Rendiconti di Matematica, Serie VII
Volume 21, Roma (2001), 143-157

Biharmonic curves on a surface

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Riassunto: *Questo articolo è dedicato allo studio delle curve biarmoniche su una superficie. Queste curve sono i punti critici del funzionale bienergia e generalizzano le curve armoniche (le geodetiche). Innanzi tutto vengono ricavate le condizioni sulla curvatura gaussiana della superficie lungo una curva biarmonica che non sia una geodetica. Vengono poi studiate le curve biarmoniche su una superficie di rivoluzione e vengono date le soluzioni esplicite nel caso in cui la superficie di rivoluzione abbia curvatura di Gauss costante.*

Abstract: *In this paper we consider the biharmonic curves on a surface. These curves are critical points of the bienergy functional, and generalize the harmonic curves (geodesics). We first find conditions on the Gaussian curvature of the surface along a nongeodesic biharmonic curve. Then we study biharmonic curves in a surface of revolution, giving the explicit solutions in the case of surfaces of revolution with constant Gaussian curvature.*

1 – Introduction

Since their first work on harmonic maps, J. Eells and J. H. Sampson in [4] suggested the idea of studying k-harmonic maps. If $k = 2$, the idea is the following. First define harmonic maps $\phi : (M, g) \to (N, h)$ between two Riemannian manifolds as critical points of the energy $E(\phi) = \frac{1}{2}\int_M |d\phi|^2\, dv$. The corresponding Euler-Lagrange equation for the energy is given by the vanishing of the tension field $\tau_\phi = \operatorname{trace} \nabla d\phi$. Then define

Key Words and Phrases: *Biharmonic curves, Surfaces of revolution.*
A.M.S. Classification: 58E20

the *bienergy* of a map ϕ by $E_2(\phi) = \frac{1}{2}\int_M |\tau_\phi|^2\, dv$, and say that ϕ is *biharmonic* if it is a critical point of the bienergy.

In [6], [7] G.Y. Jiang derived the first variation formula of the bienergy showing that the Euler-Lagrange equation for E_2 is $\tau_{2\phi} = J(\tau_\phi) = 0$, where J is the Jacobi operator of ϕ. The equation $\tau_{2\phi} = 0$ will be called the *biharmonic equation.*

We now restrict our attention to isometric immersions $\gamma : I \to (M, g)$ from an interval I to a Riemannian manifold. The image $C = \gamma(I)$ is the trace of a curve in M and γ is a parametrization of C by arc length. In this case the tension field becomes $\tau_\gamma = \nabla_{\dot\gamma}\dot\gamma$ and the biharmonic equation reduces to

$$J(\tau_\gamma) = -\nabla^2_{\dot\gamma}\tau_\gamma - R(\dot\gamma, \tau_\gamma)\dot\gamma = 0. \tag{1}$$

Note that $C = \gamma(I)$ is part of a geodesic of M if and only if γ is harmonic. Moreover, from the biharmonic equation if γ is harmonic then it is biharmonic, thus geodesics are a subclass of biharmonic curves. The converse is not true and this makes the class of biharmonic curves richer than that of geodesics. It is then natural to ask what geometric properties characterize biharmonic curves.

With this in mind, the aim of this paper is to study the geometry of nongeodesic biharmonic curves in a surface.

We first prove that along a nongeodesic biharmonic curve the Gauss curvature is constant and equal to the square of the geodesic curvature. Moreover, if the Gauss curvature of the surface is nonpositive, then the biharmonic curves are exactly the geodesics.

In the second part we concentrate on parametrized surfaces of revolution in three-dimensional Euclidean space. If the surface of revolution has nonconstant Gauss curvature, then a nongeodesic biharmonic curve is a parallel. Surfaces of revolution with constant Gauss curvature are more interesting. In this case we give the explicit solution of the biharmonic equation. We also give the classification of the surfaces of revolution all of whose parallels are biharmonic curves. Finally we describe the biharmonic curves in some special surfaces of revolution.

Notation. We shall place ourselves in the C^∞ category, i.e. manifolds, metrics, connections, maps will be assumed to be smooth. By (M^m, g) we shall mean a connected manifold, of dimension m, without boundary, endowed with a Riemannian metric g. We shall denote by ∇ the Levi-Civita connection on (M, g). For vector fields X, Y, Z on M we define the Riemann curvature operator by $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$.

2 – Biharmonic curves on a surface

Let (M^2, g) be a Riemannian surface and let $\gamma : I \to (M^2, g)$ be a differentiable curve parametrized by arc length. Let $\{T, N\}$ be an orthonormal frame field tangent to M^2 along γ, where $T = \dot{\gamma}$ is a unit vector field tangent to γ. Then we have the following Frenet equations

$$\nabla_T T = k_g N\,; \qquad \nabla_T N = -k_g T\,.$$

Here $k_g = |\tau_\gamma| = |\nabla_{\dot{\gamma}}\dot{\gamma}|$ is the geodesic curvature of γ. Using these Frenet equations the Euler-Lagrange equation of the bienergy can be written as

$$-\nabla^3_T T - R(T, k_g N)T = 3k_g\dot{k}_g T - (\ddot{k}_g - k_g^3 + k_g K)N = 0\,,$$

where $K = R(T, N, T, N)$ is the Gauss curvature of the surface. It follows that γ is a biharmonic curve if and only if

$$\begin{cases} \ddot{k}_g - k_g^3 + k_g K = 0 \\ k_g \dot{k}_g = 0 \end{cases}$$

If we look for nongeodesic solutions, that is for biharmonic curves with $k_g \neq 0$, we have

$$\begin{cases} k_g = \text{constant} \neq 0 \\ k_g^2 = K \end{cases} \tag{2}$$

Thus we have:

Proposition 2.1. *Let $\gamma : I \to (M^2, g)$ be a differentiable curve in a surface M^2. Then, if γ is a nongeodesic biharmonic curve, along γ the Gauss curvature is constant, positive and equal to the square of the geodesic curvature of γ. Therefore if M^2 has nonpositive Gaussian curvature any biharmonic curve is a geodesic of M^2.*

Remark 2.2. A curve $\gamma : I \to (M^2, g)$ is called a free elastic curve if it is a critical point for the bending energy $F(\gamma) = \frac{1}{2}\int_I k_g^2 \, |\dot\gamma|^2 \, dt$. In [8] J. Langer and D. A. Singer have shown that free elastic curves are solutions of the following differential equation (the Euler-Lagrange equation for the bending energy):

$$2\ddot{k}_g + k_g^3 + 2k_g K = 0.$$

Note that $k_g = 0$ is a solution of the above equation and thus geodesics are free elastic curves. If $\gamma : I \to (M^2, g)$ is a nongeodesic biharmonic curve, then from Proposition 2.1, we have

$$2\ddot{k}_g + k_g^3 + 2k_g K = 3k_g^3 \neq 0.$$

This says that the intersection of the set of biharmonic curves and that of free elastic curves consists of geodesics.

We point out that in general is rather difficult to find curves with constant geodesic curvature, although they always exist from the existence theorem of ordinary differential equations. In fact the corresponding system of differential equations is far more complicated than that for geodesics. Nevertheless, in some special cases it is possible to give explicit solutions. For example, on the right circular cylinder $X(u, v) = (r\cos(u), r\sin(u), \lambda v)$, the curve $\alpha(t) = X(a_1 t^2 + a_2 t + a_3, b_1 t^2 + b_2 t + b_3)$ $(a_i, b_i \in \mathbb{R},\ i = 1, 2, 3)$ has geodesic curvature $k_g = 4a_1^2 + 4b_1^2$.

3 – Biharmonic curves on a surface of revolution

Let $\alpha(u) = (f(u), 0, g(u))$ be a curve in the xz-plane and consider the surface of revolution obtained by rotating this curve around the z-axis with the standard parametrization

$$X(u, v) = (f(u)\cos v, f(u)\sin v, g(u))\,,$$

where v is the rotation angle around the z-axis. If α is parametrized by arc length, so that $\dot f^2 + \dot g^2 = 1$, the Gauss curvature of the surface of revolution is given by

$$K(u, v) = -\frac{\ddot f(u)}{f(u)}\,.$$

The Gauss curvature K depends only on u, that is K is constant along any parallel. This implies that if the Gauss curvature is constant along a curve, then either the curve is a parallel or the curve lies in a part of the surface with constant Gauss curvature. In particular a nongeodesic biharmonic curve is either a parallel or lies on a surface with constant Gauss curvature. For this reason we shall study biharmonic curves in a surface of revolution with constant Gauss curvature in a separate section.

Now we want to determine the surfaces of revolution all of whose parallels are biharmonic curves. We shall prove

THEOREM 3.1. *Let $M^2 \subset \mathbb{R}^3$ be a surface of revolution obtained by rotating the arc lenght parametrized curve $\alpha(u) = (f(u), 0, g(u))$ in the xz-plane around the z-axis. Then all parallels of M are biharmonic curves if and only if either*

(i) *f is constant and M is a right circular cylinder*

or

(ii)

$$f(u) = \pm c\sqrt{u}$$

and

$$g(u) = u\sqrt{\frac{4u - c^2}{4u}} - \frac{c^2}{8}\log\left(8u + 8\,u\sqrt{\frac{4u - c^2}{4u}} - c^2\right) + c_1\,,$$

where c and c_1 are positive constant.

PROOF. Let

$$\gamma_{\bar{u}}(t) = \left(f(\bar{u})\cos\left(\frac{v(t)}{f(\bar{u})}\right), f(\bar{u})\sin\left(\frac{v(t)}{f(\bar{u})}\right), g(\bar{u})\right)$$

be the parallel $u = \bar{u} =$ constant parametrized by arc length. Then its geodesic curvature is given by

$$k_g = \frac{\dot{f}}{f}\,. \tag{3}$$

From the Euler-Lagrange equation (2) we see that all parallels of M are biharmonic curves if and only if f is a solution of the following second order differential equation

$$\dot{f}^2 + \ddot{f} f = 0\,. \tag{3}$$

If f is constant, then all parallels are geodesics, since $k_g = (\dot{f}/f)$, and the surface of revolution is a right circular cylinder.

Let us assume that f is *nonconstant*. Then by standard arguments of ordinary differential equations the solution of (4) is $f(u) = \pm c\sqrt{u+b}, b, c \in \mathbb{R}^+$; without loss of generality we can assume that $b = 0$. It amounts to translating the profile curve along the axis of revolution so that the profile curve intersects the axis of revolution when $u = 0$. In order to find g we use $\dot{f}^2 + \dot{g}^2 = 1$. We first obtain $\dot{g} = \pm\sqrt{\frac{4u-c^2}{4u}}$ and by integration we have

$$g(u) = u\sqrt{\frac{4u-c^2}{4u}} - \frac{c^2}{8}\log\left(8u + 8\,u\sqrt{\frac{4u-c^2}{4u}} - c^2\right) + c_1\,.$$

In fig. 1 is a plot of the corresponding surface. ∎

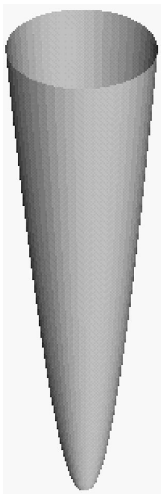

Fig. 1.

3.1 – Examples of biharmonic curves in a surface of revolution with nonconstant Gauss curvature

From the discussion in the previous section we have the following

Proposition 3.2. *Let $M^2 \subset \mathbb{R}^3$ be the surface of revolution with nonconstant Gauss curvature obtained by rotating around the z-axis the arc length parametrized curve $\alpha(u) = (f(u), 0, g(u))$. Then the biharmonic curves of M^2 are the parallels $u = \bar{u} =$ constant such that $\bar{u}$ satisfies the equation*

$$\dot{f}^2(\bar{u}) + \ddot{f}(\bar{u})f(\bar{u}) = 0.$$

Example 3.3 (The torus of revolution). The torus of revolution with its standard arc length parametrization

$$X(u, v) = ((a+r\cos(u/r))\cos v, (a+r\cos(u/r))\sin v, r\sin(u/r)), \quad a > r,$$

has Gaussian curvature

$$K = \frac{\cos(u/r)}{r(a + r\cos(u/r))}.$$

From Proposition 3.2 it follows that a parallel on the torus of revolution is a biharmonic curve if

$$\frac{1}{(a + r\cos(u/r))^2}\sin^2(u/r) = \frac{\cos(u/r)}{r(a + r\cos(u/r))},$$

and therefore for

$$u = r\arccos\left(\frac{-a \pm \sqrt{a^2 + 8r^2}}{4r}\right). \tag{5}$$

Figure 2 shows the two nongeodesic biharmonic curves on the torus given by (5), for $a = 3$ and $r = 1$.

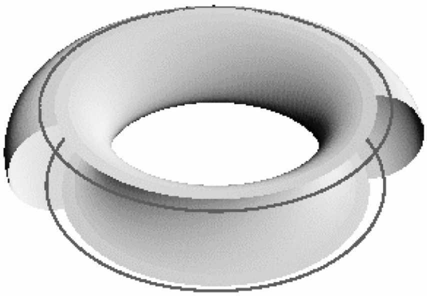

Fig. 2.

EXAMPLE 3.4 (Surfaces of Delaunay) Due to a theorem of C. Delaunay given in [3], we have that the complete immersed surfaces of revolution in $\mathbb{R}^3$ with constant mean curvature are precisely those obtained by rotating about their axes the roulettes of the conics. This is why constant mean curvature surfaces of revolution are called Delaunay surfaces.

We use a parametrization of Delaunay surfaces given by M. P. Do Carmo and M. Dajczer in [2]. First define

$$\begin{cases} \rho(u) = \dfrac{\sqrt{1+B^2+2B\sin(2Hu)}}{2H} \\ \lambda(u) = \displaystyle\int_0^u f(s)\,ds \end{cases}$$

where B is a nonnegative constant, H is the mean curvature of M, and

$$f(s) = \frac{(1+B\sin(2Hs))}{\sqrt{1+B^2+2B\sin(2Hs)}}\,.$$

A parametrization of Delaunay surfaces is obtained by rotating the arc length parametrized curve $\alpha(u)=(\rho(u),0,\lambda(u))$ around the z-axis, that is

$$X(u,v) = (\rho(u)\cos v, \rho(u)\sin v, \lambda(u))\,.$$

Depending on the value of B we get the four types of Delaunay surfaces, that is:

- the right circular cylinder ($B = 0$);

- the unduloids ($0 < B < 1$);
- the round sphere ($B = 1$);
- the nodoids ($B > 1$).

From Proposition 3.2 we find that a parallel $u = \textit{constant}$ of a Delaunay surface is a biharmonic curve if u satisfies the following equation

$$\frac{-4BH^2 \sin(2Hu)}{1 + B^2 + 2B\sin(2Hu)} = 0,$$

that is for $u = k\pi/2H,\ k \in \mathbb{Z}$. Note that the value of u does not depend on B but only on the mean curvature H. The following figure shows the biharmonic parallels in an unduloid ($B = 1/2$) and in a nodoid ($B = 2$) with mean curvature $H = 1$.

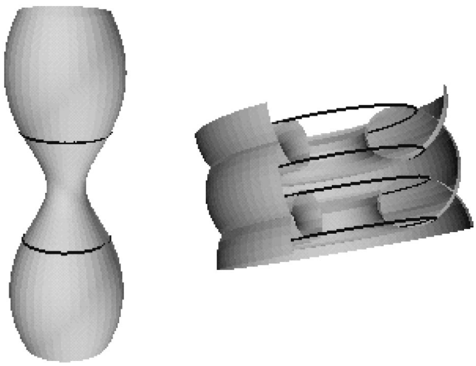

Fig. 3. Biharmonic parallels on a nodoid and on an unduloid.

In general it is complicated to parametrize the profile curve by arc length. Sometimes it is better to calculate esplicitly the geodesic curvature of a parallel and then to use the biharmonic equation $K = k_g^2$, as it is shown in the following

Example 3.5 (The ellipsoid of revolution). Let $M^2 \subset \mathbb{R}^3$ be an ellipsoid of revolution parametrized by

$$X(u, v) = (a \cos u \cos v, a \cos u \sin v, c \sin u)\,.$$

The Gauss curvature of M^2 is given by

$$K = \frac{c^2}{(a^2 \sin^2 u + c^2 \cos^2 u)^2} \,.$$

A parallel of M^2 parametrized by arc length has geodesic curvature

$$k_g^2 = \frac{\sin^2 u}{\cos^2 u (a^2 \sin^2 u + c^2 \cos^2 u)} \,,$$

thus the biharmonic equation becomes

$$\frac{c^2}{(a^2 \sin^2 u + c^2 \cos^2 u)^2} = \frac{\sin^2 u}{\cos^2 u (a^2 \sin^2 u + c^2 \cos^2 u)} \,.$$

Replacing $\cos^2 u$ with $1 - \sin^2 u$ we get that a parallel $u = \mathit{constant}$ is a biharmonic curve if u satisfies the following trigonometric equation

$$(a^2 - c^2) \sin^4 u + 2c^2 \sin^2 u - c^2 = 0 \,.$$

If M^2 is not a sphere, i.e. for $a \neq c$, then the latter has solutions

$$\sin^2 u = \begin{cases} \dfrac{c}{c+a} \\ \dfrac{c}{c-a} \end{cases}$$

The second equation is never satisfied, while the first equation gives always the two biharmonic parallels $u = \arcsin(\pm\sqrt{\frac{c}{c+a}})$.

4 – Surfaces of revolution with constant positive Gauss curvature

Let $M^2 \subset \mathbb{R}^3$ be a surface of revolution with positive constant Gauss curvature $K = \frac{1}{a^2}$. The surface M^2 can be parametrized by

$$X(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$$

where

$$f(u) = b\cos\frac{u}{a} \quad , \quad g(u) = \int_0^u \sqrt{\left(1 - \frac{b^2}{a^2}\sin^2\frac{s}{a}\right)}ds,$$

with $b \in \mathbb{R}^+$.

Let $\alpha(t) = X(u(t), v(t))$ be a curve on M^2 parametrized by arc length. We now calculate the Darboux basis along α. The normal vector to the surface restricted to the curve is

$$N(t) = \left(\frac{dg}{du}\cos v, \frac{dg}{du}\sin v, -\frac{df}{du}\right)_{|\alpha(t)}.$$

Writing the tangent vector field $\dot\alpha(t)$ with respect to the coordinate frame $\{X_u,\, X_v\}$ we have

$$\dot\alpha(t) = \dot u X_u + \dot v X_v$$

with $\dot v^2 f^2 + \dot u^2 = 1$. The Darboux basis along α is then

$$(6) \qquad \begin{aligned} b_1 &= \dot\alpha(t) = \dot u X_u + \dot v X_v \\ b_2 &= b_3 \wedge b_1 = \dot v f X_u - \frac{\dot u}{f} X_v \\ b_3 &= N(t) = \left(\frac{dg}{du}\cos v, \frac{dg}{du}\sin v, -\frac{df}{du}\right) \end{aligned}$$

We now recall the Darboux equations (see, for example, [5], page 518):

$$\begin{aligned} \dot{b_1} &= k_g b_2 + k_n b_3 \\ \dot{b_2} &= -k_g b_1 + \tau_g b_3 \\ \dot{b_3} &= -k_n b_1 - \tau_g b_2 \end{aligned}$$

where k_n is the normal curvature whilst τ_g is the geodesic torsion. From the Darboux equations we get

$$(7) \qquad \langle \dot{b_1}, b_2 \rangle = k_g.$$

Then using (6), (7) and $\dot v^2 f^2 + \dot u^2 = 1$ we have

$$(8) \qquad k_g = -\ddot v \dot u f + \ddot u \dot v f - \dot v \dot u^2 \frac{df}{du} - \dot v \frac{df}{du}.$$

Now we take the derivative with respect to t of $\dot{v}^2 f^2 + \dot{u}^2 = 1$, that is

$$2\dot{v}\ddot{v}f^2 + 2\dot{v}^2\dot{u}f\frac{df}{du} + 2\dot{u}\ddot{u} = 0.$$

If α is not a parallel, i.e. $\dot{u} \neq 0$, we have

$$\ddot{u} = -\frac{\dot{v}\ddot{v}f^2 + \dot{v}^2\dot{u}f\frac{df}{du}}{\dot{u}}.$$

Substituting the latter in (8), we have finally

$$k_g = \frac{-\ddot{v}f - 2\dot{u}\dot{v}\frac{df}{du}}{\dot{u}},$$

and the biharmonic equation $k_g^2 = K$ becomes

$$\frac{-\ddot{v}f - 2\dot{u}\dot{v}\frac{df}{du}}{\dot{u}} = \frac{1}{a}.$$

Multiplying both sides by f we obtain

$$-\frac{d}{dt}(\dot{v}f^2) = \frac{1}{a}f\dot{u}.$$

If we substitute $f(u) = b\cos\frac{u}{a}$ we get

$$\frac{d}{dt}\left(\dot{v}b^2\cos^2\frac{u}{a}\right) = \frac{d}{dt}\left(-b\sin\frac{u}{a}\right).$$

The condition that $\alpha(t) = X(u(t), v(t))$ is a biharmonic curve parametrized by arc length is then expressed by the system

$$(9) \qquad \begin{cases} \dot{v}b^2\cos^2\frac{u}{a} = -b\sin\frac{u}{a} + c \\ \dot{v}^2 b^2\cos^2\frac{u}{a} + \dot{u}^2 = 1 \end{cases}$$

Solving this system with respect to $\dot{u}$ we have the equation

$$\frac{du}{dt} = \pm\frac{1}{b\cos\frac{u}{a}}\sqrt{b^2 - c^2 - 2b^2\sin^2\frac{u}{a} + 2bc\sin\frac{u}{a}}.$$

Separating the variables and integrating we obtain

$$t + A = \pm\frac{a}{\sqrt{2}}\arcsin\left(\frac{c - 2b\sin\frac{u}{a}}{\sqrt{2b^2 - c^2}}\right)$$

with the condition that $2b^2 - c^2 > 0$. We can finally explicit $u(t)$ getting

$$u(t) = -a\arcsin\left(\frac{c \pm \sqrt{2b^2 - c^2}\sin(A + \frac{\sqrt{2}}{a}t)}{2b}\right).$$

The function $v(t)$ can be calculated from (9) and has the expression

$$v(t) = \int_0^t \left(\frac{-\sin\frac{u(s)}{a}}{b\cos^2\frac{u(s)}{a}} + \frac{c}{b^2\cos^2\frac{u(s)}{a}}\right) ds.$$

In this way we have found the general solution of a biharmonic curve in a surface of revolution with constant positive Gauss curvature which is not a parallel. Figure 4 shows a plot of such a curve in the three different types of constant positive Gauss curvature surfaces of revolution:

- the sphere type ($a = b$);
- the (U.S.A.) football type ($b < a$);
- the barrel type ($a < b$).

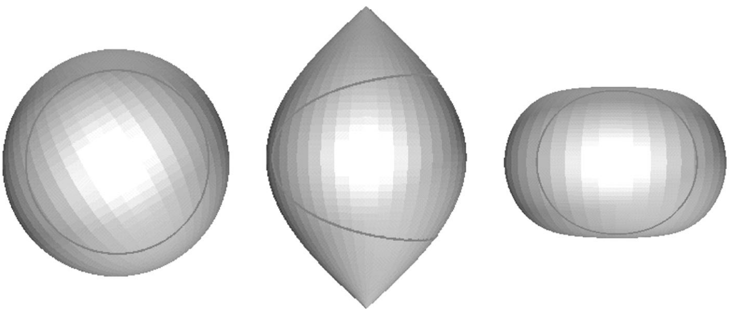

Fig. 4. Biharmonic curves in the three types of constant positive Gauss curvature surfaces of revolution.

Remark 4.1. The case of the sphere is rather special and needs a bit more attention. In fact, there is a geometric way to understand the behaviour of biharmonic curves on the sphere. Let α be a curve on the sphere of radius r parametrized by arc length and let τ and k be the torsion and curvature of α as seen from the ambient space $\mathbb{R}^3$. Then from the Frenet equations we have

$$\tau k^2 = [b_1, \dot{b_1}, \ddot{b_1}], \tag{10}$$

where $[a, b, c]$ denotes the determinant of three vectors in $\mathbb{R}^3$. For a spherical curve, $\dot{b_1}$ and $\ddot{b_1}$ can be expressed in the Darboux basis as follows:

$$\dot{b_1} = k_g b_2 - \frac{1}{r} b_3$$

$$\ddot{b_1} = -\left(\frac{1}{r^2} + k_g^2\right) b_1 + \dot{k}_g b_2.$$

Substitution of $\dot{b_1}$ and $\ddot{b_1}$ into (10) yields

$$\tau k^2 = \frac{1}{r}\dot{k}_g.$$

This equation says that if α is a biharmonic curve, then $\tau k^2 = 0$, and since $k^2 = k_n^2 + k_g^2 = \frac{2}{r^2} = constant \neq 0$ we must have that $\tau = 0$. Thus, a biharmonic curve on a sphere of radius r is the intersection of the sphere with a plane such that the resulting circle has radius $\frac{r}{\sqrt{2}}$.

If we look for biharmonic parallels, then we can apply the same calculation of the previous section to get

$$k_g^2 = \frac{\dot{f}^2}{f^2} = \frac{1}{a^2}\tan^2\frac{u}{a}.$$

The biharmonic equation is then

$$\tan^2\frac{u}{a} = 1,$$

whose solutions, $u = \pm a\frac{\pi}{4}$, give the biharmonic parallels on each of the three types of positive Gauss curvature surfaces of revolution.

Acknowledgements

The authors wish to thank the referee for comments that have improved the paper.

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Lavoro pervenuto alla redazione il 30 novembre 1999
ed accettato per la pubblicazione il 6 marzo 2001.
Bozze licenziate il 26 aprile 2001

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