Rendiconti di Matematica, Serie VII
Volume 21, Roma (2001), 1-15

L_p-estimates for solutions of the heat equation in a dihedral angle

Dedicated to Professor Maria Giovanna Garroni on the occasion of her jubilee

V.A. SOLONNIKOV

Riassunto: *Si dimostrano stime coercive negli spazi L_p con peso per le soluzioni degli problemi di Cauchy-Dirichlet e di Cauchy-Neumann per la equazione del calore nel diedro n-dimensionale con l'angolo d'apertura arbitrario: $\theta \in (0, 2\pi]$. La dimostrazione e basata sulle stime delle funzioni di Green di questi problemi.*

Abstract: *We prove coercive weighted L_p-estimates for the solutions of the Dirichlet and Neumann initial-boundary value problems for the heat equation in n-dimensional dihedral angle with arbitrary opening angle $\theta \in (0, 2\pi]$. A crucial role in the proof of this result is played by pointwise estimates of the Green functions of these problems.*

1 – Introduction

The present paper which can be considered as a complement to the article [12] is concerned with L_p-estimates for solutions of the Dirichlet and of the Neumann initial-boundary value problems in an infinite wedge

Key Words and Phrases: *Parabolic initial boundary value problems — Weighted estimates.*
A.M.S. Classification: 35K05 – 35K20

$D_\theta \subset \mathbb{R}^n$ with the opening angle $\theta \in (0, 2\pi]$:

$$\begin{cases} u_t - \Delta u = f(x,t)\,, & x \in D_\theta, \qquad t > 0\,, \\ u|_{t=0} = 0\,, \end{cases} \tag{1.1}$$

$$u|_{\partial D_\theta} = 0\,, \tag{1.2}$$

$$\frac{\partial u}{\partial n}\Big|_{\partial D_\theta} = 0\,. \tag{1.3}$$

We assume that $D_\theta = d_\theta \times \mathbb{R}^{n-2} = \{x \in \mathbb{R}^n : x' = (x_1, x_2) \in d_\theta, x'' = (x_3, ..., x_n) \in \mathbb{R}^{n-2}\}$ where d_θ is an infinite plane sector which can be given in the polar coordinates $\{r = |x'|,\ \varphi = \operatorname{arctg} x_2/x_1\}$ by the relations

$$r > 0\,, \quad 0 < \varphi < \theta\,.$$

We denote by Γ_0 and Γ_1 the faces of D_θ: $\Gamma_0 = \gamma_0 \times \mathbb{R}^{n-2}$, $\Gamma_\theta = \gamma_1 \times \mathbb{R}^{n-2}$, where $\gamma_0 = \{\varphi = 0,\ r > 0\}$ and $\gamma_1 = \{\varphi = \theta,\ r > 0\}$ are boundary lines of d_θ.

The case $n = 2$, when $D_\theta = d_\theta$, $\Gamma_i = \gamma_i$, $i = 0, 1$, is not excluded.

The main result of the paper is contained in the following two theorems.

THEOREM 1.1. *The solution of the problem* (1.1), (1.2) *satisfies the inequality*

$$\int_0^T\!\!\int_{D_\theta} \Big(\Big|\frac{\partial u}{\partial t}\Big|^p + |D_x^2 u|^p\Big)\, |x'|^{p\mu}\, dx\, dt \ \le\ c \int_0^T\!\!\int_{D_\theta} |f|^p\, |x'|^{p\mu}\, dx\, dt \tag{1.4}$$

where $p > 1$, T *is an arbitrary positive number,* c *is a constant independent of* T, $D_x^2 u = (\frac{\partial^2 u}{\partial x_i\, \partial x_j})_{i,j=1,\dots,n}$ *and* μ *is a number satisfying the condition*

$$\frac{2}{p'} - \min\Big(\frac{\pi}{\theta}, 2\Big) < \mu < \frac{2}{p'}\,, \tag{1.5}$$

$p' = \frac{p}{p-1}$.

THEOREM 1.2. *The solution of the problem* (1.1), (1.2) *satisfies the inequality*

$$\begin{aligned}\int_0^T\int_{D_\theta}\Big(\Big|\frac{\partial u}{\partial t}\Big|^p + |D_x^2 u|^p + |\nabla u|^p\,|x'|^{-p} + |u|^p\,|x'|^{-2p}\Big)\,|x'|^{\mu p}\,dx\,dt \le\\ \le c\int_0^T\int_{D_\theta}|f(x,t)|^p\,|x'|^{\mu p}\,dx\,dt\end{aligned} \tag{1.6}$$

with the constant independent of $T>0$, *provided that*

$$-\frac{\pi}{\theta} < \frac{2}{p'} - \mu < \frac{\pi}{\theta}\,. \tag{1.7}$$

Problems (1.1), (1.2) and (1.1), (1.3) may be considered as model problems arising in the study of the Cauchy-Dirichlet and Cauchy-Neumann problems for the second order parabolic equations in domains with edges at the boundary. As known from the theory of elliptic equations, in this case usual coersive L_p-estimates are not always true and it is necessary to work in weighted spaces. This was made clear by V.A. Kondrat'ev [3] who considered elliptic boundary value problems in domains with conical points at the boundaries for which he proved coercive estimates in weighted W_2^l Sobolev spaces. Much more general results for elliptic boundary value problems in domains with irregular boundaries were obtained in a series of papers of V.G.Maz'ya and B.A. Plamenevsky [6], [7], [8] (see also the book [9]). The solutions of parabolic initial-boundary value problems in general also loose regularity at the singular points of the boundaries (see [2], [4], [5]), therefore it is natural to consider these problems in some weighted (anisotropic) spaces. "A parabolic analogue" of V.A. Kondratiev's results for a certain class of parabolic initial-boundary value problems in domains with conical points was obtained by V.A. Kozlov [4].

The proof of Theorems 1.1 and 1.2 is based on the representation formulas for the solutions of problems (1.1), (1.2) and (1.1), (1.3) in terms of the corresponding Green functions $G(x,y,t)$:

$$u(x,t) = \int_0^t\int_{D_\theta} G(x,y,t-\tau)\,f(y,\tau)\,dy\,d\tau \tag{1.8}$$

and on pointwise estimates of these functions and of their derivatives obtained in [12]. Similar estimates for the Green function of initial-boundary value problem for the heat equation in D_θ with the boundary conditions

$$\frac{\partial u}{\partial n} + h_i \frac{\partial u}{\partial r}\Big|_{x\in\Gamma_i} = 0\,, \quad i=1,2,\ h_i = \text{const.}$$

are given in [1]. In the above-mentioned papers by V.A. Kozlov the Green functions for parabolic initial-boundary value problems considered there are estimated in a more precise manner than in [12], [1] (these estimates guarantee the exponential decay of $G(x,y,t)$ for large $|x|^2/t$, or $|y|^2/t$, while estimates obtained in [12], [1] give only power-like decay). However, this difference turns out inessential in the proof of coercive estimates (1.4), (1.6) and of estimates of solutions of the same problems in weighted Hölder norms obtained in [12].

The paper is written at Centro de Matemática e Aplicações Fundamentais of the University of Lisbon whose hospitality is gratefully acknowledged.

2 – Some auxiliary estimates

We present here L_p-estimates of the potential

$$v(x,t) = \int_0^t\!\!\int_{\mathbb{R}^n} K(x,y,t-\tau)\, f(y,\tau)\, dy\, d\tau \tag{2.1}$$

under different assumptions concerning the kernel $K(x,y,t)$. As for $f(x,t)$, it is sufficient to assume that it belongs to a certain class of functions which is dense in usual or weighted L_p space and for which the integral (2.1) is convergent. In most cases the assumption of the boundedness and of power-like decay at infinity of $f(x,t)$ is sufficient for the convergence of the integrals (2.1) considered in this paper. Exception should be made for the case when $K(x,y,t) = \frac{\partial^2}{\partial x_i\, \partial x_j}\,\Gamma(x-y,t)$ or $K(x,y,t) = \frac{\partial \Gamma(x-y,t)}{\partial t}$. Then the integral in (2.1) is singular and can be understood, for instance, as

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_0^{t-\varepsilon}\!\!\int_{\mathbb{R}^n} K(x,y,t-\tau)\, f(y,\tau)\, dy\, d\tau = \\ &= \int_0^t\!\!\int_{\mathbb{R}^n} K(x,y,t-\tau)\, [f(y,\tau) - f(x,\tau)]\, dy\, d\tau \end{aligned} \tag{2.2}$$

which is meaningful for Hölder continuous $f(x,t)$ (in Section 3, we consider singular intergals with $f(x,t)$ vanishing outside a certain dihedral angle which are also well defined for Hölder continuous $f(x,t)$ decaying at infinity like power functions).

We estimate the integral (2.1) in the domain $\mathbb{R}^n \times (0,T)$ with arbitrary $T > 0$ but the constants in our estimates are independent of T.

We observe first of all that the above-mentioned singular integrals satisfy the well known inequality

$$\|v\|_{L_p(\mathbb{R}^n\times(0,T))} \le c\,\|f\|_{L_p(\mathbb{R}^n\times(0,T))}\,. \tag{2.3}$$

In addition, it is easy to prove the following elementary proposition.

Proposition 2.1. *Let* $x = (x', x'')$, $x' = (x_1, x_2)$, $x'' = (x_3, ..., x_n)$. *Inequality* (2.3) *holds, if* $K(x,y,t) = K(x', y', x''- y'', t)$ *and*

$$\int_0^\infty\int_{\mathbb{R}^{n-2}} |K(x', y', z, t)|\, dz\, dt \le \frac{c}{|x'-y'|\,(|x'|+|y'|)}\,. \tag{2.4}$$

Proof. We make use of the boundedness of the integral

$$I_\delta = \int_{\mathbb{R}^2} \Big(\frac{|x'|}{|y'|}\Big)^{\delta} \frac{dy'}{|x'-y'|\,(|x'|+|y'|)}$$

with arbitrary $\delta \in (0,2)$. By virtue of the Hölder inequality, we have

$$\begin{aligned} |v(x,t)| \le \Big(\int_0^t\int_{\mathbb{R}^n} \Big(\frac{|y'|}{|x'|}\Big)^{\varepsilon p} |K(x,y,t-\tau)|\,|f(y,\tau)|^p\,dy\,d\tau\Big)^{1/p}\cdot \\ \cdot\Big(\int_0^t\int_{\mathbb{R}^n} \Big(\frac{|x'|}{|y'|}\Big)^{\varepsilon p'} |K(x,y,t-\tau)|\,dy\,d\tau\Big)^{1/p'} \end{aligned} \tag{2.5}$$

where $\varepsilon \in (0, \min(\frac{2}{p}, \frac{2}{p'}))$. The second integral in (2.5) is not greater than $c\, I_{\varepsilon\, p'}$, hence,

$$\begin{aligned} &\int_0^T\int_{\mathbb{R}^n} |v(x,t)|^p\,dx\,dt \le \\ &\le (cI_{\varepsilon p'})^{p-1}\int_0^T\int_{\mathbb{R}^n} |f(y,\tau)|^p\,dy\,d\tau \int_\tau^t\int_{\mathbb{R}^n} \Big(\frac{|y'|}{|x'|}\Big)^{\varepsilon p}|K(x,y,t-\tau)|\,dx\,dt \le \\ &\le c^p\, I_{\varepsilon p'}^{p-1}\, I_{\varepsilon p} \int_0^T\int_{\mathbb{R}^n} |f(y,\tau)|^p\,dy\,d\tau\,. \end{aligned}$$

□

The following propositions concern the estimates of (2.1) in weighted L_p-norms.

PROPOSITION 2.2. *If*

$$|K(x,y,t)| \le c(|x-y|^2 + |x'|^2 + |y'|^2 + t)^{-\frac{n+2-\lambda}{2}} |x'|^{-\lambda}, \quad 0 \le \lambda < 2, \tag{2.6}$$

then

$$\int_0^T \int_{\mathbb{R}^n} |v|^p \, |x'|^{p\mu} \, dx \, dt \le c \int_0^T \int_{\mathbb{R}^n} |f|^p \, |x'|^{p\mu} \, dx \, dt \tag{2.7}$$

with arbitrary μ *satisfying the inequality*

$$\lambda - \frac{2}{p} < \mu < \frac{2}{p'}. \tag{2.8}$$

PROOF. We choose the number μ_1 such that

$$\frac{\lambda}{p'} < \mu_1 < \frac{2}{p'}, \quad \mu < \mu_1 < \frac{2-\lambda}{p} + \mu$$

(this is possible in virtue of (2.8)) and we observe that

$$\begin{aligned} J_1 &= \int_0^\infty \int_{\mathbb{R}^n} \Big(\frac{|x'|}{|y'|}\Big)^{p'\mu_1} |K(x,y,t)| \, dy \, dt \le \\ &\le c \, |x'|^{p'\mu_1 - \lambda} \int_{\mathbb{R}^2} \frac{dy'}{(|x'|^2 + |y'|^2)^{(2-\lambda)/2} \, |y'|^{p'\mu_1}} = c_1, \\ J_2 &= \int_0^\infty \int_{\mathbb{R}^n} \Big(\frac{|y'|}{|x'|}\Big)^{p(\mu_1 - \mu)} |K(x,y,t)| \, dx \, dt \le \\ &\le c \, |y'|^{p(\mu_1 - \mu)} \int_{\mathbb{R}^2} \frac{dx'}{(|x'|^2 + |y'|^2)^{(2-\lambda)/2} \, |x'|^{\lambda + p(\mu_1 - \mu)}} = c_2. \end{aligned}$$

Hence, arguing as in the preceding proposition, we obtain

$$|v(x,t)| \le c_1^{1/p'} \Big(\int_0^t \int_{\mathbb{R}^n} \Big(\frac{|y'|}{|x'|}\Big)^{p\mu_1} |K(x,y,t-\tau)| \, |f(y,\tau)|^p \, dy \, d\tau \Big)^{1/p}$$

and

$$\int_0^T\int_{\mathbb{R}^n} |v(x,t)|^p\,|x'|^{p\mu}\,dx\,dt \le$$
$$\le c_1^{p-1}\int_0^T\int_{\mathbb{R}^n} |f(y,\tau)|^p|y'|^{p\mu}dy\,d\tau\int_\tau^T\int_{\mathbb{R}^n}\Big(\frac{|y'|}{|x'|}\Big)^{p(\mu_1-\mu)}|K(x,t,t-\tau)|dy\,d\tau\le$$
$$\le\ c_1^{p-1}\,c_2\int_0^T\int_{\mathbb{R}^n} |f(y,\tau)|^p\,|y'|^{p\mu}\,dy\,d\tau\,.$$

The proposition is proved. ∎

Proposition 2.3. *Assume that*

$$|K(x,y,t)|\le\frac{c}{(|x-y|^2+t)^{\frac{n}{2}}}\Big(\frac{|x'|}{|x'|+|y'|+\sqrt{t}}\Big)^{\lambda_1}\Big(\frac{|y'|}{|x'|+|y'|+\sqrt{t}}\Big)^{\lambda_2} \tag{2.9}$$

where $\lambda_i\ge 0,\ \lambda_1+\lambda_2>0$. *Then*

$$\int_0^T\int_{\mathbb{R}^n} |v(x,t)|^p\,|x'|^{p\mu-2p}\,dx'\,dt\ \le\ c\int_0^T\int_{\mathbb{R}^n} |f(x,t)|^p\,|x'|^{p\mu}\,dx\,dt \tag{2.10}$$

with the parameter μ *satisfying the inequalities*

$$-\lambda_2<\frac{2}{p'}-\mu<\lambda_1\,.$$

Proof. We prove (2.8) using the same scheme as in the preceding propositions. We choose the numbers σ and κ such that

$$0<\sigma<\mu+\lambda_1-\frac{2}{p'},\ \ 0<\kappa<\frac{2}{p'}+\lambda_2-\mu\,,\quad 0<\sigma+\kappa<\lambda_1+\lambda_2,$$

and we estimate $v(x,t)$ with the help of the Hölder inequality in the following way:

$$|v(x,t)|\le c\Big(\int_0^t\int_{\mathbb{R}^n} |f(y,\tau)|^p\Big(\frac{|y'|}{|x'|}\Big)^{p\mu}R_{x'}^{\sigma p}R_{y'}^{\kappa p}\,\frac{dy\,d\tau}{(|x-y|^2+t-\tau)^{n/2}}\Big)^{1/p}L^{1/p'}\,,$$

$R_{x'} = |x'|/(|x'|+|y'|+\sqrt{t-\tau})$, $R_{y'} = |y'|/(|x'|+|y'|+\sqrt{t-\tau})$. Here

$$\begin{aligned} L &= \int_0^t\!\!\int_{\mathbb{R}^n} \Big(\frac{|x'|}{|y'|}\Big)^{p'\mu} R_{x'}^{(\lambda_1-\sigma)p'}\, R_{y'}^{(\lambda_2-\kappa)p'}\, \frac{dy\,d\tau}{(|x-y|^2+t-\tau)^{n/2}} \le \\ &\le \int_0^t\!\!\int_{\mathbb{R}^n} \frac{|x'|^{p'(\mu+\lambda_1-\sigma)}\,|y'|^{p'(\lambda_2-\kappa-\mu)}\,dy\,d\tau}{(|x'|+|y'|+\sqrt{t-\tau})^{(\lambda_1+\lambda_2-\sigma-\kappa)p'}\,(|x-y|^2+t-\tau)^{n/2}} \le \\ &\le c\int_{\mathbb{R}^2} \frac{|x'|^{p'(\mu+\lambda_1-\sigma)}\,dy'}{|y'|^{p'(\mu-\lambda_2+\kappa)}\,(|x'|+|y'|)^{(\lambda_1+\lambda_2-\sigma-\kappa)p'-\delta}\,|x'-y'|^{\delta}} \le c_1\,|x'|^2\,, \end{aligned}$$

if $\delta>0$ is small enough. Hence,

$$\begin{aligned} &\int_0^T\!\!\int_{\mathbb{R}^n} |v(x,t)|^p\,|x'|^{\mu p-2p}\,dx\,dt \le \\ &\le c\int_0^T\!\!\int_{\mathbb{R}^n} |f(y,\tau)|^p\,|y'|^{p\mu}\,dy\,d\tau \int_\tau^T\!\!\int_{\mathbb{R}^n} \frac{L^{p-1}\,R_{x'}^{\sigma p}\,R_{y'}^{\kappa p}\,dx\,dt}{|x'|^{2p}\,(|x-y|^2+t-\tau)^{n/2}}\,, \end{aligned}$$

and since the last integral does not exceed

$$\begin{aligned} &c_1^{p-1}\,|y'|^{p\kappa}\int_\tau^\infty\!\!\int_{\mathbb{R}^n} \frac{dx\,dt}{|x'|^{2-p\sigma}\,(|x'|+|y'|+\sqrt{t-\tau})^{(\sigma+\kappa)p}(|x-y|^2+t-\tau)^{n/2}} \le \\ &\le c\,|y'|^{p\kappa}\int_{\mathbb{R}^2} \frac{dx'}{|x'|^{2-p\sigma}\,(|x'|+|y'|)^{(\sigma+\kappa)p-\delta_1}\,|x'-y'|^{\delta_1}} \le c_2\,, \end{aligned}$$

if $\delta_1>0$ is small, it follows that

$$\int_0^T\!\!\int_{\mathbb{R}^n} |v|^p\,|x'|^{\mu p-2p}\,dx\,dt \le c\,c_2\int_0^T\!\!\int_{\mathbb{R}^n} |f|^p\,|x'|^{\mu p}\,dx\,dt\,,$$

and (2.10) is proved. □

3 – Proof of Theorem 1.1

Let $D_x^2 G(x,y,t)$ be an arbitrary second derivative $\frac{\partial^2 G}{\partial x_i\,\partial x_j}$. The integral

$$\begin{aligned} w(x,t) &= \int_0^t d\tau \int_{D_\theta} D_x^2 G(x,y,t-\tau)\,f(y,\tau)\,dy \equiv \\ &\equiv \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} d\tau \int_{D_\theta} D_x^2 G(x,y,t-\tau)\,f(y,\tau)\,dy \end{aligned}$$

can be represented as a sum

$$\begin{aligned} w(x,t) &= \int_0^t d\tau \int_{S_1(x,t-\tau)} D_x^2 G(x,y,t-\tau)\, f(y,\tau)\, dy + \\ &+ \int_0^t d\tau \int_{S_2} D_x^2 G(x,y,t-\tau)\, f(y,\tau)\, dy \equiv \\ &\equiv w_1(x,t) + w_2(x,t) \end{aligned}$$

where

$$\begin{aligned} S_1(x,t-\tau) &= \{y \in D_\theta\colon\ |x-y|^2 + t - \tau \le \frac{1}{4}\,|y'|^2\}\,, \\ S_2(x,t-\tau) &= \{y \in D_\theta\colon\ |x-y|^2 + t - \tau \ge \frac{1}{4}\,|y'|^2\}\,. \end{aligned}$$

It was proved in [12] that in the case $y \in S_2(x,t-\tau)$ the function $D_x^2 G(x,y,t-\tau)$ satisfies the inequality (2.6) where $\lambda = 0$, if $\pi/\theta > 2$, $\lambda = 2 - \pi/\theta$, if $\pi/\theta < 2$, λ arbirarily small, positive, if $\pi/\theta = 2$. Hence,

$$|w_2(x,t)| \le c \int_0^t \int_{D_\theta} \frac{|f(y,\tau)|\, dy\, d\tau}{(|x-y|^2 + |x'|^2 + |y'|^2 + t - \tau)^{(n+2-\lambda)/2}\, |x'|^\lambda}\,,$$

and by the proposition 2.2

$$\int_0^T \int_{D_\theta} |w_2(x,t)|^p\, |x'|^{p\mu}\, dx\, dt \le c \int_0^T \int_{D_\theta} |f(y,\tau)|^p\, |y'|^{p\mu}\, dy\, d\tau \tag{3.1}$$

for arbitrary μ satisfying (1.5).

Further, in the case $y \in S_1(x,t-\tau)$ the function $D_x^2 G(x,y,t-\tau)$ was represented in the form

$$D_x^2 G(x,y,t-\tau) = D_x^2 G_0(x,y,t-\tau) + H(x,y,t-\tau)$$

with $H(x,y,t-\tau)$ satisfying the inequality

$$|H(x,y,t-\tau)| \le c\, |y'|^{-(n+2)} \le c(|x-y|^2 + |y'|^2 + |x'|^2 + t - \tau)^{-\frac{n+2}{2}}$$

and with G_0 depending on the location of the point y. Let us consider D_θ as a union of three dihedral angles

$$D_\theta = D^{(1)}_{\theta/3} \cup D^{(2)}_{\theta/3} \cup D^{(3)}_{\theta/3}\,,$$

where $D^{(i)}_{\theta/3} = d^{(i)}_{\theta/3} \times \mathbb{R}^{n-1}$,

$$d^{(1)}_{\theta/3} = \{r>0,\, 0<\varphi\le\theta/3\}\,,$$
$$d^{(2)}_{\theta/3} = \{r>0,\, 2\theta/3\le\varphi<\theta\}\,,$$
$$d^{(3)}_{\theta/3} = \{r>0,\, \theta/3\le\varphi<2\theta/3\}\,,$$

let $\mathbb{R}^{n-1}_i$, $i=1,2$, be a $n-1$-dimensional subspace of $\mathbb{R}^n$ containing Γ_i and let $\mathbb{R}^n_i$ be a half-space with $\partial\mathbb{R}^n_i = \mathbb{R}^{n-1}_i$ such that $\mathbb{R}^n_i \supset D^{(i)}_{\theta/3}$. If $y\in D^{(i)}_{\theta/3}$, then $G_0 \equiv G^{(i)}_0$ is the Green function of the Neumann initial-boundary value problem for the heat equation in the half-space $\mathbb{R}^n_i$, i.e.,

$$G^{(i)}_0(x,y,t-\tau) = \Gamma(x-y,t-\tau) + \Gamma(x-y^*_i,t-\tau)$$

where y^*_i is a point symmetric to y with respect to $\mathbb{R}^{n-1}_i$. Finally, for $y\in D^{(3)}_{\theta/3}$

$$G_0(x,y,t-\tau) = \Gamma(x-y,t-\tau)\,.$$

Hence, the function $w_1(x,t)\,|x'|^\mu$ can be represented in the form

$$\begin{aligned} w_1(x,t)\,|x'|^\mu =& \int_0^t\int_{S_1} D^2_x G f|y'|^\mu\,dyd\tau + \int_0^t\int_{S_1} D^2_x G f(|x'|^\mu - |y'|^\mu)dyd\tau = \\ =& \sum_{i=1}^3 \Big(\int_0^t\int_{D^{(i)}_{\theta/3}} D^2_x G^{(i)}_0(x,y,t-\tau)\,f(y,\tau)\,|y'|^\mu\,dy\,d\tau + \\ &-\int_0^t\int_{D^{(i)}_{\theta/3}\setminus S_1} D^2_x G^{(i)}_0(x,y,t-\tau)\,f(y,\tau)\,|y'|^\mu\,dy\,d\tau + \\ &+\int_0^t\int_{S_{i,1}} H_i(x,y,t-\tau)\,f(y,\tau)\,|y'|^\mu\,dy\,d\tau\Big) + \\ &+\int_0^t\int_{S_1} f(y,\tau)\,|y'|^\mu\,D^2_x G(x,y,t-\tau)\,\frac{|x'|^\mu - |y'|^\mu}{|y'|^\mu}\,dy\,d\tau \end{aligned}$$

where $S_{i,1} = S_1 \cap D^{(i)}_{\theta/3}$ and $H_i(x,y,t-\tau) = H(x,y,t-\tau)$, $y\in D^{(i)}_{\theta/3}$.

We observe that

$$|D^2_x G^{(i)}_0(x,y,t-\tau)| \le c(|x-y|^2 + |x'|^2 + |y'|^2 + t-\tau)^{-\frac{n+2}{2}}$$

if $y \in D^{(i)}_{\theta/3} \backslash S_1$, and

$$|D^2_x G(x,y,t-\tau)|\Big||x'|^{\mu} - |y'|^{\mu}\Big||y'|^{-\mu} \le c(|x-y|^2 + t-\tau)^{-\frac{n+1}{2}}|y'|^{-1} \le$$
$$\le c(|x-y|^2 + t-\tau)^{-\frac{n+1}{2}}(|x-y|^2 + |x'|^2 + t-\tau)^{-\frac{1}{2}}\,,$$

if $y \in S_1$. Hence,

$$|w_1(x,t)|\,|x'|^{\mu} \le \sum_{i=1}^{3} \Big| \int_0^t d\tau \int_{D^{(i)}_{\theta/3}} D^2_x G^{(i)}_0(x,y,t-\tau)\, f(y,\tau)\, |y'|^{\mu}\, dy\, d\tau \Big| +$$
$$+ \int_0^t \int_{D_\theta} |K(x,y,t-\tau)|\, |f(y,\tau)|\, |y'|^{\mu}\, dy\, d\tau$$

where K is a function satisfying inequality (2.4). Making use of (2.3) and of the Proposition 2.1, we obtain

$$\int_0^T \int_{D_\theta} |w_1(x,t)|^p\, |x'|^{p\mu}\, dx\, dt \le c \int_0^T \int_{D_\theta} |f(y,\tau)|^p\, |y'|^{p\mu}\, dy\, d\tau\,,$$

and, taking account of (3.1), we conclude that

$$\int_0^T \int_{D_\theta} |w(x,t)|^p\, |x'|^{p\mu}\, dx\, dt \le c \int_0^T \int_{D_\theta} |f(y,\tau)|^p\, |y'|^{p\mu}\, dy\, d\tau\,.$$

Since the function (1.7) satisfies the equation $\frac{\partial v}{\partial t} - \Delta v = f$, the same estimate holds for $\frac{\partial v}{\partial t}$. Theorem 1.1 is proved. ∎

4 – Proof of Theorem 1.2 and some generalizations

The estimate (1.6) for the solution of the problem (1.1), (1.2) is simpler because it reduces to the estimate of $\int_0^T \int_{D_\theta} |u|^p\, |x'|^{p\mu-2p}\, dx\, dt$, which makes it unnecessary to deal with singular integrals. The Green function $G(x,y,t)$ for the Dirichlet problem satisfies the inequality (2.9) with arbitrary $\lambda_i \in (0, \frac{\pi}{\theta})$, $i = 1, 2$ (in the case of the Neumann problem $\lambda_i = 0$). The reason of this is that for the solution of the Dirichlet problem the estimate in the spaces $H^{2k,k}_{\mu}(D_\theta \times (0,T))$ of V.A.Kondrat'iev's type holds (it is similar to the inequality (1.6) with $p = 2$ in [1] for the solution of the oblique derivative problem), and it can be used in the estimate of $G(x,y,t)$ (see (3.14) in [1]).

Hence, by virtue of Proposition 2.3,

$$\int_0^T\int_{D_\theta} |u(x,t)|^p\, |x'|^{p\mu-2p}\, dx\, dt \;\le\; c\int_0^T\int_{D_\theta} |f(x,t)|^p\, |x'|^{p\mu}\, dx\, dt$$

for arbitrary μ satisfying (1.7).

Inequality (1.6) for the derivatives of $u(x,t)$ follows from local estimates of the solution. Let $\xi = (\xi_3, ..., \xi_n) \subset \mathbb{R}^{n-2}$ and

$$B_p(\xi) = \{x = (x', x'') \in D_\theta \colon\; \frac{\rho}{2} < |x'| < \rho,\; |x_i - \xi_i| < \rho,\; i = 3, ..., n\}\,.$$

As shown, for instance, in [10] (see §19),

$$\int_0^T\int_{B_\rho(\xi)} \Big(\Big|\frac{\partial u}{\partial t}\Big|^p + |D^2u|^p + \rho^{-p}\,|\nabla u|^p\Big) dx\, dt \le$$
$$\le c\Big(\rho^{-2p}\int_0^T\int_{B_{2\rho}} |u|^p\, dx\, dt \,+\, \int_0^T\int_{B_{2\rho}} |f|^p\, dx\, dt\Big)\,.$$

Making use of this inequality and of appropriate partition of unity in D_θ, we easily obtain

$$\int_0^T\int_{D_\theta} \Big(\Big|\frac{\partial u}{\partial t}\Big|^p + |D^2u|^p\Big)\, |x'|^{p\mu}\, dx\, dt \,+\, \int_0^T\int_{D_\theta} |\nabla u|^p\, |x'|^{p\mu-p}\, dx\, dt \;\le$$
$$\le\; c\Big(\int_0^T\int_{D_\theta} |u|^p\, |x'|^{p\mu-2p}\, dx\, dt \,+\, \int_0^T\int_{D_\theta} |f|^p\, |x'|^{p\mu}\, dx\, dt\Big) \le$$
$$\le\; c\int_0^T\int_{D_\theta} |f|^p\, |x'|^{p\mu}\, dx\, dt$$

which concludes the proof of Theorem 1.2. ∎

Inequalities similar to (1.4), (1.6) hold also for higher order norms of the solutions of problems (1.1), (1.2) and (1.1), (1.3).

THEOREM 4.1. *The solution of the problem* (1.1), (1.3) *satisfies the* *inequalities*

$$\sum_{|j|+2\ell=2(k+1)} \int_0^T\!\!\int_{D_\theta} |D_x^j D_t^\ell u(x,t)|^p \, |x'|^{p\mu} \, dx \, dt \le \tag{4.1}$$
$$\le c \sum_{|j|+2\ell=2k} \int_0^T\!\!\int_{D_\theta} |D_x^j D_t^\ell f(x,t)|^p \, |x'|^{p\mu} \, dx \, dt ,$$

$k = 0, 1, ...,$ *provided that*

$$f(x,0) = 0 , \quad \dots , \quad \frac{\partial^{k-1} f}{\partial t^{k-1}}\Big|_{t=0} = 0 \tag{4.2}$$

and

$$\mu > -\frac{2}{p} , \quad 0 < \frac{2}{p'} + k - \mu < \frac{\pi}{\theta} . \tag{4.3}$$

Theorem 4.2. *The solution of the problem* (1.1), (1.2) *satisfies the inequalities*

$$\sum_{|j|+2\ell\le 2(k+1)} \int_0^T\!\!\int_{D_\theta} |D_x^j D_t^\ell u(x,t)|^p \, |x'|^{p\mu-2p(k+1)+(|j|+2\ell)p} \, dx \, dt \le \tag{4.4}$$
$$\le c \sum_{|j|+2\ell\le 2k} \int_0^T\!\!\int_{D_\theta} |D_x^j D_t^\ell f(x,t)|^p \, |x'|^{p\mu-2pk+(|j|+2\ell)p} \, dx \, dt ,$$

provided that (4.2) *holds and*

$$-\frac{\pi}{\theta} < \frac{2}{p'} + k - \mu < \frac{\pi}{\theta} . \tag{4.5}$$

Inequalities (4.1), (4.4) follow easily from (1.4) and (1.6) applied to derivatives of u with respect to t and x_j, $j > 2$, and from the estimates

$$\sum_{|j|=2(k+1)} \int_{d_\theta} |D_{x'}^j u_1(x')|^p \, |x'|^{p\mu} \, dx' \le c \sum_{|j|=2k} \int_{d_\theta} |D_{x'}^j f_1(x')|^p \, |x'|^{p\mu} \, dx' ,$$
$$\sum_{|j|\le 2(k+1)} \int_{d_\theta} |D_{x'}^j u_2(x')|^p \, |x'|^{p\mu-2p(k+1)+|j|p} \, dx' \le$$
$$\le c \sum_{|j|\le 2k} \int_{d_\theta} |D_{x'}^j f_2(x')|^p \, |x'|^{p\mu-2pk+|j|p} \, dx'$$

for the solutions of two-dimensional elliptic problems

$$\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\,u_1=f_1(x')\,,\quad x'\in d_\theta,\ \frac{\partial u_1}{\partial n}\Big|_{\partial d_\theta}=0\,,$$
$$\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\,u_2=f_2(x')\,,\quad x'\in d_\theta,\ u_2|_{\partial d_\theta}=0\,.$$

These estimates hold under conditions (4.3) and (4.5), respectively (see [11], [6]; for $p=2$ the estimate of the Neumann problem is obtained in [13]).

We note finally that the case of non-homogeneous initial and boundary conditions can be reduced to the case considered here by construction of auxiliary function satisfying these conditions.

REFERENCES

[1] M.G. Garroni – V.A. Solonnikov – M.A. Vivaldi: *Green function for the heat equation with oblique boundary conditions in an angle*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci., **27** (1999), 455-485.

[2] P. Grisvard: *Singularities in boundary value problems*, Springer, 1992.

[3] V.A. Kondrat'ev: *Boundary value problems for elliptic equations in domains with conical or angular points*, Trudy Mosk. Mat. Obshch., **16** (1967), 209-292.

[4] V.A. Kozlov: *On the asymptotics of the Green function and of the Poisson kernels for parabolic initial boundary value problem in a cone*, I: Zeitschr. Anal. Anw., **8** (1989), 131-151, II: ibid, **9** (1991), 27-42.

[5] V.A. Kozlov – V.G. Maz'ya: *On singularities of solutions of the first boundary value problem for the heat equation in domains with conical points*, Izv. Vyssh. Uch. Zaved., Matem., **3** (1987), 37-44.

[6] V.G. Maz'ya – B.A. Plamenevsky: *Estimates in L_p and in Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary*, Math. Nachr., **81** (1978), 25-82.

[7] V.G. Maz'ya – B.A. Plamenevsky: *L_p-estimates of solutions of elliptic boundary value problems in domains with edges*, Trudy Mosk. Mat. Obshch.**37** (1978), 49-93.

[8] V.G. Maz'ya – B.A. Plamenevsky: *Estimates of Green's functions and Schauder estimates for solutions of elliptic boundary value problems in a dihedral angle*, Sib. Mat. Zh., **19** (1978), 1065-1082.

[9] S.A. Nazarov – B.A. Plamenevsky: *Elliptic problems in domains with piecewise smooth boundaries*, De Gruyter, 1994.

[10] V.A. Solonnikov: *On boundary value problems for linear parabolic system of differential equations of general form*, Trudy Mat. Inst. Steklov, **83** (1965), 3-162.

[11] V.A. Solonnikov: *Estimates of the solution of the Neumann problem for the elliptic equation of the second order in domains with edges at the boundary*, Preprint LOMI P-4-83 (1983), p. 31.

[12] V.A. Solonnikov: *On the solvability of classical initial-boundary value problems for the heat equation in dihedral angle*, Zap. Nauchn. Semin. Lningr. Otdel. Mat. Inst. Steklova **138** (1984), 1323-1372.

[13] W. Zajączkowski – V.A. Solonnikov: *On the Neumann problem for second order elliptic equations in domains with edges at the boundaries*, Zap. Nauchn. Semin. Leningr. Otdel. Mat. Inst. Steklova, **127** (1983), 7-48.

Lavoro pervenuto alla redazione il 27 settembre 1999
ed accettato per la pubblicazione il 21 dicembre 1999.
Bozze licenziate il 17 luglio 2000

Indirizzo dell'Autore:

V.A. Solonnikov – V.A. Steklov Mathematical Institute – S. Petersburg Department – Fontanka 27 – 191011 S. Petersburg, Russia
E-mail: solonnik@pdmi.ras.ru