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# Morrey spaces and local regularity of minimizers of variational integrals

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Riassunto: *Si studia la regolarità (hölderianità, BMO, maggiore sommabilità) dei minimi locali di funzionali integrali del Calcolo delle Variazioni nel caso scalare. Si assume che la funzione integranda soddisfi una condizione di crescita in cui è presente una funzione che appartiene a un certo spazio di Morrey. Non si fanno ipotesi di differenziabilità. Risultati analoghi sono dimostrati per soluzioni deboli di equazioni ellittiche non lineari del tipo del p-laplaciano.*

Abstract: *We study the regularity (Hölder continuity, BMO, higher summability) of local minimizers of integral functionals of the Calculus of Variations in the scalar case. We assume that the integrand satisfies a growth condition involving a function which belongs to a certain Morrey space. No differentiability assumption is required. Analogous results are proved for weak solutions of nonlinear elliptic equations of p-Laplacian type.*

## 1 – Introduction

We consider an integral functional of the type

$$\mathcal{F}(v;\Omega) := \int_\Omega F(x, v(x), Dv(x))\, dx\,, \tag{1.1}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying the

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following growth assumption:

$$|z|^p \le F(x,v,z) \le L(|z|^p + |v|^p) + \varphi(x) \tag{1.2}$$

with $N > p > 1$, $L > 1$. We are concerned with the regularity of local minimizers in $W^{1,p}(\Omega)$ of such functional.

It is well known that if $\varphi$ is in $L^r_{\rm loc}(\Omega)$ with $r > \frac{N}{p}$ then a minimizer (or even a $Q$-minimizer) of $\mathcal{F}$ is locally Hölder continuous (see [6] and [11]). On the other hand, if $r \le \frac{N}{p}$ simple examples show that the minimizer needs not to be either continuous or locally bounded. In the case $r = \frac{N}{p}$, and under slightly more restrictive assumptions on $F$, in [8] it is proved that any minimizer $u$ is locally in $VMO(\Omega)$. If $1 < r < \frac{N}{p}$ summability results are stated in [4] for solutions of elliptic equations and in [9] for the case of minimizers; more precisely in [9] the authors prove that if

$$|z|^p \le F(x,v,z) \le L|z|^p + \varphi(x)$$

and $\varphi \in L^r_{\rm loc}(\Omega)$, then a local minimizer $u$ is in $L^{(pr)^*}_{\rm loc}(\Omega)$ (see also Remark 3.5 below).

As a further step in the study of regularity one can investigate the case when $\varphi$ belongs to intermediate spaces with respect to the $L^p$ spaces. In the framework of Lorentz spaces the problem has been studied in [8]. In particular it is proved that if $\varphi$ is in the Lorentz space $L^{\frac{N}{p},\frac{1}{p}}_{\rm loc}(\Omega)$ then $u$ is continuous (see Theorem 3.3 below); this result is sharp in the sense that examples exist proving that the continuity of a minimizer $u$ is no more guaranteed if $\varphi \in L^{\frac{N}{p},\frac{1}{p}+\varepsilon}_{\rm loc}(\Omega) \setminus L^{\frac{N}{p},\frac{1}{p}}_{\rm loc}(\Omega)$, for any $\varepsilon > 0$. Summability results for solutions of elliptic equations in this framework are stated in [2].

In this paper we study the regularity and the summability properties of local minimizers of a functional $\mathcal{F}$ satisfying assumptions (1.1) and (1.2) as $\varphi$ varies in a Morrey space $M^{r,\gamma}_{\rm loc}(\Omega)$ (see Definition 2.1 below), with $1 < r < \frac{N}{p}$. Our first result deals with the case when $\gamma \ge N - pr$.

THEOREM 1.1. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of a functional $\mathcal{F}$ of the type* (1.1) *satisfying* (1.2) *with $\varphi \in M^{r,\gamma}_{\rm loc}(\Omega)$, where $1 < r < \frac{N}{p}$ and $0 < \gamma < N$.*

(i) *If* $N - pr < \gamma < N$, *then* $u$ *is locally Hölder continuous in* $\Omega$*;*
(ii) *if* $\gamma = N - pr$, *then* $u$ *is locally in* $BMO(\Omega)$.

In other words this result states that, differently from what happens when $\gamma = 0$ (in this case the Morrey space $M^{r,0}_{\mathrm{loc}}(\Omega)$ reduces to the Lebesgue space $L^r_{\mathrm{loc}}(\Omega)$), we still have locally Hölder continuous minimizers when $\varphi \in L^r_{\mathrm{loc}}(\Omega)$ with $r < \frac{N}{p}$, provided that the decay of the integral of $|\varphi|^r$ in a cube $Q_\varrho$ is of order $\varrho^\gamma$ with $\gamma$ large enough. Moreover, in view of the Sobolev imbedding Theorem 2.5 (ii) (which is sharp), also the conclusion (ii) is sharp. In order to compare this result with what is proved in [8], notice also that if $r < \frac{N}{p}$ then $L^{\frac{N}{p},\frac{1}{p}}_{\mathrm{loc}}(\Omega) \subset L^{\frac{N}{p}}_{\mathrm{loc}}(\Omega) \subset M^{r,N-pr}_{\mathrm{loc}}(\Omega)$.

In the case $0 < \gamma < N - pr$ we may only expect, in the same spirit of [4], [2] and [9], to improve the summability properties of the minimizer $u$. To state the result let us set $q_\gamma = \frac{q(N-\gamma)}{N-\gamma-q}$ whenever $q$, $\gamma \geq 0$ and $N - \gamma - q > 0$.

THEOREM 1.2. *Let* $u$ *and* $\mathcal{F}$ *be as in Theorem* 1.1 *above. If* $\varphi \in M^{r,\gamma}_{\mathrm{loc}}(\Omega)$, *with* $1 < r < \frac{N}{p}$ *and* $0 < \gamma < N - pr$, *then* $u$ *is locally in* $M^{(pr)_\gamma(1-\delta),\gamma+\delta(N-\gamma)}(\Omega)$ *for any* $\delta > 0$ *such that* $(pr)_\gamma(1-\delta) \geq 1$.

We cannot expect $u$ to be in a Morrey space $M^{q,\lambda}_{\mathrm{loc}}(\Omega)$ with $q > (pr)_\gamma$ or with $q = (pr)_\gamma$ and $\lambda > \gamma$. In fact, in Section 4 we give an example showing that $u$ is neither in the Lebesgue space $L^{(pr)_\gamma+\varepsilon}_{\mathrm{loc}}(\Omega)$ nor in the Morrey space $M^{(pr)_\gamma,\gamma+\varepsilon}_{\mathrm{loc}}(\Omega)$, for any $\varepsilon > 0$. However, the question whether $u$ may belong or not to the borderline Morrey space $M^{(pr)_\gamma,\gamma}_{\mathrm{loc}}(\Omega)$ remains open. As far as we know the only case in which this inclusion holds is the special case $\gamma = 0$, proved in [9], in which the borderline Morrey space reduces to $L^{(pr)^*}_{\mathrm{loc}}(\Omega)$.

All the results which are proved here have a natural and almost straightforward counterpart in the case of nonlinear elliptic equations of $p$-Laplacian type. These results are stated at the end of Section 3.

## 2 – Notations and preliminaries

In the sequel $\Omega$ denotes a bounded open set in $\mathbb{R}^N$, $Q_R(x)$ is the $N$-dimensional cube centered in $x$ with sides of length $2R$ parallel to the axes, that is $\{y \in \mathbb{R}^N : |y_i - x_i| < R$, for any $1 \le i \le N\}$. We write $\Omega(x_0, R)$ instead of $\Omega \cap Q_R(x_0)$.

If $f$ is an integrable function we set

$$f_{Q_R(x_0)} = f_{x_0,R} = \fint_{Q_R(x_0)} f(x)\; dx = \frac{1}{|Q_R|} \int_{Q_R(x_0)} f(x)\; dx\,.$$

We omit $x_0$ if no confusion may arise. If $a < N$ then we set $a^* = \frac{Na}{N-a}$.

In the sequel the letter $c$ stands for a generic constant which may vary from line to line.

We recall now the definition of some functional spaces.

Definition 2.1. Let us consider a measurable function $f : \Omega \to \mathbb{R}$. For any $p \ge 1$, $\lambda \ge 0$ and $\mu > 0$ we say that

(i) $f$ is in the *Lorentz space* $L^{p,\mu}(\Omega)$ if

$$[f]_{L^{p,\mu}(\Omega)} := \left[\int_0^{+\infty} \left(f^*(s) s^{\frac{1}{p}}\right)^{\mu} \frac{ds}{s}\right]^{\frac{1}{\mu}} < \infty\,,$$

where $f^* : [0, +\infty) \to [0, +\infty)$ is the decreasing rearrangement of $f$ in $\Omega$, that is

$$f^*(s) := \sup\{t \ge 0 : |\{x \in \Omega : |f(x)| > t\}| > s\}\,;$$

(ii) $f$ is in the *Campanato space* $\mathcal{L}^{p,\lambda}(\Omega)$ if $f \in L^p(\Omega)$ and

$$[f]^p_{\mathcal{L}^{p,\lambda}(\Omega)} := \sup_{\substack{x\in\Omega \\ \varrho>0}} \varrho^{-\lambda} \int_{\Omega(x,\varrho)} |f - f_{\Omega(x,\varrho)}|^p\, dy < \infty\,;$$

(iii) $f$ is in the *Morrey space* $M^{p,\lambda}(\Omega)$ if $f \in L^p(\Omega)$ and

$$||f||^p_{M^{p,\lambda}(\Omega)} := \sup_{\substack{x\in\Omega \\ \varrho>0}} \varrho^{-\lambda} \int_{\Omega(x,\varrho)} |f|^p\, dy < \infty\,;$$

(iv) $f$ is in the *Sobolev-Morrey space* $W^{1,(p,\lambda)}(\Omega)$ if $f \in W^{1,p}(\Omega)$ and

$$||f||^p_{W^{1,(p,\lambda)}(\Omega)} := ||f||^p_{M^{p,\lambda}(\Omega)} + ||Df||^p_{M^{p,\lambda}(\Omega)} < \infty\,;$$

(v) $f$ is in $BMO(\Omega)$, where $\Omega$ is a cube in $\mathbb{R}^N$, if $f \in L^1(\Omega)$ and

$$[f]_{BMO(\Omega)} := \sup_{\tilde{Q}\in\mathcal{A}} \fint_{\tilde{Q}} |f - f_{\tilde{Q}}|\, dy \; < \infty\,,$$

where $\mathcal{A}$ is the family of cubes included in $\Omega$ with sides parallel to those of $\Omega$.

In particular we recall the following lemma (see [12], [13]) which will be used in the sequel.

LEMMA 2.2. *Let $\Omega$ be a bounded open set of $\mathbb{R}^N$ and let $Q$ be a cube in $\mathbb{R}^N$ with sides parallel to the coordinate axes. Then*
(I) *$f \in M^{p,\lambda}(\Omega)$ if and only if there exists $\delta > 0$ such that*

$$\sup_{\substack{x\in\Omega \\ 0<\varrho<\delta}} \varrho^{-\lambda} \int_{\Omega(x,\varrho)} |f|^p\, dy < \infty\,;$$

(II) *$||f||^p_{M^{p,\lambda}(Q)}$ is equivalent to*

$$\sup_{Q_\varrho(x)\subset Q} \varrho^{-\lambda} \int_{Q_\varrho(x)} |f|^p\, dy\,.$$

In the following proposition we briefly recall some properties of these spaces (see [1], [12], [13] for proofs and more details).

PROPOSITION 2.3. *Let $\Omega$ and $Q$ be as in Lemma 2.2. Then for any $p \geq 1$*
(a) *$M^{p,0}(\Omega) \equiv L^p(\Omega)$;*
(b) *$M^{p,\lambda}(\Omega) \subset \mathcal{L}^{p,\lambda}(\Omega)$ and $M^{p,\lambda}(Q) \equiv \mathcal{L}^{p,\lambda}(Q)$ if $0 \leq \lambda < N$;*
(c) *$M^{p,N}(\Omega) \equiv L^\infty(\Omega)$;*
(d) *$\mathcal{L}^{p,N}(Q) \equiv BMO(Q)$;*
(e) *$\mathcal{L}^{p,\lambda}(Q) \equiv C^{0,\frac{\lambda-N}{p}}(\overline{Q})$ if $N < \lambda \leq N + p$;*
(f) *$L^r(\Omega) \subset L^{p,\lambda}(\Omega) \subset L^{p,p}(\Omega) = L^p(\Omega) \subset L^{p,\mu}(\Omega) \subset L^q(\Omega)$ if $1 \leq q < p < r$ and $0 < \lambda < p < \mu$;*
(g) *$M^{p,\lambda}(\Omega) \subset M^{q,\mu}(\Omega)$ if and only if $1 \leq q \leq p$, $0 \leq \lambda, \mu \leq N$ and $\frac{N-\lambda}{p} \leq \frac{N-\mu}{q}$,*
*where the symbol $\equiv$ means that the spaces coincide and have equivalent norms.*

We will also make use of the following theorem, due to F. Riesz (see [15] and [16]).

THEOREM 2.4. *Let $Q$ be a cube in $\mathbb{R}^N$, $f \in L^1(Q)$ and $p > 1$. Then*

$$\int_Q |f|^p \, dy = \sup_P \sum_{Q_i \in P} \left[ |Q_i|^{-\left(1-\frac{1}{p}\right)} \int_{Q_i(x)} |f| \, dy \right]^p , \tag{2.1}$$

*where $P = \{Q_i\}$ is any decomposition of $Q$ in a finite number of subcubes with pairwise disjoint interiors and sides parallel to those of $Q$.*

Finally we prove an imbedding theorem for Sobolev-Morrey spaces. See also [5], [16], [13] and [12] for related results. In the sequel if $p, \lambda \geq 0$ and $N - \lambda - p > 0$, we set

$$p_\lambda := \frac{p(N-\lambda)}{N-\lambda-p} \, . \tag{2.2}$$

THEOREM 2.5. *Let $Q = Q_R(x_0)$ be a cube in $\mathbb{R}^N$. Let $f \in W^{1,(p,\lambda)}(Q)$ with $p \geq 1$, $0 \leq \lambda < N$.*
(i) *If $\lambda > N - p$ then $f \in C^{0,\frac{\lambda-(N-p)}{p}}(\overline{Q})$;*
(ii) *if $\lambda = N - p$ then $f \in BMO(Q)$;*
(iii) *if $\lambda < N - p$ then $f \in M^{p_\lambda,\lambda}(Q)$.*

PROOF. Poincaré inequality implies that for any $Q_\varrho(x) \subset \mathbb{R}^N$

$$\frac{1}{\varrho^p} \int_{Q \cap Q_\varrho(x)} |f - f_{Q\cap Q_\varrho(x)}|^p \, dy \leq c \int_{Q\cap Q_\varrho(x)} |Df|^p \, dy \leq c \varrho^\lambda$$

with $c$ not depending on $x$ or $\varrho$, so that $f \in \mathcal{L}^{p,\lambda+p}(Q)$. Using (e) and (d) of Proposition 2.3, we respectively obtain (i) and (ii).

Let us finally consider the case $\lambda < N - p$.

Let $P = \{Q_i\}$ be a decomposition of a cube $Q_\varrho(x) \subset Q$ in a finite number of subcubes having pairwise disjoint interiors and sides parallel to those of $Q_\varrho(x)$. Let us denote by $\mathcal{S}$ the family of such decompositions $P$ of

$Q_\varrho(x)$. Using Theorem 2.4, Hölder inequality and the Sobolev imbedding theorem, we have

$$\int_{Q_\varrho(x)} |f|^{p_\lambda}\, dy = \sup_{P\in\mathcal{S}} \sum_{Q_i\in P} \left[ |Q_i|^{-\left(1-\frac{1}{p_\lambda}\right)} \int_{Q_i} |f|\, dy \right]^{p_\lambda} \leq$$

$$\leq \sup_{P\in\mathcal{S}} \sum_{Q_i\in P} \left[ |Q_i|^{\frac{1}{p_\lambda}-\frac{1}{p^*}} \left( \int_{Q_i} |f|^{p^*}\, dy \right)^{\frac{1}{p^*}} \right]^{p_\lambda} \leq$$

$$\leq c \sup_{P\in\mathcal{S}} \sum_{Q_i\in P} \left[ |Q_i|^{\frac{1}{p_\lambda}-\frac{1}{p^*}} \left( \int_{Q_i} (|Df|^p + |f|^p)\, dy \right)^{\frac{1}{p}} \right]^{p_\lambda}$$

where $p_\lambda$ is defined as in (2.2). Since $f \in W^{1,(p,\lambda)}(Q_\varrho(x))$, it follows that

$$\int_{Q_\varrho(x)} |f|^{p_\lambda}\, dy \leq c \sup_{P\in\mathcal{S}} \sum_{Q_i\in P} \left[ |Q_i|^{-\frac{\lambda}{N}} \int_{Q_i} (|Df|^p + |f|^p)\ dy \right]^{\frac{p_\lambda}{p}-1} \times$$

$$\times \int_{Q_i} (|Df|^p + |f|^p)\ dy \leq$$

$$\leq c||f||^{p_\lambda - p}_{W^{1,(p,\lambda)}(Q_\varrho(x))} \int_{Q_\varrho(x)} (|Df|^p + |f|^p)\ dy\,.$$

From this inequality we have

$$\sup_{Q_\varrho(x)\subset Q} |Q_\varrho|^{-\frac{\lambda}{N}} \int_{Q_\varrho(x)} |f|^{p_\lambda}\, dy \leq$$

$$\leq c \sup_{Q_\varrho(x)\subset Q} ||f||^{p_\lambda - p}_{W^{1,(p,\lambda)}(Q_\varrho(x))} \left[ |Q_\varrho|^{-\frac{\lambda}{N}} \int_{Q_\varrho(x)} (|Df|^p + |f|^p)\ dy \right] \leq$$

$$\leq c||f||^{p_\lambda}_{W^{1,(p,\lambda)}(Q)}$$

and the theorem is proved because of (II) of Lemma 2.2. $\square$

## 3 – Proof of the main results

In this section we give the proof of the regularity results Theorems 1.1 and 1.2 stated in the introduction. Before that let us recall the following definition.

DEFINITION 3.1. We say that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a *Q-minimizer* of $\mathcal{F}$ if there exists a constant $Q$ such that

$$\mathcal{F}(u; K) \leq Q\mathcal{F}(v; K)$$

for any $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$, with $K = \mathrm{spt}(u-v) \subset\subset \Omega$.

In particular if $Q = 1$ we say that $u$ is a *local minimizer* of $\mathcal{F}$.

As we deal with local results it is not restrictive to assume global regularity assumptions on $\varphi$ (see (1.2)), instead of the corresponding local assumptions. Hence, from now on we will assume that $\varphi$ is in $L^r(\Omega)$, $M^{r,\lambda}(\Omega)$, $L^{s,\mu}(\Omega)$ etc. instead of $L^r_{\mathrm{loc}}(\Omega)$, $M^{r,\lambda}_{\mathrm{loc}}(\Omega)$, $L^{s,\mu}_{\mathrm{loc}}(\Omega)$.

Let us state an a priori estimate that will be crucial in the proofs.

PROPOSITION 3.2. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the functional* (1.1) *with the integrand function $F$ satisfying* (1.2).

*If $\varphi \in L^r(\Omega)$, with $1 < r \leq \frac{N}{p}$, then for any $\varepsilon > 0$ and for any $Q_\varrho(x) \subset Q_R(x) \subset \Omega$, with $R \leq 1$,*

$$\int_{Q_\varrho(x)} (|Du|^p + |u|^p)\, dy \leq c\left[\left(\frac{\varrho}{R}\right)^{N-p+p\sigma} + \varepsilon + R^p\right] \int_{Q_R(x)} (|Du|^p + |u|^p)\, dy +$$

$$+ c_\varepsilon R^{N(1-\frac{1}{r})}\left[\int_{Q_R(x)} |\varphi|^r\, dy\right]^{\frac{1}{r}},$$

*for some $0 < \sigma \leq 1$ and $c, c_\varepsilon > 0$ not depending on $x$, $\varrho$ or $R$.*

PROOF. The proof of the result closely follows the one of Proposition 3.6 in [8] . Henceforth we shall only indicate the necessary changes.

Define $F_0, F_1 : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$,

$$F_0(x, u, z) := \min\{F(x, u, z), L(|z|^p + |u|^p)\}\,,$$
$$F_1(x, u, z) := F(x, u, z) - F_0(x, u, z)\,.$$

By (1.2) it follows that

$$
\begin{aligned}
|z|^p \le F_0(x,u,z) &\le L(|z|^p + |u|^p)\,,\\
0 \le F_1(x,u,z) &\le \varphi(x)\,.
\end{aligned}
$$

Fixed $Q_R(x_0) \subset \Omega$, with $R \le 1$, we set $V = u + W_0^{1,1}(Q_R(x_0))$. We define the functional $\mathcal{F}_0 : V \to \mathbb{R} \cup \{+\infty\}$, by

$$
\mathcal{F}_0(w) := \int_{Q_R(x_0)} F_0(x,w,Dw)\,dx\,.
$$

For any $\delta > 0$ there exists $u_\delta \in V$ such that

$$
\mathcal{F}_0(u_\delta) \le \inf_{w\in V} \mathcal{F}_0(w) + \delta R^N\,.
$$

This inequality, together with the minimality of $u$, implies

$$
\begin{aligned}
\mathcal{F}_0(u) = \mathcal{F}(u) - &\int_{Q_R(x_0)} F_1(x,u,Du)\,dx \le\\
&\le \mathcal{F}_0(u_\delta) + \int_{Q_R(x_0)} [F_1(x,u_\delta,Du_\delta) - F_1(x,u,Du)]\,dx \le\\
&\le \inf_{w\in V} \mathcal{F}_0(w) + \delta R^N + H(R)\,,
\end{aligned}
$$

with

$$
H(R) := \int_{Q_R(x_0)} \varphi(x)\,dx\,.
$$

From now on, the proof goes as in [8] with obvious variations. $\square$

Using Proposition 3.2 instead of Proposition 3.6 of [8] , we can restate Theorems 3.9 and 3.1 of [8] under slightly more general assumptions. More precisely:

THEOREM 3.3. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of a functional $\mathcal{F}$ of the type* (1.1) *satisfying* (1.2), *with $1 < p < N$.*

(i) *If $\varphi \in L^{\frac{N}{p}}(\Omega)$ then $u \in VMO_{\mathrm{loc}}(\Omega)$;*

(ii) *if $\varphi \in L^{\frac{N}{p},\frac{1}{p}}(\Omega)$ then $u$ is continuous in $\Omega$.*

We are in position to prove Theorem 1.1, that is if $\varphi \in M^{r,\gamma}(\Omega)$, with $\gamma \in (N-pr, N)$, then a local minimizer $u$ of the functional (1.1) is in $C^{0,\alpha}_{\mathrm{loc}}(\Omega)$; if instead $\gamma = N - pr$, then $u \in BMO_{\mathrm{loc}}(\Omega)$.

Proof of Theorem 1.1. Let $Q_{2R_0}(x_0) \subset \Omega$. Proposition 3.2 implies that for any $\varepsilon > 0$, $x \in Q_{R_0}(x_0)$ and $Q_\varrho(x) \subset Q_R(x) \subset Q_{2R_0}(x_0)$, with $R \leq 1$,

$$\int_{Q_\varrho(x)} (|Du|^p + |u|^p)\, dy \leq c \left[ \left(\frac{\varrho}{R}\right)^{N-p+p\sigma} + \varepsilon + R^p \right] \int_{Q_R(x)} (|Du|^p + |u|^p)\, dy + \\ + c_\varepsilon R^{N-\frac{N}{r}+\frac{\gamma}{r}} ||\varphi||_{M^{r,\gamma}(\Omega)}\,,$$

where $c$ and $c_\varepsilon$ are constants not depending on $x$ or $R$. By an iteration argument (see *e.g.* [8] , Proposition 3.7) it follows that there exists $R_1 \leq \min\{1, R_0\}$ such that for any $x \in Q_{R_0}(x_0)$ and $\varrho < R \leq R_1$

$$\int_{Q_\varrho(x)} (|Du|^p + |u|^p)\, dy \leq c \left(\frac{\varrho}{R}\right)^{N-p+p\sigma'} \int_{Q_R(x)} (|Du|^p + |u|^p)\, dy + \\ + c\varrho^{N-\frac{N}{r}+\frac{\gamma}{r}} ||\varphi||_{M^{r,\gamma}(\Omega)}\,,$$

with $0 < \sigma' < \sigma$ and $c$ not depending on $x$ or $\varrho$. In particular for any $\varrho < R_1$

$$\int_{Q_\varrho(x)} (|Du|^p + |u|^p)\, dy \leq c \left(\frac{\varrho}{R_1}\right)^{N-p+p\sigma'} \int_{Q_{2R_0}(x_0)} (|Du|^p + |u|^p)\, dy + \\ + c\varrho^{N-\frac{N}{r}+\frac{\gamma}{r}} ||\varphi||_{M^{r,\gamma}(\Omega)}\,. \tag{3.2}$$

If $\gamma > N - pr$, taking $\alpha = \frac{1}{p}\min\{p\sigma', N - \frac{N}{r} + \frac{\gamma}{r} - (N-p)\}$, from (3.2) we have that

$$\frac{1}{\varrho^{N-p+p\alpha}} \int_{Q_\varrho(x)} (|Du|^p + |u|^p)\, dy \leq c$$

for any $x \in Q_{R_0}(x_0)$ and $0 < \varrho < R_1$, with $c$ not depending on $x$ or $\varrho$. This implies $u \in W^{1,(p,N-p+p\alpha)}(Q_{R_0}(x_0))$. By Theorem 2.5 (i) we have $u \in C^{0,\alpha}(\overline{Q}_{R_0}(x_0))$.

If $\gamma = N - pr$, from (3.2) we analogously have

$$\frac{1}{\varrho^{N-p}} \int_{Q_\varrho(x)} (|Du|^p + |u|^p)\, dy \leq c\,,$$

that implies $u \in W^{1,(p,N-p)}(Q_{R_0}(x_0))$. Theorem 2.5 (ii) yields that $u$ is in $BMO(Q_{R_0}(x_0))$. ∎

We turn now to the proof of Theorem 1.2. As a starting point we state some preliminary result.

LEMMA 3.4. *Let $u \in W^{1,p}(\Omega)$ be a $Q$-minimizer of $\mathcal{F}$, with $F$ satisfying assumptions* (1.1) *and* (1.2). *If $\varphi \in L^r(\Omega)$, with $1 < r < \frac{N}{p}$, then*

$$\int_{Q_\tau(x)\cap\{|u|\geq k\}} |Du|^p \, dy \leq c \int_{Q_t(x)\cap\{|u|\geq k\}} |\varphi| \, dy + \\ + c \int_{Q_t(x)\cap\{|u|\geq k\}} \frac{|u|^p}{(t-\tau)^p} \, dy$$

*holds for any $0 \leq \tau < t \leq R_0$, $Q_{R_0}(x) \subset \Omega$ and $k \geq 0$, where $c$ is a constant not depending on $x$ or $R_0$.*

A proof of this lemma for local minimizers of $\mathcal{F}$ with $F$ satisfying the growth condition

$$|z|^p \leq F(x, v, z) \leq L|z|^p + \varphi$$

can be found *e.g.* in [9]. The same proof actually holds for $Q$-minimizers of $\mathcal{F}$ with $F$ satisfying the growth condition (1.2).

REMARK 3.5. We notice that, using Lemma 3.4 above in place of Theorem 4.1 of [9], the regularity result stated in Theorem 2.1 of [9] can be extended, with no other changes in the proof, also to $Q$-minimizers. More precisely we have that *if $u$ is a $Q$-minimizer of a functional $\mathcal{F}$ with $F$ satisfying* (1.2) *and $\varphi \in L^r(\Omega)$, then $u$ is locally in $L^{(pr)^*}(\Omega)$.*

The next lemma is proved following an idea contained in [4].

LEMMA 3.6. *Let $\varphi \in L^r(Q_R(x))$ and $u \in W^{1,p}(Q_R(x)) \cap L^{pq}(Q_R(x))$, with $q > 1$, be two functions such that for any $k \geq 0$*

$$\int_{Q_\varrho(x)\cap\{|u|\geq k\}} |Du|^p \, dy \leq c \int_{Q_{2\varrho}(x)\cap\{|u|\geq k\}} \left(|\varphi| + \frac{|u|^p}{\varrho^p}\right) dy \tag{3.3}$$

*with $0 < \varrho \leq \frac{R}{2}$.*

(i) *If* $q < r$ *then*

$$\int_{Q_\varrho(x)} (|D|u|^q|^p + |u|^{pq})\, dy \leq$$
$$\leq c\varrho^{\frac{N}{q}-\frac{N}{r}} \left(\int_{Q_{2\varrho}(x)} |\varphi|^r\, dy\right)^{\frac{1}{r}} \left(\int_{Q_{2\varrho}(x)} (|u|+1)^{pq}\, dy\right)^{1-\frac{1}{q}} +$$
$$+ c\varrho^{-p} \int_{Q_{2\varrho}(x)} (|u|+1)^{pq}\, dy\,;$$

(ii) *if* $q \geq r$ *then*

$$\int_{Q_\varrho(x)} \left(|D|u|^{q-\frac{q}{r}+1}|^p + |u|^{p(q-\frac{q}{r}+1)}\right) dy \leq$$
$$\leq c\left(\int_{Q_{2\varrho}(x)} |\varphi|^r\, dy\right)^{\frac{1}{r}} \left(\int_{Q_{2\varrho}(x)} (|u|+1)^{pq}\, dy\right)^{1-\frac{1}{r}} +$$
$$+ c\varrho^{\frac{N}{r}-\frac{N}{q}-p} \left(\int_{Q_{2\varrho}(x)} (|u|+1)^{pq}\, dy\right)^{1-\frac{1}{r}+\frac{1}{q}}$$

*where* $c$ *is a constant not depending on* $x$, $\varrho$ *or* $R$.

Proof. Let us write $Q_R$ instead of $Q_R(x)$ and let $Q(n) = \{x \in Q_R : n \leq |u| < n+1\}$. Let $m$ be a positive number to be chosen later.

By (3.3) it follows that

$$\sum_{k=0}^{\infty}(k+1)^{pm-1}\sum_{n=k}^{\infty}\int_{Q_\varrho\cap Q(n)} |Du|^p\, dy \leq$$
$$\leq c\sum_{k=0}^{\infty}(k+1)^{pm-1}\sum_{n=k}^{\infty}\left[\int_{Q_{2\varrho}\cap Q(n)}\left(|\varphi| + \frac{|u|^p}{\varrho^p}\right) dy\right].$$

Exchanging the summation order we get

$$\sum_{n=0}^{\infty}\int_{Q_\varrho\cap Q(n)} |Du|^p \sum_{k=0}^{n}(k+1)^{pm-1}\, dy \leq$$
$$\leq c\sum_{n=0}^{\infty}\left[\int_{Q_{2\varrho}\cap Q(n)}\left(|\varphi| + \frac{|u|^p}{\varrho^p}\right)\sum_{k=0}^{n}(k+1)^{pm-1}\, dy\right].$$

Since $m > 0$, there exist two positive constants $c_1$, $c_2$ not depending on $n$ such that

$$c_1(n+1)^{pm} \leq \sum_{k=0}^{n}(k+1)^{pm-1} \leq c_2(n+1)^{pm}\,,$$

so that the previous inequality yields

$$\int_{Q_\varrho} |D|u|^{m+1}|^p\,dy \leq c\int_{Q_{2\varrho}} |\varphi|(|u|+1)^{pm}\,dy + c\varrho^{-p}\int_{Q_{2\varrho}} |u|^p(|u|+1)^{pm}dy.$$

Adding $\int_{Q_\varrho} |u|^{p(m+1)}\,dy$ to both sides we get

$$\begin{aligned}\int_{Q_\varrho}(|D|u|^{m+1}|+|u|^{m+1})^p\,dy &\leq c\int_{Q_\varrho}|u|^{p(m+1)}\,dy+\\ &+c\int_{Q_{2\varrho}}|\varphi|(|u|+1)^{pm}\,dy + c\varrho^{-p}\int_{Q_{2\varrho}}(|u|+1)^{p(m+1)}\,dy \leq\\ &\leq c\int_{Q_{2\varrho}}|\varphi|(|u|+1)^{pm}\,dy + c\varrho^{-p}\int_{Q_{2\varrho}}(|u|+1)^{p(m+1)}\,dy\,.\end{aligned}$$

Since $\varphi \in L^r(Q_R)$, using Hölder inequality we obtain that

$$\begin{aligned}\int_{Q_\varrho}(|D|u|^{m+1}|+|u|^{m+1})^p dy \leq c\left(\int_{Q_{2\varrho}}|\varphi|^r dy\right)^{\frac{1}{r}}\left(\int_{Q_{2\varrho}}(|u|+1)^{\frac{pmr}{r-1}}dy\right)^{1-\frac{1}{r}}&+\\ +c\varrho^{-p}\int_{Q_{2\varrho}}(|u|+1)^{p(m+1)}\,dy\,.&\end{aligned} \tag{3.4}$$

If $q < r$ we choose $m$ such that $m+1 = q$, thus $m\frac{r}{r-1} = r\frac{q-1}{r-1} < q$. Using Hölder inequality in (3.4), we get (i).

Analogously, if $q \geq r$ we choose $m$ such that $m\frac{r}{r-1} = q$ and so $m+1 = q - \frac{q}{r} + 1 \leq q$. Hölder inequality again gives (ii). □

The following proposition gives a first information on the summability of $u$.

Proposition 3.7. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of a functional $\mathcal{F}$ of the type* (1.1) *satisfying* (1.2) *where $\varphi \in M^{r,\gamma}(\Omega)$ with $1 < r < \frac{N}{p}$ and $0 < \gamma < N - pr$. Then $u$ is locally in $M^{pr,\gamma+pr}(\Omega)$.*

Proof. As in the proof of Theorem 1.1, (3.2) holds; since $\gamma < N - pr$ we have

$$\frac{1}{\varrho^{N-\frac{N}{r}+\frac{\gamma}{r}}}\int_{Q_\varrho(x)} (|Du|^p + |u|^p)\, dy \le c$$

where $c$ is a constant not depending on $x$ or $\varrho$. This implies that $u$ is locally in $W^{1,(p,N-\frac{N}{r}+\frac{\gamma}{r})}(\Omega)$ with $N - \frac{N}{r} + \frac{\gamma}{r} < N - p$. By Theorem 2.5 (iii) we have that $u$ is locally in $M^{pq_0,\lambda_0}(\Omega)$, with $\lambda_0 = N - \frac{N}{r} + \frac{\gamma}{r}$ and $q_0 = \frac{N-\lambda_0}{N-\lambda_0-p}$.

If $q_0 = r$ the thesis is proved. If $q_0 > r$ Proposition 2.3 (g) implies that $u \in M^{pr,\gamma+pr}_{\rm loc}(\Omega)$. If $q_0 < r$ let us define

$$\lambda_i := \lambda_0 - ip\,,$$
$$q_i := \frac{N-\lambda_i}{N-\lambda_0-p}$$

for any $i \in \mathbb{N}$. Let $n \in \mathbb{N}\cup\{0\}$ be such that $q_n < r \le q_{n+1}$. By Lemma 3.4 and Lemma 3.6 (i) if $u \in M^{pq_i,\lambda_i}_{\rm loc}(\Omega)$ and $i \le n$ then

$$\int_{Q_\varrho(x)} (|D|u|^{q_i}|^p + |u|^{pq_i})\, dy \le c\varrho^{\frac{N}{q_i}-\frac{N}{r}+\frac{\gamma}{r}+\lambda_i(1-\frac{1}{q_i})} + c\varrho^{\lambda_i-p} \le c\varrho^{\lambda_{i+1}}\,,$$

so that $|u|^{q_i}$ is locally in $W^{1,(p,\lambda_{i+1})}(\Omega)$. Theorem 2.5 (iii) implies that $|u|^{q_i}$ is in $M^{p\frac{q_{i+1}}{q_i},\lambda_{i+1}}_{\rm loc}(\Omega)$ and henceforth $u \in M^{pq_{i+1},\lambda_{i+1}}_{\rm loc}(\Omega)$. Iterating this argument we get $u \in M^{pq_{n+1},\lambda_{n+1}}_{\rm loc}(\Omega)$.

As before, if $q_{n+1} = r$ we have the result; if $q_{n+1} > r$ we use Proposition 2.3 (g). ∎

We are now in position to prove Theorem 1.2.

Proof of Theorem 1.2. For any $i \in \mathbb{N}\cup\{0\}$ let

$$\gamma_i := \gamma + pr\left(1-\frac{1}{r}\right)^i,$$
$$q_i := r\frac{N-\gamma_i}{N-\gamma-pr}\,.$$

The sequence $\gamma_i$ is decreasing and converging to $\gamma$ and the sequence $q_i$ is increasing and converging to $r\frac{N-\gamma}{N-\gamma-pr}$.

We will prove by induction that $u \in M^{pq_i,\gamma_i}_{\rm loc}(\Omega)$ for any $i \in \mathbb{N}$.

As $\gamma_0 = \gamma + pr$ and $q_0 = r$, by Proposition 3.7 $u \in M^{pq_0,\gamma_0}_{\mathrm{loc}}(\Omega)$ and the first step of the induction is proved.

Suppose now that $u \in M^{pq_n,\gamma_n}_{\mathrm{loc}}(\Omega)$. Since $q_n \geq r$ we can apply Lemma 3.6 (ii) with $q = q_n$. Noting that $N/r - N/q_n - p + \gamma_n/q_n = \gamma/r$ and $\gamma/r + \gamma_n(1 - 1/r) = \gamma_{n+1}$, we have

$$\int_{Q_\varrho(x)} \left(|D|u|^{q_n - \frac{q_n}{r}+1}|^p + |u|^{p(q_n - \frac{q_n}{r}+1)}\right) dy \leq$$
$$\leq c\varrho^{\frac{\gamma}{r}+\gamma_n(1-\frac{1}{r})} + c\varrho^{\frac{N}{r}-\frac{N}{q_n}-p+\gamma_n(1-\frac{1}{r}+\frac{1}{q_n})} \leq c\varrho^{\gamma_{n+1}},$$

that is $|u|^{q_n - \frac{q_n}{r}+1} \in W^{1,(p,\gamma_{n+1})}_{\mathrm{loc}}(\Omega)$. Since

$$q_n - \frac{q_n}{r} + 1 = r\frac{N - \gamma_{n+1} - p}{N - \gamma - pr},$$

Theorem 2.5 (iii) implies $u \in M^{pq_{n+1},\gamma_{n+1}}_{\mathrm{loc}}(\Omega)$.

For any $\varepsilon > 0$ let $n \in \mathbb{N}$ such that $q_n > r\frac{N-\gamma-\varepsilon}{N-\gamma-pr}$. As $u \in M^{pq_n,\gamma_n}_{\mathrm{loc}}(\Omega)$, Proposition 2.3 (g) implies $u \in M^{(pr)\gamma(1-\delta),\gamma+\delta(N-\gamma)}_{\mathrm{loc}}(\Omega)$ where $\delta = \frac{\varepsilon}{N-\gamma}$. $\square$

As in [8] and [9] analogous regularity results can be proved in the case of weak solutions of nonlinear partial differential equations.

Let us consider the equation

$$\operatorname{div}(A(x,u,Du)) + H(x,u) = \operatorname{div} f \tag{3.5}$$

where $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $H : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying the following assumptions:

(I) $|A(x,\eta,\xi)| \leq L(1 + |\xi|^{p-1}) \quad \forall (x,\eta,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$;

(II) For all $(x,\eta) \in \Omega \times \mathbb{R}$ and $\xi_1, \xi_2 \in \mathbb{R}^N$

$$\begin{cases} \langle A(x,\eta,\xi_1) - A(x,\eta,\xi_2), \xi_1 - \xi_2 \rangle \geq \nu|\xi_1 - \xi_2|^p & \text{if } p \geq 2 \\ \langle A(x,\eta,\xi_1) - A(x,\eta,\xi_2), \xi_1 - \xi_2 \rangle \geq & \\ \qquad \geq \nu|\xi_1 - \xi_2|^2(|\xi_1|^2 + |\xi_2|^2)^{\frac{p-2}{2}} & \text{if } 1 < p < 2; \end{cases}$$

(III) $|H(x,\eta)| \leq L(1 + |\eta|^{p-1}) \quad \forall (x,\eta) \in \Omega \times \mathbb{R}$.

THEOREM 3.8. *Let $u \in W^{1,p}(\Omega)$, $1 < p < N$, be a weak solution of* (3.5) *with $A$ and $H$ satisfying assumptions* (I), (II), (III) *and $|f|^{\frac{p}{p-1}} \in M^{r,\gamma}(\Omega)$, with $1 < r < \frac{N}{p}$ and $0 < \gamma < N$.*

(i) *If $N - pr < \gamma < N$ then $u$ is locally Hölder continuous in $\Omega$;*

(ii) *if $\gamma = N - pr$ then $u$ is locally in $BMO(\Omega)$;*

(iii) *if $0 < \gamma < N - pr$ then $u$ is locally in $M^{(pr)_\gamma(1-\delta),\gamma+\delta(N-\gamma)}(\Omega)$ for any $\delta > 0$ such that $(pr)_\gamma(1-\delta) \geq 1$.*

PROOF. Under the assumptions of the theorem an analogous result to Proposition 3.2 is proved in [8] with (3.1) replaced by

$$\int_{Q_\varrho(x)} (|Du|^p + |u|^p)\, dy \leq c\left[\left(\frac{\varrho}{R}\right)^{N-p+p\sigma} + \varepsilon\right]\int_{Q_R(x)} (|Du|^p + |u|^p)\, dy + \\ + c_\varepsilon \int_{Q_R(x)} \left(|f|^{\frac{p}{p-1}} + 1\right) dy\,.$$

On the other hand it is easy to prove that Lemma 3.4 still holds (see the proof of Theorem 5.1 in [9]). Arguing as in the proofs of Theorems 1.1 and 1.2 with $\varphi = |f|^{\frac{p}{p-1}}$, the thesis follows. ∎

A different approach has been used in [14] (see also [7]) to prove analogous regularity results for weak solutions of linear elliptic differential equations in the case $p = 2$ and $r = 1$.

## 4 – Example

We now give the example mentioned in the introduction. This example is inspired by the one given in Section 10 of [13].

In the sequel $N = 3$, $p = 2$, $1 < r < \frac{3}{2}$ and $0 < \gamma < 3 - 2r$. Let

$$r_i = (2^i)^{-\frac{1}{\gamma}}\,, \quad d_i = \left(e^{2^i}\right)^{-\frac{1}{3-\gamma}}\,, \quad \psi_i = \left[\left(\frac{{r_i}^\gamma}{{d_i}^3}\frac{1}{e^{2^i}}\frac{1}{i^2}\right)^{\frac{1}{3}}\right]^3\,,$$

where $i \in \mathbb{N}$ and $[\cdot]$ stands for the integer part. Let $S_i = \{(x,y,z) \in Q_{r_i}(0) : r_{i+1} < z < r_i\}$. In each $S_i$ let us consider $\psi_i$ cubes $D_{ij} =$

$Q_{d_i/2}(c_{ij})$ where the centers $c_{ij} = (c_{ij_1}, c_{ij_2}, c_{ij_3})$ are defined as follows:

$$
\begin{aligned}
c_{ij_1} &= -r_i + \frac{d_i}{2} + j_1 \frac{2r_i - d_i}{\psi_i^{1/3} - 1}\,, \\
c_{ij_2} &= -r_i + \frac{d_i}{2} + j_2 \frac{2r_i - d_i}{\psi_i^{1/3} - 1}\,, \\
c_{ij_3} &= r_i - \frac{d_i}{2} - j_3 \frac{r_i - r_{i+1} - d_i}{\psi_i^{1/3} - 1}\,,
\end{aligned}
$$

for $j_1, j_2, j_3 = 0, 1, \ldots, \psi_i^{1/3} - 1$. Let

$$
i_0 = \min \left\{ i \in \mathbb{N} \,:\, \frac{r_i - r_{i+1} - d_i}{\psi_i^{1/3} - 1} \geq 2d_i \right\}.
$$

We notice that if $i \geq i_0$ the cubes $D_{ij}$ are disjoint.

We divide each $D_{ij}$ into six pyramids. In particular we define

$$
\begin{aligned}
D_{ij}^{x^+} &= \{(x,y,z) \in D_{ij} : x - c_{ij_1} > |y - c_{ij_2}|, |z - c_{ij_3}|\}\,, \\
D_{ij}^{x^-} &= \{(x,y,z) \in D_{ij} : c_{ij_1} - x > |y - c_{ij_2}|, |z - c_{ij_3}|\}\,, \\
D_{ij}^{y^+} &= \{(x,y,z) \in D_{ij} : y - c_{ij_2} > |x - c_{ij_1}|, |z - c_{ij_3}|\}\,, \\
D_{ij}^{y^-} &= \{(x,y,z) \in D_{ij} : c_{ij_2} - y > |x - c_{ij_1}|, |z - c_{ij_3}|\}\,, \\
D_{ij}^{z^+} &= \{(x,y,z) \in D_{ij} : z - c_{ij_3} > |x - c_{ij_1}|, |y - c_{ij_2}|\}\,, \\
D_{ij}^{z^-} &= \{(x,y,z) \in D_{ij} : c_{ij_3} - z > |x - c_{ij_1}|, |y - c_{ij_2}|\}\,.
\end{aligned}
$$

Let $\Omega = Q_{r_{i_0}}(0)$. Let us now define the function $u : \Omega \to \mathbb{R}$, such that

$$
u(x,y,z) = \begin{cases}
0 & \text{if } (x,y,z) \in \Omega \setminus \cup_{i,j} \bar{D}_{ij} \\
\left(e^{2^i}\right)^{1/2r} (d_i - |x - c_{ij_1}|) & \text{if } (x,y,z) \in D_{ij}^{x^+} \cup D_{ij}^{x^-} \\
\left(e^{2^i}\right)^{1/2r} (d_i - |y - c_{ij_2}|) & \text{if } (x,y,z) \in D_{ij}^{y^+} \cup D_{ij}^{y^-} \\
\left(e^{2^i}\right)^{1/2r} (d_i - |z - c_{ij_3}|) & \text{if } (x,y,z) \in D_{ij}^{z^+} \cup D_{ij}^{z^-}\,.
\end{cases}
$$

Let $f : \Omega \to \mathbb{R}^3$ be the vector valued function $(f_1, f_2, f_3)$ such that

$$f(x,y,z) = \begin{cases} (0,0,0) & \text{if } (x,y,z) \in \Omega \setminus \cup_{i,j} \bar{D}_{ij} \\ \left(\mp \left(e^{2^i}\right)^{1/2r}, 0, 0\right) & \text{if } (x,y,z) \in D_{ij}^{x\pm} \\ \left(0, \mp \left(e^{2^i}\right)^{1/2r}, 0\right) & \text{if } (x,y,z) \in D_{ij}^{y\pm} \\ \left(0, 0, \mp \left(e^{2^i}\right)^{1/2r}\right) & \text{if } (x,y,z) \in D_{ij}^{z\pm} . \end{cases}$$

We have that $u$ is a $W^{1,2}(\Omega)$ solution of $\Delta u = \mathrm{div} f$. Hence $u$ is a local minimizer of the integral functional

$$\mathcal{F}(v;\Omega) = \int_\Omega \left( \frac{1}{2} |Dv(x)|^2 - \langle Dv(x), f(x) \rangle \right) dx .$$

Setting $\varphi = |f|^2$, that is

$$\varphi(x,y,z) = \begin{cases} 0 & \text{if } (x,y,z) \in \Omega \setminus \cup_{i,j} \bar{D}_{ij} \\ (e^{2^i})^{1/r} & \text{if } (x,y,z) \in D_{ij} , \end{cases}$$

we prove that

1) $\varphi$ is in $M^{r,\gamma}(\Omega)$, but neither in $L^{r+\varepsilon}(\Omega)$ nor in $M^{r,\gamma+\varepsilon}(\Omega)$.
2) $u$ is in $M^{(2r)_\gamma,\gamma}_{\mathrm{loc}}(\Omega)$, but neither in $L^{(2r)_\gamma+\varepsilon}_{\mathrm{loc}}(\Omega)$ nor in $M^{(2r)_\gamma,\gamma+\varepsilon}_{\mathrm{loc}}(\Omega)$.

1) *i)* Let us prove that $\varphi$ is in $L^r(\Omega)$, but neither in $L^{r+\varepsilon}(\Omega)$ nor in $M^{r,\gamma+\varepsilon}(\Omega)$.

For any $i \geq i_0$ is

$$\int_{Q_{r_i}(0)} |\varphi|^{r+\varepsilon} = \sum_{k \geq i} \int_{S_k} |\varphi|^{r+\varepsilon} = \sum_{k \geq i} \psi_k d_k^3 \left(e^{2^k}\right)^{1+\frac{\varepsilon}{r}} \sim \sum_{k \geq i} \frac{1}{2^k} \frac{1}{k^2} \left(e^{2^k}\right)^{\frac{\varepsilon}{r}} .$$

So, for any $\varepsilon > 0$, $\varphi$ is not in $L^{r+\varepsilon}(\Omega)$. Observing that for any $\mu \geq 0$

$$\frac{1}{r_i^{\gamma+\mu}} \sum_{k \geq i} \frac{1}{2^k} \frac{1}{k^2} \geq \frac{1}{i^2} (2^i)^{\frac{\mu}{\gamma}}$$

we have that $\varphi$ is not in $M^{r,\gamma+\mu}(\Omega)$ if $\mu > 0$.

*ii*) Let us prove that $\varphi$ is in $M^{r,\gamma}(\Omega)$.

Denote by $\tilde{Q}$ a cube $Q_\varrho(x,y,z)$. We will consider three different cases:

Case I. $\tilde{Q} \subset S_i$ for some $i$.

If $\varrho \le d_i/2$, $\tilde{Q}$ intersecates at most one of the cubes $D_{ij}$ and we have:

$$\frac{1}{\varrho^\gamma}\int_{\tilde{Q}\cap D_{ij}} |\varphi|^r \le \frac{1}{\varrho^\gamma}\int_{Q_\varrho(c_{ij})} |\varphi|^r = c\varrho^{3-\gamma}e^{2^i} \le cd_i^{3-\gamma}e^{2^i}\,.$$

If $\varrho > d_i/2$, the number of cubes $D_{ij}$ intersecated by $\tilde{Q}$ is less than $c\frac{\varrho^3}{r_i^3}\psi_i$, so that

$$\frac{1}{\varrho^\gamma}\int_{\tilde{Q}} |\varphi|^r \le c\frac{\varrho^{3-\gamma}}{r_i^3}\psi_i d_i^3 e^{2^i} \sim c\frac{\varrho^{3-\gamma}}{r_i^{3-\gamma}}\frac{1}{i^2}$$

as $\varrho < r_i$.

Case II. $\tilde{Q}\cap S_i \ne \emptyset$, $\tilde{Q}\cap S_{i+1} \ne \emptyset$, $\tilde{Q}\cap S_j = \emptyset$ for $j \ne i, i+1$.

If $\varrho \le (r_{i+1} - r_{i+2})/2$

$$\begin{aligned}\frac{1}{\varrho^\gamma}\int_{\tilde{Q}} |\varphi|^r &\le \frac{1}{\varrho^\gamma}\left(\int_{Q_\varrho(x,y,r_{i+1}+\varrho)} |\varphi|^r + \int_{Q_\varrho(x,y,r_{i+1}-\varrho)} |\varphi|^r\right) \le \\ &\le \frac{c}{\varrho^\gamma}\left(\frac{\varrho^3}{r_i^3}\psi_i d_i^3 e^{2^i} + \frac{\varrho^3}{r_{i+1}^3}\psi_{i+1}d_{i+1}^3 e^{2^{i+1}}\right) \le \frac{c}{i^2}.\end{aligned}$$

If $\varrho > (r_{i+1} - r_{i+2})/2$ we have $\varrho > \left(\frac{1}{2}\right)^{1+\frac{1}{\gamma}}\left[1-\left(\frac{1}{2}\right)^{\frac{1}{\gamma}}\right] r_i$ and

$$\begin{aligned}\frac{1}{\varrho^\gamma}\int_{\tilde{Q}} |\varphi|^r &\le \frac{1}{\varrho^\gamma}\left(\int_{S_i} |\varphi|^r + \int_{S_{i+1}} |\varphi|^r\right) \le \\ &\le \frac{c}{r_i^\gamma}\left(\psi_i d_i^3 e^{2^i} + \psi_{i+1}d_{i+1}^3 e^{2^{i+1}}\right) \le \frac{c}{i^2}\end{aligned}$$

with $c$ depending only on $\gamma$.

Case III. $\tilde{Q}\cap S_j \ne \emptyset$, for $j = i, i+1, \ldots, i+n$ with $n \ge 2$.

It follows again that $\varrho > (r_{i+1} - r_{i+2})/2 > \left(\frac{1}{2}\right)^{1+\frac{1}{\gamma}}\left[1-\left(\frac{1}{2}\right)^{\frac{1}{\gamma}}\right] r_i$ and

$$\frac{1}{\varrho^\gamma}\int_{\tilde{Q}} |\varphi|^r \le \frac{1}{\varrho^\gamma}\sum_{j\ge i}\int_{S_j} |\varphi|^r \le c\frac{1}{\varrho^\gamma}\sum_{j\ge i}\psi_j d_j^3 e^{2^j} \le c\frac{r_i^\gamma}{\varrho^\gamma} \le c^\gamma$$

with $c$ depending only on $\gamma$.

2) Let us prove that $u$ is in $M^{(2r)_\gamma,\gamma}_{\rm loc}(\Omega)$, but not in $L^{(2r)_\gamma+\varepsilon}_{\rm loc}(\Omega)$ nor in $M^{(2r)_\gamma,\gamma+\varepsilon}_{\rm loc}(\Omega)$.

Simple calculations show that

$$\int_{D_{ij}} |u|^{(2r)_\gamma} = c\left(e^{2^i}\right)^{\frac{1}{2r}(2r)_\gamma} d_i^{3+(2r)_\gamma}$$

where $c$ does not depend on $i$. By definition of $d_i$ we have that

$$\int_{D_{ij}} |u|^{(2r)_\gamma} = ce^{2^i}{d_i}^3 = c\int_{D_{ij}} |\varphi|^r$$

and the thesis follows arguing as in 1).

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