

Stability and attractivity for a class of dissipative phenomena

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RIASSUNTO: *Si considerano problemi di valori iniziali e al contorno per una classe di equazioni non lineari del terzo ordine aventi un termine forzante non autonomo, e si ottengono nuovi risultati riguardanti la stabilità asintotica delle soluzioni usando il secondo metodo di Liapunov. La classe include equazioni che si incontrano in Teoria della Superconduttività, Meccanica Quantistica e Teoria dei Materiali Viscoelastici.*

ABSTRACT: *We consider initial-boundary-value problems for a class of nonlinear third order equations having non-autonomous forcing terms and get new asymptotic stability results by means of the Liapunov second method. The class includes equations arising in Superconductor Theory, Quantum Mechanics and in the Theory of Viscoelastic Materials.*

1 – Introduction

In this paper we study a large class of initial-boundary-value problems of the form

$$(1.1) \quad \begin{cases} -\varepsilon u_{xxt} + u_{tt} - c^2 u_{xx} = f(x, t, u, u_x, u_{xx}, u_t), & x \in]0, 1[, \quad t > t_0, \\ u(0, t) = 0, & u(1, t) = 0 \end{cases}$$

KEY WORDS AND PHRASES: *Nonlinear higher order PDE - Stability, boundedness - Boundary value problems.*

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($t_0 \geq 0$, ε, c positive constants), with initial conditions

$$(1.2) \quad u(x, t_0) = u_0(x), \quad u_t(x, t_0) = u_1(x).$$

Many papers [1]-[4], [7]-[10] have been devoted to the analysis of the operator $L = -\varepsilon \partial_{xxt} + \partial_{tt} - c^2 \partial_{xx}$, which plays a significant role because it characterizes noteworthy dissipative phenomena. When $f = -b \sin u - au_t + F(x, t)$, where a, b are positive constants, we deal with the perturbed Sine-Gordon equation related to the classical Josephson effect in the Theory of Superconductors [6], [11]. On the other hand it is well known [12] that equation (1.1)₁ describes the evolution of the displacement $u(x, t)$ of the section of a rod from its rest position x in a Voigt material when an external force f is applied; in this case $c^2 = E/\rho$, $\varepsilon = 1/(\rho\mu)$, where ρ is the (constant) linear density of the rod at rest, and E, μ are respectively the elastic and viscous constants of the rod, which enter the stress-strain relation $\sigma = E\nu + \partial_t \nu/\mu$, where σ is the stress, ν is the strain.

Now we suppose that (1.1) admits the null solution $u(x, t) \equiv 0$ and look for new conditions for the stability and attractivity of the latter, so as to improve some results found in [2]. As done there, the distance between the null and a nonnull solution $u(x, t)$ of the problem (1.1)-(1.2) is introduced as the functional $d(u, u_t)$, where for any $(\varphi, \psi) \in C_0^2([0, 1]) \times C_0([0, 1])$ we define

$$(1.3) \quad d^2(\varphi, \psi) = \int_0^1 (\varphi^2 + \varphi_x^2 + \varphi_{xx}^2 + \psi^2) dx.$$

The notions of stability, attractivity and exponential-asymptotic stability are formulated using this distance. Imposing the condition that φ, ψ vanish in $0, 1$ one easily derives that $|\varphi(x)|, |\varphi_x(x)| \leq d(\varphi, \psi)$ for any x ; therefore a convergence in the norm d implies also a uniform pointwise convergence of φ, φ_x .

Using the Liapunov second method, we first obtain some preliminary results involving a class of auxiliary functionals depending on a parameter we will choose according to the examined problem. Supposing the forcing term satisfies suitable conditions, we get a theorem of exponential stability. The assumptions we shall make at the beginning of Section 3

will make the cancellation of the term $-c^2u_{xx}$ by f impossible (this guarantees that the nature of the problem cannot be changed by the choice of f). We emphasize that these new hypotheses are now less restrictive than in [2] because they allow the forcing term f to be in a certain sense unbounded as a function of t . Moreover, this result holds with respect to a metric stronger than d , which can be introduced when the solutions are more regular. After that, we consider the case that f is specialized as a sum of a non-analytic term depending only on u and of u_t times a bounded function. Suitably modifying the above mentioned auxiliary functionals, a theorem of asymptotic stability in the large is obtained. For each theorem we give an example of an application.

2 – Preliminaries

We shall say that the null solution is

- *uniformly stable* if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) \in]0, \varepsilon[$ such that for any $t_0 \in J = [0, +\infty[$ the inequality $d(u_0, u_1) < \delta$ implies $d(u, u_t) < \varepsilon$ for any $t \geq t_0$;
- *attractive* if for any t_0 there exist a $\sigma(t_0) > 0$ such that the inequality $d(u_0, u_1) < \sigma$ implies $d(u, u_t) \rightarrow 0$ as $t \rightarrow +\infty$; in particular *attractive in the large* if $\sigma(t_0) = +\infty$ for any t_0 ;
- *uniformly asymptotically stable (in the large)* if it is uniformly stable and attractive (in the large);
- *exponential-asymptotically stable* if there exist constants $C, D(t_0), \delta'(t_0) > 0$ such that $d(u_0, u_1) < \delta'$ implies $d(u, u_t) < D d(u_0, u_1)e^{-C(t-t_0)}$ for any $t \geq t_0$.

As shown in [2], a set of sufficient conditions for the existence and uniqueness of the solution of the problem (1.1)-(1.2) in the time interval $[0, T]$ is the following:

(2.1) $f(x, t, u, p, q, r)$ is defined and continuous on the set

$$\{(x, t, u, p, q, r) \mid 0 \leq x \leq 1, 0 \leq t \leq T, -\infty < u, p, q, r < +\infty, T > 0\};$$

(2.2) there exists a constant $\mu > 0$ such that

$$|f(x, t, u_1, p_1, q_1, r_1) - f(x, t, u_2, p_2, q_2, r_2)| \leq \mu\{|u_1 - u_2| + |p_1 - p_2| + |q_1 - q_2| + |r_1 - r_2|\};$$

(2.3) u_0, u'_0, u''_0, u_1 continuous on $0 \leq x \leq 1$

$$\text{and such that } u_0(0) = u_0(1) = u_1(0) = u_1(1) = 0.$$

In particular, (2.2) is essential to ensure the uniqueness of the solution.

We shall assume that they are all fulfilled for the class of problems considered in Section 3. The function f considered in Section 4 does not satisfy condition (2.2), but however we are able to obtain the stability properties of the null solution.

To prove our theorems we shall use the Liapunov direct method. We introduce the Liapunov functional

$$(2.4) \quad V(\varphi, \psi) = \frac{1}{2} \int_0^1 \{(\varepsilon\varphi_{xx} - \psi)^2 + \gamma\psi^2 + c^2(1 + \gamma)\varphi_x^2\} dx,$$

where γ is an arbitrary positive constant. It turns out that

$$V \leq \frac{1}{2} \int_0^1 \{\varepsilon^2\varphi_{xx}^2 + \psi^2 + \varepsilon\varphi_{xx}^2 + \varepsilon\psi^2 + \gamma\psi^2 + c^2(1 + \gamma)\varphi_x^2\} dx.$$

Setting

$$(2.5) \quad c_2^2 = \max\{c^2(1 + \gamma)/2, \varepsilon(1 + \varepsilon)/2, (1 + \varepsilon + \gamma)/2\},$$

we thus derive

$$(2.6) \quad V(\varphi, \psi) \leq c_2^2 d^2(\varphi, \psi).$$

Moreover, it is known that

$$(2.7) \quad \varphi(0) = 0 \implies \int_0^1 \varphi_x^2(x) dx \geq \int_0^1 \varphi^2(x) dx,$$

and [5]

$$(2.8) \quad \varphi(0) = 0, \varphi(1) = 0 \implies \int_0^1 \varphi_{xx}^2(x) dx \geq \int_0^1 \varphi_x^2(x) dx.$$

Using (2.7), (2.8) and an argument employed in [2], we get

$$(2.9) \quad V(\varphi, \psi) \geq c_1^2 d^2(\varphi, \psi).$$

where

$$(2.10) \quad c_1^2 = \min\{\varepsilon^2/16, c^2(1+\gamma)/2, (\gamma-1/2)/2\}, \quad (\gamma > 1/2),$$

Therefore, from (2.6) and (2.9) we find

$$(2.11) \quad \frac{V}{c_2^2} \leq d^2 \leq \frac{V}{c_1^2}.$$

On the other hand, choosing $\gamma = 1$ in (2.4) it turns out

$$(2.12) \quad \begin{aligned} \dot{V} &= \int_0^1 \left\{ -\frac{\varepsilon c^2}{2} u_{xx}^2 - \varepsilon u_{xt}^2 + \frac{\varepsilon}{2} u_t^2 + \right. \\ &\quad \left. - \frac{\varepsilon}{2} (c u_{xx} + f/c)^2 - \frac{\varepsilon}{2} (u_t - 2f/\varepsilon)^2 + A f^2 \right\} dx \leq \\ &\leq - \int_0^1 \left\{ \frac{\varepsilon c^2}{6} (u^2 + u_x^2 + u_{xx}^2) + \frac{\varepsilon}{2} u_t^2 + A f^2 \right\} dx \leq \\ &\leq -c_3^2 d^2(u, u_t) + \int_0^1 A f^2 dx \end{aligned}$$

where we have set

$$(2.13) \quad A := (\varepsilon/2c^2) + 2/\varepsilon, \quad c_3^2 := \min\left\{\frac{\varepsilon c^2}{6}, \frac{\varepsilon}{2}\right\},$$

and we have used (2.7), (2.8). In the sequel we shall set also $p := c_3^2/c_2^2$.

3 – Stability and attraction region for (1.1)

We introduce the following

Hypothesis 1- Assume that

$$(3.1) \quad A \int_0^1 f^2 dx \leq \hat{g}(t, d^2) c_1^2 d^2$$

where f is the function of (1.1) and $\hat{g}(t, \eta)$ ($t > t_0$, $\eta > 0$) is continuous, nonnegative, non-decreasing in η and such that the limit

$$(3.2) \quad \lim_{t \rightarrow +\infty} \frac{\int_0^t \hat{g}(\tau, \eta/c_1^2) d\tau}{t} =: q(\eta)$$

defines a continuous, non-decreasing function $q : \eta \in J \rightarrow J$ with $q(0) < p$.

The assumption that $\hat{g}(t, \eta)$ is non-decreasing in η is no real loss of generality; if originally this is not the case, we just need to replace $\hat{g}(t, \eta)$ by $\max_{0 \leq \theta \leq \eta} \hat{g}(t, \theta)$ to fulfill this condition.

THEOREM 1. *Under these assumptions the null solution of the problem (1.1) is exponential-asymptotically stable and the region of attraction related to the initial time t_0 includes the set*

$$d(u_0, u_1) < \left[\sup_{r \in]0, \bar{r}[} \frac{r}{c_2^2} e^{-M(t_0, r)} \right]^{1/2},$$

where \bar{r} and $M(t_0, r)$ are defined by (3.6) and (3.9).

PROOF. From (2.12), using (3.1), (2.11) and the monotonicity in η of $g(t, \eta) := \hat{g}(t, \frac{\eta}{c_1^2})$, we find

$$\begin{aligned} \dot{V}(u, u_t) &\leq -c_3 d^2 + \hat{g}(t, d^2) c_1^2 d^2 \leq \\ &\leq [-p + g(t, V)]V. \end{aligned}$$

By the “comparison principle” (Lemma 24.3 of [13]) V is bound from above

$$(3.3) \quad 0 \leq V(t) \leq y(t), \quad t \geq t_0$$

by the solution $y(t)$ of the Cauchy problem

$$(3.4) \quad \dot{y} = [-p + g(t, y)]y, \quad y(t_0) = y_0 \equiv V(t_0) > 0.$$

We therefore study the latter. Problem (3.4) is equivalent to the integral equation

$$(3.5) \quad y(t) = y_0 e^{-p(t-t_0) + \int_{t_0}^t g(\tau, y(\tau)) d\tau}.$$

Let

$$(3.6) \quad \bar{r} := \sup\{\rho \geq 0 \mid q(\rho) < p\}.$$

The inequality $q(0) < p$ implies $\bar{r} > 0$. Chosen a $r \in]0, \bar{r}[$, for any $\eta \leq r$ and $t_0 \in J$ condition (3.2) implies

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{\int_{t_0}^t g(\tau, \eta) d\tau}{t - t_0} &\leq \lim_{t \rightarrow +\infty} \frac{\int_{t_0}^t g(\tau, r) d\tau}{t - t_0} = \\ &= \lim_{t \rightarrow +\infty} \frac{\int_0^t g(\tau, r) d\tau - \int_0^{t_0} g(\tau, r) d\tau}{t} \frac{t}{t - t_0} = q(r), \end{aligned}$$

and therefore $\forall \sigma > 0 \exists t'(\sigma, r, t_0) > t_0$ such that $\forall t > t'$

$$\frac{\int_{t_0}^t g(\tau, \eta) d\tau}{t - t_0} < q(r) + \sigma.$$

Choosing $\sigma \equiv \frac{p-q(r)}{2}$ and, denoting by $t'_0(r, t_0)$ the corresponding value of $t'(\sigma(r), r, t_0)$, we find that for any $t > t'_0$, $\eta \leq r$

$$\frac{\int_{t_0}^t g(\tau, \eta) d\tau}{t - t_0} < q(r) + \sigma = \frac{p + q(r)}{2},$$

whence

$$(3.7) \quad -p(t - t_0) + \int_{t_0}^t g(\tau, \eta) d\tau < -\frac{p - q(r)}{2}(t - t_0).$$

On the other hand, as $q(r) > 0$, for $t \in]t_0, t'_0]$ and $\eta \leq r$

$$(3.8) \quad -p(t - t_0) + \int_{t_0}^t d\tau g(\tau, \eta) < -\frac{p - q(r)}{2}(t - t_0) + M(t_0, r),$$

where we have set

$$(3.9) \quad M(t_0, r) := \max \left\{ 0, \max_{t_0 \leq t \leq t'_0} \left[-\frac{p - q(r)}{2}(t - t_0) + \int_{t_0}^t g(\tau, r) d\tau \right] \right\}$$

Looking back at (3.7) we realize that (3.8) actually holds for any $t > t_0$, because by definition $M(t_0, r) \geq 0$. Therefore, given any $t > t_0$, if the solution of (3.4) satisfies $y(\tau) \leq r$ for any $\tau \in [t_0, t[$ then (3.5) and (3.8) imply

$$(3.10) \quad y(t) < y_0 e^{M(t_0, r)} e^{-\frac{p - q(r)}{2}(t - t_0)}.$$

Now it is easy to show first that, indeed,

$$(3.11) \quad 0 < y_0 < r e^{-M(t_0, r)} \quad \Rightarrow \quad y(t) < r \quad \forall t \geq t_0.$$

In fact, if *per absurdum* there existed $t_1 > t_0$ such that

$$(3.12) \quad y(\tau; t_0, y_0) < r \quad \text{for } t_0 \leq \tau < t_1$$

$$(3.13) \quad y(t_1; t_0, y_0) = r$$

then (3.10), (3.12) would imply

$$y(t_1) < y_0 e^{M(t_0, r)} e^{-\frac{p - q(r)}{2}(t_1 - t_0)} < r e^{-\frac{p - q(r)}{2}(t_1 - t_0)} < r,$$

against (3.13). Having proved the bound (3.11), now we can immediately improve it. We can reconsider the first part of the previous inequality chain based on (3.10) for *any* $t > t_0$

$$y(t) < y_0 e^{M(t_0, r)} e^{-\frac{p - q(r)}{2}(t - t_0)}$$

and thus find the implication

$$(3.14) \quad 0 < y_0 < r e^{-M(t_0, r)} \quad \Rightarrow \quad y(t) < y_0 e^{M(t_0, r)} e^{-\frac{p-q(r)}{2}(t-t_0)}$$

for any $t > t_0$. Now (3.3), (3.14) imply

$$V(t) < V(t_0) e^{M(t_0, r)} e^{-\frac{p-q(r)}{2}(t-t_0)},$$

provided $V(t_0) < r e^{-M(t_0, r)}$. With the short-hand notation $d^2(t) \equiv d^2(u, u_t)$, we thus find that the assumption

$$(3.15) \quad d^2(t_0) < r \frac{e^{-M(t_0, r)}}{c_2^2}$$

implies, because of (2.9) and (2.6),

$$(3.16) \quad d^2(t) < d^2(t_0) \frac{c_2^2}{c_1^2} e^{M(t_0, r)} e^{-\frac{p-q(r)}{2}(t-t_0)},$$

i.e. the exponential-asymptotical stability. Finally, from (3.15) we derive the attraction region includes the set

$$(3.17) \quad d^2(u_0, u_1) < \sup_{r \in]0, \bar{r}[} \frac{r}{c_2^2} e^{-M(t_0, r)}.$$

REMARK 1. This is an alternative to Theorem 3.2 B) of [2], which gives sufficient conditions for the exponential-asymptotical stability of the null solution. The hypothesis (3.1) considered here is much weaker than the one considered there, where it was required that there exists a positive constant M such that

$$f^2(x, t, \varphi, \varphi_x, \varphi_{xx}, \psi) \leq M(\varphi^2 + \varphi_x^2 + \varphi_{xx}^2 + \psi^2),$$

in that f may well be an unbounded function of t and nonetheless fulfill (3.1). This is the case for the following

EXAMPLE 1. Let $f = b(t) \sin \varphi$, with a function $b(t)$ such that the limit $\lim_{t \rightarrow +\infty} (\int_0^t b^2(\tau) d\tau) / t$ be finite and smaller than p ; then we can

set $\hat{g}(t, \eta) \equiv b^2(t)$. For instance we could take b^2 a continuous function that vanishes everywhere except in intervals centered at equally spaced points, where it takes linearly increasing maxima but keeps the integral constant, e.g.

$$(3.18) \quad b^2(t) = b_0 \begin{cases} n^2(t - n + 1/n) & \text{if } t \in [n - 1/n, n], \\ n - n^2(t - n) & \text{if } t \in]n, n + 1/n], \\ 0 & \text{otherwise,} \end{cases}$$

with $b_0 < p$ and $n = 2, 3, \dots$

REMARK 2. Under the assumption that the problem (1.1), (1.2) admits solutions $u(x, t)$ having also continuous derivative u_{xtt} , then one can replace (1.3) by the metric

$$(3.19) \quad d_1^2(\varphi, \psi) = d^2(\varphi, \psi) + \int_0^1 \psi_x^2 dx,$$

(2.4) by the functional

$$(3.20) \quad V_1(\varphi, \psi) = V(\varphi, \psi) + \frac{\varepsilon}{2} \int_0^1 \{\varepsilon \psi_x^2 - 2c^2 \psi \varphi_{xx}\} dx$$

and verify that Theorem 1 holds with respect to the metric d_1 .

4 – Stability and attractivity for a non-analytic forcing term

We now specialize the function f of (1.1) as

$$f = F(u) - a(x, t, u, u_x, u_t, u_{xx})u_t,$$

where $F \in C(\mathbb{R})$ and $a \in C(]0, 1[\times]0, +\infty[\times \mathbb{R}^4)$, and examine the particular problem

$$(4.1) \quad \begin{cases} Lu = F(u) - a(x, t, u, u_x, u_t, u_{xx})u_t, & x \in]0, 1[, t > t_0 \\ u(0, t) = 0, u(1, t) = 0, \end{cases}$$

with initial conditions (1.2). We shall use use the modified Liapunov functional

$$(4.2) \quad W(\varphi, \psi) = \frac{1}{2} \int_0^1 \{(\varepsilon\varphi_{xx} - \psi)^2 + \gamma\psi^2 + c^2(1+\gamma)\varphi_x^2\} dx + (1+\gamma) \int_0^1 \left(\int_0^{\varphi(x)} F(z) dz \right) dx$$

where $\gamma > 1/2$ for the moment is an unspecified parameter.

THEOREM 2. *The null solution of the problem (4.1) is uniformly asymptotically stable in the large under the following assumptions:*

(4.3) *there exist $\tau \in [0, 1[$ and $D > 0$ such that, for any φ, ψ*

$$0 \leq - \int_0^1 \left(\int_0^{\varphi(x)} F(z) dz \right) dx \leq \frac{D}{\gamma + 1} d^{\tau+1}(\varphi, \psi);$$

(4.4) $\int_0^1 F(\varphi(x))\varphi_{xx}(x)dx \geq 0$ *for any $\varphi \in C_0^2([0, 1])$;*

(4.5) *the function a satisfies $\inf a > -\varepsilon, \sup a < +\infty$.*

PROOF. Reasoning as in section 3.4 of reference [2], we get

$$(4.6) \quad \begin{aligned} W(\varphi, \psi) &\geq \frac{1}{2} \int_0^1 \{(\varepsilon\varphi_{xx} - \psi)^2 + \gamma\psi^2 + c^2(1 + \gamma)\varphi_x^2\} dx = \\ &= \frac{1}{2} \int_0^1 \{(\varepsilon\varphi_{xx} - 2\psi)^2/4 + (\varepsilon\varphi_{xx} - \psi)^2/2 + (\gamma - 1/2)\psi^2 + \\ &\quad + c^2(1 + \gamma)\varphi_x^2 + \varepsilon^2\varphi_{xx}^2/4\} dx \geq \\ &\geq \frac{1}{2} \int_0^1 \{(\gamma - 1/2)\psi^2 + c^2(1 + \gamma)\varphi_x^2 + \varepsilon^2\varphi_{xx}^2/4\} dx \geq \\ &\geq \frac{1}{2} \int_0^1 \{(\gamma - 1/2)\psi^2 + c^2(1 + \gamma)(\varphi^2 + \varphi_x^2)/2 + \varepsilon^2\varphi_{xx}^2/4\} dx \geq \\ &\geq k_1 d^2(\varphi, \psi), \end{aligned}$$

where

$$(4.7) \quad k_1 := \frac{1}{2} \min \left\{ \gamma - \frac{1}{2}, \frac{\varepsilon^2}{4}, \frac{c^2(1 + \gamma)}{2} \right\}.$$

Moreover, taking the derivative of W and reasoning as we have done for (2.12), we obtain

$$\dot{W}(u, u_t) = - \int_0^1 \{c^2 \varepsilon u_{xx}^2 + \varepsilon \gamma u_{xt}^2 + a(1 + \gamma)u_t^2 + \varepsilon F(u)u_{xx} - \varepsilon a u_{xx} u_t\} dx.$$

From this, considering inequalities (2.7), (2.8) it follows

$$(4.8) \quad \dot{W}(u, u_t) \leq - \int_0^1 \{(3/4)c^2 \varepsilon u_{xx}^2 + [\varepsilon \gamma + a(1 + \gamma - \varepsilon a/c^2)]u_t^2 + \varepsilon F(u)u_{xx} + \varepsilon[(c/2)u_{xx} - (a/c)u_t]^2\} dx.$$

Because of (4.4), the third and fourth terms at the right-hand side are nonnegative and therefore by (2.7), (2.8)

$$\begin{aligned} \dot{W}(u, u_t) &\leq - \int_0^1 \{(3/4)c^2 \varepsilon u_{xx}^2 + [\varepsilon \gamma + a(1 + \gamma - \varepsilon a/c^2)]u_t^2\} dx \leq \\ &\leq - \int_0^1 \{c^2/4\varepsilon(u_{xx}^2 + u_x^2 + u^2) + [\varepsilon \gamma + a(1 + \gamma - \varepsilon a/c^2)]u_t^2\} dx. \end{aligned}$$

Owing to (4.5) we choose

$$\gamma = [1 + \sup |a(a\varepsilon/c^2 - 1)|]/(\varepsilon + \inf a) + \frac{1}{2},$$

so that the coefficient of u_t^2 at the right-hand side becomes ≥ 1 and we find

$$(4.9) \quad \dot{W}(u, u_t) \leq -k_3 d^2(u, u_t),$$

where $k_3 := \min\{c^2/4\varepsilon, 1\}$.

Finally, taking into account formula (4.2), assumption (4.3), and noting that $(\varepsilon\varphi_{xx} - \psi)^2 \leq \varepsilon^2\varphi_{xx}^2 + \psi^2 + \varepsilon(\varphi_{xx}^2 + \psi^2)$, it follows

$$(4.10) \quad W(\varphi, \psi) \leq c_2^2 d^2(\varphi, \psi) + Dd^{\tau+1}(\varphi, \psi).$$

Hence we find

$$(4.11) \quad d^2 \geq \min \left\{ \frac{W}{2c_2^2}, \left(\frac{W}{2D} \right)^{\frac{2}{\tau+1}} \right\},$$

which considered in (4.9) gives

$$(4.12) \quad \dot{W}(u, u_t) \leq -k_3 \min \left\{ \frac{W}{2c_2^2}, \left(\frac{W}{2D} \right)^{\frac{2}{\tau+1}} \right\} \leq 0.$$

The right-hand side is smaller than zero for any nonnull choice of the initial conditions, what we shall assume in the sequel. By (4.12) W is a decreasing function of t . This implies in particular that $W(t) < W(t_0)$ for $t > t_0$, whence, by (4.6), (4.10),

$$(4.14) \quad d^2(t) \leq \frac{W(t)}{k_1} < \frac{W(t_0)}{k_1} \leq 2 \max\{c_2^2 d^2(t_0), Dd^{\tau+1}(t_0)\},$$

implying the uniform stability of the null solution.

We now show that for any choice of the initial conditions W decreases to zero (at least) as a power of t for $t \rightarrow +\infty$. If at $t = t_0$

$$(4.14) \quad \frac{W}{2c_2^2} \geq \left(\frac{W}{2D} \right)^{\frac{2}{\tau+1}},$$

then, by the monotonicity of $W(t)$, for all $t \geq t_0$ this will be true and (4.12) will become

$$(4.15) \quad \dot{W}(u, u_t) \leq -k_3 \left(\frac{W}{2D} \right)^{\frac{2}{\tau+1}};$$

by the comparison principle it will follow

$$W(t) \leq y(t),$$

where $y(t)$ is the solution of the Cauchy problem

$$\dot{y}(t) = -k_3 \left(\frac{y}{2D} \right)^{\frac{2}{\tau+1}}, \quad y(t_0) = W(t_0),$$

namely

$$(4.16) \quad W(t) \leq y(t) = \frac{1}{[W(t_0) + E(t - t_0)]^{\frac{1+\tau}{1-\tau}}},$$

where $E := \frac{k_3}{(2D)^{\frac{2}{\tau+1}}} \frac{1-\tau}{1+\tau} > 0$. Clearly $y(t)$ is decreasing and goes to zero as $1/t^{\frac{1+\tau}{1-\tau}}$ when $t \rightarrow +\infty$. Recalling (4.6), we find

$$(4.17) \quad d^2(t) \leq \frac{W(t)}{k_1} \leq \frac{1}{k_1[W(t_0) + E(t - t_0)]^{\frac{1+\tau}{1-\tau}}}$$

for $t \geq t_0$, implying the attractivity of the null solution in this case. If on the contrary

$$\frac{W(t_0)}{2c_2^2} < \left(\frac{W(t_0)}{2D} \right)^{\frac{2}{\tau+1}},$$

(4.12) will imply for some time

$$\dot{W}(u, u_t) \leq -k_3 W$$

and by the comparison principle an (at least) exponential decrease of W . Hence there will exist a $T > t_0$ such that

$$\frac{W(T)}{2c_2^2} = \left(\frac{W(T)}{2D} \right)^{\frac{2}{\tau+1}},$$

after which (4.12) will take again the form (4.15) and thus imply

$$(4.18) \quad d^2(t) \leq \frac{W(t)}{k_1} \leq \frac{1}{k_1[W(T) + E(t - T)]^{\frac{1+\tau}{1-\tau}}}$$

for $t \geq T$. This implies the attractivity in the large of the null solution.

EXAMPLE 2. An example of a forcing term fulfilling the conditions (4.3), (4.4) is the non-analytic one

$$(4.19) \quad F(u) = -k \operatorname{sign}(u)|u|^\tau \quad 0 < \tau \leq 1, k = \operatorname{const} > 0.$$

In fact, in this case (4.3) is fulfilled since

$$-\int_0^1 \left(\int_0^{\varphi(x)} F(z) dz \right) dx = \frac{k}{\tau+1} \int_0^1 |\varphi(x)|^{\tau+1} dx \geq 0,$$

and, by Schwarz inequality and (2.7), (2.8)

$$\begin{aligned} \int_0^1 |u|^{\tau+1} dx &\leq \left(\int_0^1 u^2 dx \right)^{\frac{\tau+1}{2}} \leq \\ &\leq \left(\frac{1}{3} \int_0^1 (u^2 + u_x^2 + u_{xx}^2) dx \right)^{\frac{\tau+1}{2}} \leq \frac{1}{3^{\frac{\tau+1}{2}}} d^{\tau+1}(u, u_t); \end{aligned}$$

(4.4) is fulfilled since, integrating by parts,

$$(4.20) \quad \int_0^1 F(u) u_{xx}(x) dx = -k \int_0^1 \text{sign}(u) |u|^\tau u_{xx}(x) dx = \tau k \int_0^1 \frac{u_x^2}{|u|^{1-\tau}} dx \geq 0.$$

REMARK 3. This result should be compared with Theorem 3.3. in reference [2]: the claim is the same, but the hypotheses are adapted to cover the case of a non-analytic forcing term.

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