But, $\varphi_\lambda(1/2, 1/2) = e^{\lambda/2} \sin(\lambda/2)$, hence

$$\frac{\varphi_\lambda(1/2, 1/2)}{\sup\limits_B |\Delta\varphi_\lambda|} \to \infty$$

as $\lambda \to \infty$ through a suitable sequence. Moreover, all φ_λ vanish on $B \cap \{x_1 \leq -1/2\}$. So, the stronger version of (4.2) is violently false!

Proof of Theorem 4.1. There is clearly no loss of generality in assuming $\varphi(0) = 0$. Now, by Corollary 2.25 (used in a very weak way!) there is a constant $C(n, a)$ such that

$$\max_{x \in \overline{B}} |p(x)| \leq C(n, a) \max_{x \in K} |\operatorname{grad} p(x)| \tag{4.4}$$

holds for all p in HP_2 with $p(0) = 0$. By duality (see [14; Section 3] for details) this implies that for each y in $\overline{B}$ there is a distribution $\sigma = \sigma_y$ of order 1 supported in K such that

$$p(y) = \langle p, \delta_y \rangle = \langle p, \sigma \rangle \tag{4.5}$$

holds for every p in HP_2 with $p(0) = 0$, and the norm of σ is at most $C(n, a)$. Here δ_y denotes the Dirac measure at the point y. More concretely, this means there are measures $\mu_1, \ldots, \mu_n$ such that, in (4.5)

$$\langle p, \sigma \rangle = \sum_{i=1}^{n} \int (\partial p/\partial x_i) d\mu_i$$

with $\sum_i \|\mu_i\|_M \leq C(n,a)$. Taking account of the normalization we can write in place of (4.5)

$$f(y) = \langle f, \sigma\rangle + \langle f, \delta_0\rangle \tag{4.6}$$

for every f in HP_2 (we have let $p(y) = f(y) - f(0)$ in (4.5)). Another way to formulate (4.6) is that the distribution

$$\tau = \tau_y := \sigma_y + \delta_0$$

has the same action on HP_2 as δ_y. This is the "quasi-balayage": we have "swept" δ_y to a distribution τ_y living on $K \cup \{0\}$, having the same action on a (to be sure, very limited!) class of harmonic functions, in this case HP_2.

After these preliminaries, we can now complete the proof of the theorem. By Proposition 10.6 of [14] the potential of $\delta_y - \tau_y$ (which we denote simply by U^λ, $\lambda := \delta_y - \tau_y$) is integrable over $\mathbb{R}^n$ with L^1 norm at most $C(n)$ times the norm of λ, hence at most $C_1(n,a)$. We have

$$\varphi(y) = \langle \varphi, \delta_y - \tau_y\rangle = \langle \varphi, \lambda\rangle = -\int_{\mathbb{R}^n} (\Delta\varphi) U^\lambda dx. \tag{4.7}$$

(It is important to note here that τ_y annihilates φ because $\operatorname{grad}\varphi$ vanishes on $\operatorname{supp}\sigma_y$; this aspect of quasi-balayage, sweeping on to a distribution of order 1 rather than a measure, was discussed theoretically in [14] but is used for the first time here.)

From (4.7),

$$|\varphi(y)| \leq \|\Delta\varphi\|_\infty \int_{\mathbb{R}^n} |U^\lambda(x)| dx \leq C_1(n,a)\|\Delta\varphi\|_\infty$$

and the proof is finished. □

Remark 1. In this theorem, the capacity hypothesis entered only very weakly, all we need to impose on K is some condition ensuring that if $\operatorname{grad} p$ vanishes on K when $p \in HP_2$, then p is a constant on B (see Proposition 2.14). It is easy to construct explicitly finite sets with this property, but even for this simple situation there does not seem to exist a general theory (in case $n = 2$ it is of course easy to get sharp estimates). In our next theorem, we shall have to do quasi-balayage for HP_m with m in principle large, but even so use of capacity conditions, although convenient, seems like "overkill", and it should be a worthwhile undertaking to seek the basic metrical properties of finite sets which really are at the root of estimates like (2.26).

REMARK 2. By an almost identical argument, using Proposition 10.1 of [14] we obtain, under the hypotheses of theorem (4.1):

$$|\operatorname{grad}\varphi(y)| \le C_2(n,a)\|\Delta\varphi\|_\infty \qquad \text{when } y \in B\,. \tag{4.8}$$

COROLLARY 4.9. *Let Ω be an open bounded set such that*

$$\liminf_{r\to 0}\Big(\inf_{x\in\partial\Omega}\frac{\operatorname{cap}(\overline{B(x,r)}\cap(\mathbb{R}^n\setminus\Omega))}{\operatorname{cap}(\overline{B(0,r)})}\Big) > 0\,. \tag{4.10}$$

Then for any φ in $C_c^\infty(\Omega)$,

$$|\varphi(y)| \le C_1 d^2(y)\|\Delta\varphi\|_\infty\,, \tag{4.11}$$

and

$$|\operatorname{grad}\varphi(y)| \le C_2 d(y)\|\Delta\varphi\|_\infty\,, \tag{4.12}$$

where $d(y)$ is the distance from y to $\mathbb{R}^n\setminus\Omega$, and C_1 and C_2 are constants depending only on n and the value of the lim inf *in* (4.10).

PROOF. By the assumptions, there is a positive constant a and a positive r_0 such that

$$\frac{\operatorname{cap}(\overline{B(y_0,r)}\cap(\mathbb{R}^n\setminus\Omega))}{\operatorname{cap}(\overline{B(0,r)})} \ge a \tag{4.13}$$

for all y_0 in $\partial\Omega$ and for all $r \le r_0$. For each y in Ω with $d(y) \le r_0$, let y_0 be the nearest point to y on $\partial\Omega$. Set $\psi(x) = \varphi(rx+y_0)$, then $\psi(0) = 0$ and $\operatorname{grad}\psi$ vanishes on

$$K_{r,y_0} := \{x \in \overline{B(0,1)}:\ rx + y_0 \in \mathbb{R}^n\setminus\Omega\}\,.$$

By an argument used in Section 3 and by (4.13),

$$\frac{\operatorname{cap}(K_{r,y_0})}{\operatorname{cap}(\overline{B}(0,1))} \geq a \qquad \text{for } r \leq r_0 \,.$$

Hence, by applying the estimate (4.2) we have

$$\sup_{B(0,1)} |\psi(x)| \leq |\psi(0)| + C(n,a)\|\Delta\psi\|_\infty = C(n,a)r^2\|\Delta\varphi\|_\infty \,,$$

or

$$\sup_{B(y_0,r)} |\varphi(y)| \leq C(n,a)r^2\|\Delta\varphi\|_\infty \,.$$

Taking $r = d(y)$ we obtain (4.11) when $d(y) \leq r_0$. Since $\varphi(y)$ is bounded by $\|\Delta\varphi\|_\infty$, (4.11) holds for all y in Ω with a larger constant. The proof of (4.12) is similar. ∎

By duality, every estimate like (4.11) implies an existence theorem:

COROLLARY 4.14. *Suppose that Ω satisfies the hypotheses of Corollary* 4.9*. Then, for every measure μ in Ω satisfying*

$$\int_\Omega d(x)^2 d|\mu| < \infty \tag{4.15}$$

there is u in $L^1(\Omega)$ satisfying (*distributionally*) *$\Delta u = \mu$ in Ω.*

PROOF. Applying Proposition 5.1 of [14] to (4.11) gives the existence of a measure ν in Ω with $\|\nu\|_M < \infty$ satisfying $\Delta\nu = \mu$. By elliptic regularity theory $\nu \in L^p_{\mathrm{loc}}$ for $p < \frac{n}{n-2}$, so $d\nu$ can be identified as an element $u \in L^1(\Omega) \cap L^p_{\mathrm{loc}}$ (we also get an estimate for $\|u\|_1$, which we omit). ∎

Corollaries 4.9 and 4.14 were proved by BRUNA and ORTEGA-CERDÁ [2], but instead of (4.10) they assumed that Ω has a smooth boundary. They also showed that for nonnegative measures, (4.15) is a necessary condition. In a similar way we can consider also the case when Ω is unbounded and satisfies the capacitary density condition at infinity.

COROLLARY 4.16. *Suppose that Ω is an open set and $K = \mathbb{R}^n \setminus \Omega$ satisfies*

$$\liminf_{R\to 0} \frac{\operatorname{cap}(\overline{B(0,r)} \cap K)}{\operatorname{cap}(\overline{B(0,R)})} > 0\,. \tag{4.17}$$

Then, for every measure μ in Ω satisfying

$$\int_\Omega (1+|x|)^2 d|\mu|(x) < \infty$$

there is u in $L^1(\Omega)$ satisfying $\Delta u = \mu$.

A very interesting use of a priori estimates is in estimating the size of zero sets of holomorphic functions, in one or several variables, as well as the inverse problem of constructing holomorphic functions in various classes with prescribed zero-sets (cf. [2] for examples, and references). We illustrate the idea with an example, in one complex dimension. Suppose that f is holomorphic and non-constant on a domain Ω in the complex plane with zeros precisely at the points $\{z_j\}$, m_j denoting the multiplicity of z_j. Then, in the sense of distributions, $\Delta u = 2\pi\mu$ where μ is the measure putting mass m_j at z_j, and elsewhere zero, and $u := \log|f|$. This is equivalent to

$$2\pi \int \varphi\, d\mu = \int \varphi \Delta u = \int u\, \Delta\varphi \quad \textit{for all } \varphi \in C_c^\infty(\Omega)\,. \tag{4.18}$$

Suppose for example, for some positive continuous w on Ω,

$$J(f) := \int_\Omega |\log|f(z)||\, w(z) dA(z) < \infty$$

where dA denotes area measure. Then from (4.18)

$$2\pi \Big| \int \varphi d\mu \Big| \le \int |u\Delta\varphi| dA \le J(f) \sup_{z\in\Omega} \frac{|\Delta\varphi(z)|}{w(z)}\,.$$

Thus, a necessary condition for the existence of a nontrivial f with $J(f) \le M$ having the zero-set described by μ is that the “a priori inequality”

$$2\pi \Big| \int \varphi d\mu \Big| \le M \sup_{z\in\Omega} \frac{|\Delta\varphi(z)|}{w(z)} \tag{4.19}$$

hold for every $\varphi \in C_c^\infty(\Omega)$. Moreover, with certain additional assumptions, the reasoning can be reversed, and from (4.19) follows the existence of a nontrivial f having $J(f)$ at most M, and with the prescribed zeros. Estimates of type (4.19) can be obtained using quasi-balayage, although we won't pursue that here.

Generally speaking, quasi-balayage is a useful tool for estimating functions *vanishing with their gradient* outside the set of interest. It does not seem useful for estimation problems lacking this feature. Consider, e.g. the *Dirichlet problem* to construct a solution u to $\Delta u = f$ in some domain Ω such that $u = 0$ on $\partial\Omega$ (we may assume Ω has a smooth boundary, but may be unbounded, and that f is in some reasonable class, say $L^\infty(\Omega)$). The solvability of this problem is equivalent to

$$\int_\Omega f\,\varphi\,dx = \int_\Omega (\Delta u)\varphi\,dx = \int_\Omega u\Delta\varphi\,dx \tag{4.20}$$

where the test class can now be taken as the set of smooth, compactly supported functions on $\mathbb{R}^n$ which vanish on $\partial\Omega$. If, for example a solution u exists satisfying $|u(x)| \le w(x)$, for every f in $L^\infty(\Omega)$ with $\|f\|_\infty \le 1$, then (4.20) implies the estimate

$$\int_\Omega |\varphi|dx \le \int_\Omega w(x)|\Delta\varphi(x)|dx \tag{4.21}$$

for all φ in the test class just described and, conversely, by standard functional analysis (see e.g. [14; Proposition 5.1]) (4.21) would imply the solvability of the Dirichlet problem as formulated. There are two difficulties in trying to prove (4.21) by the use of quasi-balayage. The first step is always to express $\varphi(y)$, for y in Ω as

$$-\int_{\mathbb{R}^n} (\Delta\varphi)(x)\,U^{\delta_y-\nu_y}(x)dx \tag{4.22}$$

where ν_y is a quasi-balayage measure of δ_y (relative to HP_2) living on $\partial\Omega$, the set where φ is known to vanish. This is already a difficulty, because $\partial\Omega$ may well not permit such quasi-balayage e.g. it could be a halfplane, or more generally a level set of a nontrivial polynomial in HP_2. But, apart from this, a more serious difficulty is that the estimation of (4.22) involves bounding $|(\Delta\varphi)(x)|$ for all x in $\mathbb{R}^n$, not merely in Ω, so we can't get an estimate of type (4.21) unless we can first show that $\varphi|_{\overline{\Omega}}$ can be redefined outside of Ω so as to remain in $C^2(\mathbb{R}^n)$, with appropriate bounds on $\Delta\varphi$ outside Ω; and such an extension problem is probably harder than the Dirichlet problem itself. The important point to be made here is that, since φ vanishes only *simply* on $\partial\Omega$ (i.e. grad φ need not be zero there) redefining it as 0 on the exterior domain is not permissible, since then the distribution $\Delta\varphi$ would pick up an unwanted contribution on $\partial\Omega$. In some very special situations the extension problem is easily dealt with, e.g. if Ω is a half-space (say, $\{x_n > 0\}$) and $\varphi = 0$ on $\partial\Omega$, then φ may be redefined on $\mathbb{R}^n \setminus \Omega$ so as to have *odd symmetry* w.r.t. the hyperplane $\{x_n = 0\}$. In this way, the estimate (4.21) with $\Omega = \mathbb{R}^n$ and $w(x) = C(1+|x|))^2 \cdot \log(2+|x|)$ due essentially to KARP [8], valid for φ in $C_c^\infty(\mathbb{R}^n)$, can be proved also for a half-space Ω and test functions φ vanishing on $\partial\Omega$—which leads, as already described, to solvability with bounds of a Dirichlet problem in the half-space. It seems an interesting problem to try to extend this (by whatever method) to other domains. For a good survey of the Dirichlet problem in unbounded domains, see [15].

Our final result in this section deals with non-compactly supported functions. This, which can be regarded as a theorem of Phragmén-Lindelöf type for solutions of Cauchy's problem, has significant applications to the regularity of free boundaries, which will be given elsewhere.

THEOREM 4.23. *Suppose that $u \in C^1(\mathbb{R}^n)$ and that u satisfies*

(i) $\Delta u \in L^\infty(\mathbb{R}^n)$,
(ii) $\operatorname{grad} u$ *vanishes on a set K satisfying* (4.17),
(iii) $|u(x)| = O(|x|^k)$, $|x| \to \infty$ *for some positive k.*

Then

$$|u(y)| \le |u(0)| + C_1 \|\Delta u\|_\infty \,(1+|y|)^2 \tag{4.24}$$

$$|\operatorname{grad} u(y)| \le C_2 \|\Delta u\|_\infty (1+|y|) \tag{4.25}$$

hold for all y in $\mathbb{R}^n$. Here C_1, C_2 are constants depending only on n and the lim inf *in* (4.17).

PROOF. The proof follows the same pattern as those of theorem 4.1 and Corollary 4.9, so we only indicate where something additional is needed. First of all, it is enough to obtain "local" estimates for $u(y)$ and $\operatorname{grad} u(y)$ when $y \in \overline{B}$, analogous to (4.2) since then the global result follows by a scaling argument. And, in the proof of the local result only one novelty arises: to justify the partial integration step

$$u(y) = \langle u, \delta_y \rangle = \int_{\mathbb{R}^n} u \Delta\, U^{\lambda_y} dx = \int_{\mathbb{R}^n} (\Delta\, u) U^{\lambda_y} dx$$

where λ_y is δ_y minus its quasi-balayage distribution onto K, we have to make sure U^{λ_y} decays at ∞ fast enough so that, for large $|x|$,

$$|u(x)| \cdot |\operatorname{grad} U^{\lambda_y}(x)| + |\operatorname{grad} u(x)| \cdot |U^{\lambda_y}(x)| = o(|x|^{-n+1})\,. \tag{4.26}$$

Now, hypotheses (i) and (iii) imply, by standard elliptic estimates, $|\operatorname{grad} u(x)| = O(|x|^{k-1})$ for large $|x|$. Thus, to ensure (4.26) we only need choose the "order of quasi-balayage", that is, do quasi-balayage w.r.t. HP_m, where m is so large that for distributions λ_y annihilating HP_m we have

$$|\operatorname{grad} U^{\lambda_y}(x)| = o(|x|^{-n-k}), |U^{\lambda_y}(x)| = o(|x|^{-n-k+1})$$

and these follow for large enough m, see [14; Section 4 and 10]. □

REMARK. The estimates

$$|u(y)| \leq |u(y_0)| + C_1 \|\Delta u\|_\infty |y - y_0|^2 \tag{4.27}$$

$$|\operatorname{grad} u(y)| \leq C_2 \|\Delta\, u\|_\infty |y - y_0| \tag{4.28}$$

near a finite point y_0, analogous to (4.24) and (4.25), hold under the analogous (to (4.17)) condition

$$\liminf_{r \to 0} \frac{\operatorname{cap}(K \cap \overline{B(y_0, r)})}{\operatorname{cap}\, \overline{B(0, r)}} > 0\,.$$

The proofs are similar to the preceding. Note that in this case it suffices to do the quasi-balayage w.r.t. HP_2. See Remark 1 below the proof of Theorem 4.1.

5 – Approximation by rapidly decreasing harmonic functions, continued

We present here concisely another approach to the approximation problem dealt with in Section 3.

Definition 5.1. For an open connected set Ω of $\mathbb{R}^n$, and a positive integer m, $H_m(\Omega)$ will in this section, denote the set of functions h harmonic in Ω, and such that $h(x) = O(|x|^{-m-1})$, $|\operatorname{grad} h(x)| = O(|x|^{-m-2})$ as $|x| \to \infty$.

Theorem 5.2. *Suppose that $\overline{\Omega} \neq \mathbb{R}^n$ and that k is a positive integer. Then the following are equivalent*

(a) *$H_{n+k}\,(\Omega)$ is dense in $HL^1(\Omega)$.*

(b) *For every f in $L^\infty(\Omega)$ we define*

$$\tilde{f}(x) = \begin{cases} f(x)\,, & x \in \Omega \\ 0\,, & x \notin \Omega\,. \end{cases} \tag{5.3}$$

Then if u is a solution of

$$\Delta u = \tilde{f}\,, \qquad \text{on } \mathbb{R}^n \tag{5.4}$$

satisfying

$$u(x) = 0, |\operatorname{grad} u(x)| = 0 \qquad \text{on } \mathbb{R}^n \setminus \Omega\,, \tag{5.5}$$

and

$$u(x) = O(|x|^{k+2})\,, \text{as } |x| \to \infty \tag{5.6}$$

we also have

$$u(x) = O(|x|^2 \log |x|)\,, \qquad \text{as } |x| \to \infty\,. \tag{5.7}$$

Note that this theorem gives an *equivalent* condition for approximability, in terms of a Phragmén-Lindelöf type condition for solutions to Cauchy's problem, similar to those discussed in Section 4.

The proof will be presented elsewhere; let us here only illustrate the use of this theorem.

EXAMPLE 5.8. We define $\Omega = \{x \in \mathbb{R}^n : |x| > 1\}$. Let p denote a nontrivial homogeneous harmonic polynomial of degree m, and let ψ be a function in $C^\infty(\mathbb{R}^n)$, equal to 0 for $|x| \le 1$ and to 1 for $|x| \ge 2$. Define $u = \psi p$. Then u and $\operatorname{grad} u$ vanish on $\mathbb{R}^n \setminus \Omega$, $\Delta u \in L^\infty(\mathbb{R}^n)$ since $\Delta u(x) = 0$ for $|x| \ge 2$, but $u(x)$ is not $o(|x|^m)$ as $|x| \to \infty$. This shows that (e.g. taking $m = k + 2 = 3$), u satisfies (5.3) (for some f in $L^\infty(\Omega)$), (5.5), and (5.6), but not (5.7). Thus, by theorem 5.2 we conclude: $H_{n+1}(\Omega)$ is not dense in $HL^1(\Omega)$.

EXAMPLE 5.9. We define $G = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 > 1, -\infty < x_3 < \infty\}$. Let $u(x_1, x_2)$ be as in the preceding example (for dimension $n = 2$), and $v(x_1, x_2, x_3) = u(x_1, x_2)$. Then, it is easy to check that $\Delta v \in L^\infty(\mathbb{R}^3)$ and v, in place of u, satisfies (5.5), (5.6) (for $k = 1$, if we take $m = 3$) but not (5.7). We conclude that $H_4(G)$ is not dense in $HL^1(G)$.

To use theorem 5.2 in the opposite direction, and deduce approximation theorems, we can apply Phragmén-Lindelöf type theorems, as in the preceding section to show (for domains Ω with sufficiently large complement) that (5.7) follows from the remaining properties of u. In particular, we have:

PROPOSITION 5.10. *If the "upper capacitary density" condition*

$$\limsup_{R\to\infty} \frac{\operatorname{cap}(\overline{B(0,R)} \setminus \Omega)}{\operatorname{cap}(\overline{B(0,R)})} > 0 \tag{5.11}$$

holds, then u satisfying $\Delta u \in L^\infty(\mathbb{R}^n)$, (5.5)*, and* (5.6) *for any k, also satisfies* (5.7)*. Consequently, for Ω satisfying* (5.11)*, and $\overline{\Omega} \ne \mathbb{R}^n$, $H_m(\Omega)$ is dense in $HL^1(\Omega)$ for $m \ge n + 1$.*

If Ω satisfies the stronger condition, analogous to (5.11) but with lim inf (i.e. its complement has positive "lower capacitary density") then we can, assuming the remaining hypotheses of Proposition 5.10, deduce in place of (5.7) the stronger condition

$$u(x) = O(|x|^2)\,; \qquad x \in \Omega, |x| \to \infty \tag{5.12}$$

which, as already remarked, is useful in certain problems involving the regularity of free boundaries. The proofs of Proposition 5.10, and the analogous assertion concerning (5.12) are implicit in the results in Sections 3 and 4.

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Indirizzo degli Autori:

Walter K. Hayman – Department of Mathematics – Imperial College – 180 Queen's Gate – London SW7 2BZ – UK

Lavi Karp – Department of Applied Mathematics – Ort Braude College – P.O.Box 78 – Karmiel 21982 Israel and Department of Mathematics – Technion – 32000 Haifa – Israel
E-mail: karp@techunix.technion.ac.il

Harold S. Shapiro – Department of Mathematics – Royal Institute of Technology – S-100 44 Stockholm – Sweden
E-mail: shapiro@math.kth.se