Rendiconti di Matematica, Serie VII
Volume 21, Roma (2001), 207-221

# The limit class of Gehring type $G_\infty$ in the $n$-dimensional case

L. BASILE – L. D'APUZZO – M. SQUILLANTE

RIASSUNTO: *Si stabilisce un teorema di propagazione per una classe di funzioni verificanti una diseguaglianza limite di tipo Gehring estendendo al caso $n$-dimensionale un precedente risultato degli autori. Fondamentale per tale estensione è la caratterizzazione di tali funzioni mediante disuguaglianze inverse di tipo Chebychev; tale caratterizzazione è ottenuta utilizzando un teorema di ricoprimento stabilito da* BOJARSKI, SBORDONE *e* WIK *in* [3].

ABSTRACT: *We consider a class of functions verifying a limit case of Gehring inequalities and we state a propagation theorem that extends a previous result of the authors to the $n$-dimensional case. A crucial property to get this extension is a characterization of the functions in terms of Chebychev reverse inequalities; the main tool for obtaining this characterization is a covering lemma stated by* BOJARSKI, SBORDONE *and* WIK *in* [3].

## 1 – Introduction

The functional classes related to Gehring and Muckenhoupt conditions have been widely investigated (see, for instance [4], [7], [14], [17]; these conditions and the limit cases have been considered in some recent papers ([1], [2], [3], [5], [10], [11], [13], [16]). Let us recall some definitions

KEY WORDS AND PHRASES: *Reverse Hölder inequality – Muckenhoupt weight – Chebychev inequality.*
A.M.S. CLASSIFICATION: 42B25 – 42B26 – 26D15 – 46E30

and notations; through the paper, interval stands for an open bounded rectangle with sides parallel to the coordinate axes; furthermore, for a given real function defined over a set $X$, $f_Y$ stands for the restriction of $f$ to $Y$, for every $Y \subseteq X$.

In the sequel, we will consider classes of non negative measurable functions defined over open bounded intervals of $R^n$. The class $A_p = A_p(I_0; k)$, $k \geq 1$, is the class of functions that verify the inequality

$$\Big(⨍_I f(x)dx\Big)\Big(⨍_I f(x)^{-\frac{1}{p-1}}dx\Big)^{p-1} \leq k\,, \tag{1.1}$$

for every interval $I$ contained in the open bounded interval $I_0$, where $⨍_I f\,dx$ stands for the mean value of $f$ over $I$: $\frac{1}{|I|}\int_I f\,dx$.

The Muckenhoupt class $A_1 = A_1(I_0; c)$, $c \geq 1$ is the class of the functions that verify, for every interval $I \subseteq I_0$, the inequality

$$\Big(⨍_I f(x)dx\Big) \leq c \operatorname*{ess\,inf}_I f(x)\,.$$

In [2] it has been introduced the Gehring limit class $G_\infty = G_\infty(I_0; c)$, $c > 1$, of the functions $f$ verifying the inequality

$$\operatorname*{ess\,sup}_I f(x) \leq c ⨍_I f(x)dx\,, \tag{1.2}$$

for every interval $I \subseteq I_0$. We have proved a theorem concerning, in the one dimensional case, the propagation of the inequalities (1.2) to inequalities of kind (1.1). The main tool in the proof is the increasing rearrangement $f_*$ of a function $f$: indeed, if $f$ is in $G_\infty$, then $f_*$ is in $G_\infty$ too, with the same constant. This argument fails to be true in the $n$-dimensional case ($n \geq 2$) as can be shown by a counter-example that uses an argument of [3] and the implication $f \in A_1 \Rightarrow 1/f \in G_\infty$.

Our goal in this paper is to extend the above result to the $n$-dimensional case. Indeed we prove the following

THEOREM 1. *Let $f$ be in $G_\infty(I_0; c)$, that is $f$ verifies the inequality* (1.2) *for every interval $I \subseteq I_0$; then $f$ is in $A_p(I_0; \frac{1}{c}(\frac{p-1}{p-c})^{p-1})$ for every $p > c$. The constant $\frac{1}{c}(\frac{p-1}{p-c})^{p-1}$ and the lower bound $c$ for the exponents cannot be improved.*

This theorem states for the class $G_\infty$ a result corresponding to one stated for the class $A_1$ in [3] in the one-dimensional case and extended in [10] to the $n$-dimensional case.

The class $G_\infty$ is related to the class $A_1$ by the implication $f \in A_1 \Rightarrow 1/f \in G_\infty$, but the reverse implication does not hold (see remark (2.8) in [2]): this justifies the interest for investigating the properties of the class $G_\infty$ as in the above theorem.

To get our result, in the same line of thinking of [10], but applying a covering lemma proved in [3], we obtain a characterization of $G_\infty$ in terms of a reverse Chebychev type inequality.

The method we follow doesn't yield, immediately, the propagation when the inequalities (1.2) are satisfied over cubes, because we should need a suitable covering lemma and a related characterization in terms of reverse Chebychev inequalities that are not available at the moment; propagation results, without the optimality for the exponent, related to the condition $A_1$ over spheres and cubes have been obtained in [11]; the similar problem for the class $G_\infty$ is an open question.

## 2 – Characterization of $G_\infty$

In the following, $\mu^{(n)}(\cdot)$ stands for the $n$-dimensional measure of a measurable subset of $R^n$; if $f$ is a non negative measurable function defined over a set $X$, the notation $\{f < \lambda\}$ stands for the set $\{x \in X : f(x) < \lambda\}$.

The following lemmas will be useful to obtain the characterization of $G_\infty$ in terms of Chebychev like inequalities.

Lemma 1 [3]. *Let $E$ be a measurable bounded set of $R$; then for every $\varepsilon > 0$ there exists a sequence $(I_\nu)_{\nu=1}^\infty$ of subintervals with mutually disjoint interiors such that*

i) $|E \cap \bigcup_\nu I_\nu| = |E|$

ii) $(1-\varepsilon)|I_\nu| \le |I_\nu \cap E| < |I_\nu| \qquad \nu = 1, 2, \dots$

Lemma 2. *Let $f$ be a nonnegative measurable function over a measurable set $X$ of $R^n$ of finite measure. Then the inequality*

$$\lambda |\{f < \lambda\}| \le c \int_{\{f<\lambda\}} f\,dx \qquad c > 1 \tag{2.1}$$

*holds for every* $\lambda \leq \operatorname{ess\,sup} f$ *if and only if the inequality*

$$f(x)|\{f < f(x)\}| \leq c \int_{\{f<f(x)\}} f\,dt$$

*holds a.e. in* $X$.

PROOF. Fix a representative of $f(x)$, call it $f$ and denote by $f(X)$ its range. Of course, it is enough to prove the "if" part of the statement, by choosing $\lambda \notin f(X)$, $\lambda \in [0, \operatorname{ess\,sup} f]$.

Suppose $\lambda \neq \operatorname{ess\,sup} f$ and set $\lambda^* = \sup\{y \in [\lambda, \operatorname{ess\,sup} f] : t \notin f(X)$ $\forall\, t \in [\lambda, y]\}$.

If $\lambda^* \in f(X)$ then

$$\lambda|\{x : f(x) < \lambda\}| \leq \lambda^*|\{x : f(x) < \lambda^*\}| \leq c \int_{\{f<\lambda^*\}} f\,dx = c \int_{\{f<\lambda\}} f\,dx\,.$$

If $\lambda^* \notin f(X)$ then we can consider a sequence $(\lambda_n)_{n\in N}$ such that $\lambda_n \in f(X)$ and $\lambda_n \to \lambda^*$: indeed it is enough to choose $\lambda_n \in [\lambda^*, \lambda^* + 1/2^n] \cap f(X)$ $\forall\, n \in N$.

Then we get

$$\begin{aligned}\lambda|\{x : f(x) < \lambda\}| \leq \lambda^*|\{x : f(x) < \lambda^*\}| = \lim_n \lambda_n|\{x : f(x) < \lambda_n\}| \leq \\ \leq c \lim_n \int_{\{f<\lambda_n\}} f\,dt = c \int_{\{f\leq\lambda^*\}} f\,dx = c \int_{\{f<\lambda\}} f\,dx\,,\end{aligned}$$

from which (2.1) immediately follows.

Suppose now $\lambda = \operatorname{ess\,sup} f \notin f(X)$. If $\lambda < +\infty$ then consider a sequence $(\lambda_n)_{n\in N}$ such that $\lambda_n \in [\lambda - 1/2^n, \lambda] \cap f(X)$ $\forall\, n \in N$.

Obviously $\lambda_n \to \lambda$ and $\{x : f(x) < \lambda\} = \bigcup_n \{x : f(x) < \lambda_n\}$. Then we get

$$\begin{aligned}\lambda|\{x : f(x) < \lambda\}| = \lim_n \lambda_n|\{x : f(x) < \lambda_n\}| \leq \\ \leq c \lim_n \int_{\{f<\lambda_n\}} f\,dx \leq c \int_{\{f<\lambda\}} f\,dx\,.\end{aligned}$$

If $\lambda = +\infty$ (2.1) follows in a similar way by choosing a suitable sequence $(\lambda_n)_{n\in N}$ in $f(X)$. □

Lemma 3. *Let $I_0 = (a_1, b_1) \times \cdots \times (a_n, b_n)$ be an open interval of $R^n$ and $f$ in $G_\infty(I_0; c)$. Then, for a.e. $x_n \in (a_n, b_n)$, the function $f(\cdot, x_n)$ lies in $G_\infty(J_0; c)$, where $J_0 = (a_1, b_1) \times \cdots \times (a_{n-1}, b_{n-1})$.*

Proof. Let $J$ be an open $(n-1)$-dimensional subinterval of $J_0$ and $\bar{x}_n \in (a_n, b_n)$. Choose $\delta > 0$ such that $I_\delta = (\bar{x}_n - \delta, \bar{x}_n + \delta) \subseteq (a_n, b_n)$. As $f$ lies in $G_\infty(I_0; c)$ it is

$$\operatorname*{ess\,sup}_{J\times I_\delta} f \le \frac{c}{|J \times I_\delta|} \int_{J\times I_\delta} f\,dx = \frac{c}{|J \times I_\delta|} \int_{I_\delta} \int_J f(\cdot, x_n) d\mu^{(n-1)} dx_n = \\ = c ⨍_{I_\delta} ⨍_J f(\cdot, x_n) d\mu^{(n-1)} dx_n\,;$$

therefore for a.e. $\bar{x}_n \in (a_n, b_n)$

$$\operatorname*{ess\,sup}_{J} f(\cdot, \bar{x}_n) \le \operatorname*{ess\,sup}_{J\times I_\delta} f \le c ⨍_{I_\delta} ⨍_J f(\cdot, x_n) d\mu^{(n-1)} dx_n$$

and, by Lebesgue differentiation theorem, we get, for $\delta \to 0$,

$$\operatorname*{ess\,sup}_{J} f(\cdot, \bar{x}_n) \le c ⨍_J f(\cdot, \bar{x}_n) d\mu^{(n-1)} \qquad \forall\, \bar{x}_n \in (a_n, b_n) \text{ a.e.} \qquad \square$$

The following theorem give a characterization of the functional class $G_\infty$ in terms of reverse Chebychev inequalities; it states a new result also in the one-dimensional case.

Theorem 2. *Let $I_0 = (a_1, b_1) \times \cdots \times (a_n, b_n)$ be an open interval of $R^n$. Then the following propositions are equivalent:*

a) *$f$ is in $G_\infty(I_0; c)$*

b) *for every open subinterval $I$ of $I_0$ and for every $\lambda \le \operatorname*{ess\,sup}_I f$ it results*

$$\lambda |\{f_I < \lambda\}| \le c \int_{\{f_I < \lambda\}} f\,dx \tag{2.2}$$

c) *for every open subinterval $I$ of $I_0$ and for a.e. every $x \in I$ it results*

$$f(x) |\{f_I < f(x)\}| \le c \int_{\{f_I < f(x)\}} f\,dt$$

Proof. The equivalence between b) and c) immediately follows by Lemma 2; then we have just to prove that a) is equivalent to b). Let $I$ be a subinterval of $I_0$; by (2.2) we get

$$\begin{aligned} c\int_I f\,dx = c\Big[\int_{\{f_I\geq\lambda\}} f\,dx + \int_{\{f_I<\lambda\}} f\,dx\Big] \geq \\ \geq c\lambda|\{f_I\geq\lambda\}| + \lambda|\{f_I<\lambda\}| \geq \lambda|I| \end{aligned}$$

and this ensures, for $\lambda = \operatorname{ess\,sup}_I f$, that $f$ lies in $G_\infty(I_0;c)$.

Conversely, let us suppose that $f$ belongs to $G_\infty(I_0;c)$; we shall prove the validity of (2.2) by induction on the dimension $n$.

In the case $n=1$, let $I$ be a subinterval of $I_0$, $\lambda \leq \operatorname{ess\,sup}_I f$, $E = \{x \in I : f(x) < \lambda\} = \{f_I < \lambda\}$ and $D$ the set of the density points of $I - E = \{x \in I : f(x) \geq \lambda\}$.

If $|E| = |I|$, then $\lambda = \operatorname{ess\,sup}_I f$ and (2.2) follows.

If $|E| < |I|$ then we apply Lemma 1 to $E$ and we get

$$\lambda|I\cap I_\nu| \leq \operatorname*{ess\,sup}_{I\cap I_\nu} f|I\cap I_\nu| \leq c⨍_{I\cap I_\nu} f\,dx\,, \qquad \text{for every } \nu\,. \tag{2.3}$$

Indeed by the strict inequality in ii) it results $|I_\nu - E| > 0$ for every $\nu = 1,2\ldots$; if it is also $|I_\nu \cap I - E| = |\{f_{I_\nu} \geq \lambda\}| > 0$ then

$$\lambda \leq \operatorname*{ess\,sup}_{I_\nu} f, \qquad \nu = 1,2\ldots,$$

and so, because $f$ is in $G_\infty(I_0;c)$, (2.3) follows.

Let now $|I_\nu \cap I - E| = |\{f_{I\cap I_\nu} \geq \lambda\}| = 0$: by the strict inequality in ii) and the maximality of the intervals $I_\nu = ]a_\nu, b_\nu[$ (see proof of Lemma 1 in [3]), if we consider, for $h \in N$, the interval $I_\nu^h = ]a_\nu - \frac{1}{2h}, b_\nu + \frac{1}{2h}[$, then we get $|I_\nu^h \cap I - E| = |\{f_{I\cap I_\nu^h} \geq \lambda\}| > 0$ and $\lambda \leq \operatorname*{ess\,sup}_{I\cap I_\nu^h} f$; so

$$\lambda|I\cap I_\nu| \leq \operatorname*{ess\,sup}_{I\cap I_\nu^h} f|I\cap I_\nu^h| \leq c⨍_{I\cap I_\nu^h} f\,dx$$

and for $h \to +\infty$ we obtain (2.3) again.

Moreover the conditions i) and ii) in the Lemma 1 ensure that

$$\begin{aligned}(1-\varepsilon)\Big|\bigcup_\nu I_\nu\Big| \le \sum_\nu |I_\nu \cap E| = \sum_\nu |I \cap I_\nu \cap E| = \\ = \Big|\bigcup_\nu I_\nu \cap E\Big| = |E| \le \Big|I \cap \bigcup_\nu I_\nu\Big|\end{aligned} \tag{2.4}$$

and from inequalities (2.4) and (2.3)

$$\begin{aligned}\lambda|E| \le \lambda\Big|I \cap \bigcup_\nu I_\nu\Big| = \lambda\sum_\nu |I\cap I_\nu| \le c\sum_\nu \Big[\int_{I\cap I_\nu \cap E} f\,dx + \int_{I\cap I_\nu - E} f\,dx\Big] = \\ = c\Big[\int_{\bigcup_\nu I_\nu \cap E} f\,dx + \int_{I\cap\bigcup_\nu I_\nu - E} f\,dx\Big] = c\Big[\int_E f\,dx + \int_{\bigcup_\nu I_\nu - E} f\,dx\Big];\end{aligned}$$

as $|\bigcup_\nu I_\nu - E| \to 0$ for $\varepsilon \to 0$ because of (2.4), the above inequality gives

$$\lambda|E| \le c\int_E f(x)dx\,,$$

that is (2.2) is verified in the one dimensional case.

Let now $I_0$ be an open interval of $R^n$, $n \ge 2$, and $f$ in $G_\infty(I_0; c)$. Let us suppose the validity of (2.2) in the dimension $n-1$. Given $I = (a_1, b_1) \times \cdots \times (a_n, b_n)$ and $\lambda \le \operatorname{ess\,sup}_I f$. We apply Lemma 1 to the set $E = \{x_n \in (a_n, b_n) : f_I(\cdot, x_n) < \lambda\}$ and we find that, for every $\nu$

$$|I_\nu - E| = |\{x_n \in I_\nu : f_I(\cdot, x_n) \ge \lambda\}| > 0\,; \tag{2.5}$$

further, if we set $I^{(n-1)} = (a_1, b_1) \times \cdots \times (a_{n-1}, b_{n-1})$, we get

$$\begin{aligned}(1-\varepsilon)\Big|\bigcup_\nu (I^{(n-1)} \times I_\nu)\Big| \le \Big|\bigcup_\nu (I^{(n-1)} \times (I_\nu \cap E))\Big| = \\ = \Big|I^{(n-1)} \times \Big(\bigcup_\nu I_\nu \cap E\Big)\Big| = |I^{(n-1)} \times E| \le \Big|\bigcup_\nu (I^{(n-1)} \times I_\nu)\Big|.\end{aligned} \tag{2.6}$$

Because of (2.5), for every $\nu$, it is $|I^{(n-1)} \times (I_\nu - E)| = |\{f_{I^{(n-1)}\times I_\nu} \ge \lambda\}| > 0$ and so $\lambda \le \operatorname{ess\,sup}_{I^{(n-1)}\times I_\nu} f$.

Then, by i) in Lemma 1 and as $f$ is in $G_\infty(I_0;c)$

$$
\begin{aligned}
&\lambda|I^{(n-1)}\times E| \le \lambda\Big|I^{(n-1)}\times\bigcup_\nu I_\nu\Big| = \lambda\sum_\nu |I^{(n-1)}\times I_\nu| \le \\
&\le \sum_\nu \operatorname*{ess\,sup}_{I^{(n-1)}\times I_\nu} f|I^{(n-1)}\times I_\nu| \le c\sum_\nu \int_{I^{(n-1)}\times I_\nu} f\,dx = \\
&= c\int_{\bigcup_\nu(I^{(n-1)}\times I_\nu)} f\,dx = c\Big[\int_{\bigcup_\nu(I^{n-1}\times(I_\nu\cap E))} f\,dx + \int_{\bigcup_\nu(I^{n-1}\times(I_\nu - E))} f\,dx\Big]\,.
\end{aligned}
$$

By i) in Lemma 1 and as $|\bigcup_\nu I^{(n-1)}\times(I_\nu - E)|\to 0$ for $\varepsilon\to 0$ because of (2.6), by the above inequality we get

$$\lambda|I^{(n-1)}\times E| \le c\int_{I^{(n-1)}\times E} f\,dx\,,$$

that is

$$\lambda\int_{I^{(n-1)}\times E}\chi_{\{f<\lambda\}}dx \le c\int_{I^{(n-1)}\times E} f\,dx\,. \tag{2.7}$$

On the other hand the formula (2.2) is true in the $n-1$ dimensional case and this fact, with Lemma 3, ensures that

$$
\begin{aligned}
&\lambda\int_{I^{(n-1)}\times((a_n,b_n)-E)}\chi_{\{f<\lambda\}}dx = \\
&\quad = \int_{(a_n,b_n)-E} dx_n\cdot\lambda\int_{I^{(n-1)}}\chi_{\{f<\lambda\}}d\mu_{n-1} \le \\
&\quad \le \int_{(a_n,b_n)-E} dx_n\cdot c\int_{I^{(n-1)}} f\chi_{\{f<\lambda\}}d\mu_{n-1} = \\
&\quad = c\int_{I^{(n-1)}\times((a_n,b_n)-E)} f\chi_{\{f<\lambda\}}dx\,.
\end{aligned} \tag{2.8}
$$

By (2.7) and (2.8) it follows

$$\lambda|\{f_I<\lambda\}| = \lambda\int_I \chi_{\{f<\lambda\}}dx \le c\int_{\{f_I<\lambda\}} f\,dx\,. \qquad \square$$

REMARK 1. An alternative way to get the implication a) $\Rightarrow$ b) in the one dimensional case is suggested by the proof of Lemma 3.4 of [10] as it follows.

Let $I$ be a subinterval of $I_0$, $\lambda = \operatorname{ess\,sup}_I f$, $E = \{x \in I : f(x) < \lambda\} = \{f_I < \lambda\}$ and $D$ the set of the density points of $I - E = \{x \in I : f(x) \geq \lambda\}$.

If $|E| = |I|$, then $\lambda = \operatorname{ess\,sup}_I f$ and (2.2) follows.

Suppose $|E| < |I|$. Then for every $\varepsilon \in ]0, |I - E|[$, we consider an open set $U_\varepsilon \subset I$ containing the set $E \bigcup [(I - E) - D]$ with $|U_\varepsilon| < |E| + \varepsilon$.

Since $U_\varepsilon$ is open there are countably many pairwise disjoint open subintervals $I_\nu$ of $I$ such that $U_\varepsilon = \bigcup_\nu I_\nu$.

For $\sigma > 1$, and for every $\nu$, set $J_\nu = \sigma I_\nu \cap I$. It results $I_\nu \subset J_\nu \subset I$ and $|J_\nu \cap (I - E)| > 0$, then $\lambda = \operatorname{ess\,sup}_{J_\nu} f$ and so

$$\lambda |J_\nu| \leq \operatorname*{ess\,sup}_{J_\nu} f \cdot |J_\nu| \leq c \int_{J_\nu} f(x)dx$$

that is

$$\lambda \sigma |I_\nu| \leq c \int_{\sigma I_\nu} f(x)dx$$

and, as $\sigma \to 1$,

$$\lambda |I_\nu| \leq c \int_{I_\nu} f(x)dx\,.$$

So we have:

$$\begin{aligned}\lambda|E| \leq \lambda|U_\varepsilon| = \lambda \sum_\nu |I_\nu| \leq c \sum_\nu \int_{I_\nu} f\,dx = c \int_{U_\varepsilon} f\,dx = \\ = c \int_E f\,dx + \int_{U_\varepsilon - E} f\,dx\,,\end{aligned}$$

and, because $|U_\varepsilon - E| \to 0$ for $\varepsilon \to 0$, we get $\lambda|E| \leq c \int_E f\,dx$.

COROLLARY. *In the hypotheses of Theorem* 2 *if $f$ is in $G_\infty(I_0; c)$, then the inequality* (2.2) *holds for every open subinterval $I$ of $I_0$ and for every $\lambda \leq \alpha_I = c⨍_I f(x)dx$.*

PROOF. If $\lambda \leq \operatorname{ess\,sup}_I f$, then the assertion follows from the implication a) $\Rightarrow$ b) of Theorem 2. If $\operatorname{ess\,sup}_I f < \lambda \leq \alpha_I = c⨍_I f(x)dx$, then it is $\{f_I < \lambda\} = I$ and so $\lambda|\{f_I < \lambda\}| \leq \alpha_I |I| = c \int_{\{f_I < \lambda\}} f\,dx$. ∎

## 3 – Propagation

The following lemmas are crucial to obtain the propagation of the inequalities (1.2), characterizing the class $G_\infty$, into the inequalities (1.1) with suitable constants and exponents.

Lemma 4. *Let $\mu$ be a $\sigma$-finite positive measure on a $\sigma$-algebra over a set $X$ and $f : X \to ]0,+\infty[$ measurable. If $\varphi : ]0,+\infty[ \to ]0,+\infty[$ is decreasing, absolutely continuous in $]0,t[$ for every $t < +\infty$, and it results $\lim_{t\to+\infty}\varphi(t) = 0$, then*

$$\int_X \varphi \circ f \, d\mu = -\int_0^{+\infty} \varphi'(t)\mu(\{f < t\})dt\,.$$

Proof. It follows by Fubini theorem and the equality $\varphi(f(x)) = -\int_{f(x)}^{+\infty} \varphi' dt$ (see also [15]). ∎

Lemma 5. *In the hypotheses of Lemma 4 it results, for every $\lambda > 0$*

$$\int_{\{f<\lambda\}} \varphi(f(x))d\mu = -\int_0^{\lambda} \varphi'(t)\mu(\{f < t\})dt + \varphi(\lambda)\mu(\{f < \lambda\})\,. \tag{3.1}$$

Proof. By applying Lemma 4 to the set $E_\lambda = \{f < \lambda\}$, we get

$$\begin{aligned}
\int_{\{f<\lambda\}} \varphi(f(x))d\mu &= -\int_0^{+\infty} \varphi'(t)\mu(\{x \in E_\lambda : f(x) < t\})dt = \\
&-\int_0^{\lambda} \varphi'(t)\mu(\{x \in E_\lambda : f(x) < t\})dt + \\
&-\int_\lambda^{+\infty} \varphi'(t)\mu(\{x \in E_\lambda : f(x) < t\})dt = \\
&-\int_0^{\lambda} \varphi'(t)\mu(\{x \in E_\lambda : f(x) < t\})dt - \mu(E_\lambda)\int_\lambda^{+\infty} \varphi'(t)dt = \\
&-\int_0^{\lambda} \varphi'(t)\mu(E_t)dt + \varphi(\lambda)\mu(E_\lambda)\,.
\end{aligned}$$

∎

Remark 2. By choosing $\varphi(t) = t^r$, $r < 0$, the equality (3.1) becames

$$\int_{\{f<\lambda\}} f(x)^r d\mu = -r\int_0^\lambda t^{r-1}\mu(\{f<t\})dt + \lambda^r\mu(\{f<\lambda\})\,. \tag{3.2}$$

The next lemma ensures the propagation of the reverse Chebychev type inequalities (2.1) and the sommability of suitable negative powers of the involved function $f$.

Lemma 6. *Let $X \subseteq R^n$ be a measurable set with $|X| < +\infty$ and $f$ a positive measurable function on $X$. If there exist $\alpha > 0$ and $c > 1$ s.t.*

$$\lambda|\{f<\lambda\}| \le c\int_{\{f<\lambda\}} f\,dx \tag{3.3}$$

*for every $\lambda \le \alpha$, then, for every $\lambda \le \alpha$ and $r : \frac{1}{1-c} < r < 0$, it results*

$$\int_{\{f<\lambda\}} f(x)^r dx \le \frac{\lambda^r}{rc - r + 1}|\{f<\lambda\}|\,. \tag{3.4}$$

Proof. Let $r$ be a negative number and $d\mu = f\,dx$; by applying the equality (3.2) with exponent $r-1$, we get

$$\begin{aligned}\int_{\{f<\lambda\}} f^r dx &= \int_{\{f<\lambda\}} f^{r-1}d\mu = -(r-1)\int_0^\lambda t^{r-2}\mu(\{f<t\})dt+\\ &\quad + \lambda^{r-1}\mu(\{f<\lambda\}) =\\ &= (1-r)\int_0^\lambda t^{r-2}\Big(\int_{\{f<t\}} d\mu\Big)dt + \lambda^{r-1}\int_{\{f<\lambda\}} d\mu =\\ &= (1-r)\int_0^\lambda t^{r-2}\Big(\int_{\{f<t\}} f\,dx\Big)dt + \lambda^{r-1}\int_{\{f<\lambda\}} f\,dx\,;\end{aligned}$$

then, by (3.3)

$$\begin{aligned}\int_{\{f<\lambda\}} f^r dx &\ge (1-r)\int_0^\lambda t^{r-2}\frac{t}{c}|\{f<t\}|dt + \lambda^{r-1}\frac{\lambda}{c}|\{f<\lambda\}| =\\ &= \frac{1-r}{c}\int_0^\lambda t^{r-1}|\{f<t\}|dt + \frac{\lambda^r}{c}|\{f<\lambda\}|\,;\end{aligned}$$

by the above inequality and representing $\int_0^\lambda t^{r-1}|\{f<t\}|dt$ by means of the formula (3.2), we obtain the inequality

$$\int_{\{f<\lambda\}} f^r dx \geq \frac{r-1}{rc}\Big[\int_{\{f<\lambda\}} f^r dx - \lambda^r|\{f<\lambda\}|\Big] + \frac{\lambda^r}{c}|\{f<\lambda\}|$$

or, in other terms

$$\frac{r-1-rc}{rc}\int_{\{f<\lambda\}} f^r dx \leq \lambda^r \frac{r-1}{rc}|\{f<\lambda\}| - \frac{\lambda^r}{c}|\{f<\lambda\}| = -\frac{\lambda^r}{rc}|\{f<\lambda\}|\,;$$

for $\frac{r-1-rc}{rc} > 0$, that is $r > \frac{1}{1-c}$, the above inequality is equivalent to (3.4). ∎

Remark 3. The inequality (3.4), written for $r = -\frac{1}{p-1}$, holds for $p > c$:

$$\int_{\{f<\lambda\}} f^{-\frac{1}{p-1}} dx \leq \lambda^{-\frac{1}{p-1}} \frac{p-1}{p-c}|\{f<\lambda\}|\,. \tag{3.5}$$

Remark 4. The lower bound $\frac{1}{1-c}$ for the exponent r and the constant $\frac{\lambda^r}{rc-r+1}$ in (3.4) are the best possible. Indeed for the function

$$f : x = (x_1, x_2, \cdots, x_n) \in (0,1)^n \to x_1^{c-1} \qquad c > 1$$

and for $\lambda \leq 1$, it results

$$\{f<\lambda\} = \{x : x_1^{c-1} < \lambda\} = \{x : x_1 < \lambda^{\frac{1}{c-1}}\} = (0, \lambda^{\frac{1}{c-1}}) \times (0,1)^{n-1}$$

and

$$\int_{\{f<\lambda\}} f\,dx = \int_0^{\lambda^{\frac{1}{c-1}}} x_1^{c-1} dx_1 = \Big[\frac{x_1^c}{c}\Big]_0^{\lambda^{\frac{1}{c-1}}} = \frac{1}{c}\lambda\lambda^{\frac{1}{c-1}}\,;$$

finally

$$c\int_{\{f<\lambda\}} f\,dx = \lambda|\{f<\lambda\}|\,;$$

so $f$ verifies the condition (3.3) for every $\lambda \le 1$. Furthermore $f$ is not integrable for $r \le \frac{1}{1-c}$ and

$$\int_{\{f<\lambda\}} f^r dx = \int_{\{f<\lambda\}} x_1^{r(c-1)} dx = \int_0^{\lambda^{\frac{1}{c-1}}} x_1^{rc-r} dx_1 = \Big[\frac{x_1^{rc-r+1}}{rc-r+1}\Big]_0^{\lambda^{\frac{1}{c-1}}} = \\ = \frac{\lambda^{\frac{rc-r+1}{c-1}}}{rc-r+1} = \frac{\lambda^r}{rc-r+1}\lambda^{\frac{1}{c-1}}$$

that is

$$\int_{\{f<\lambda\}} f^r dx = \frac{\lambda^r}{rc-r+1}|\{f<\lambda\}|\,. \qquad \square$$

Once the above results have been acquired, we are able to exhibit the proof of Theorem 1 (see the Introduction)

Proof of Theorem 1. By applying the corollary of Theorem 2 and Lemma 6 with $r = -\frac{1}{p-1}$ (see Remark 3) we get, for every interval $I \subseteq I_0$, $\lambda \le c⨍_I f(x)dx$ and $p > c$

$$\begin{aligned}\int_I f^{-\frac{1}{p-1}} dx &= \int_{\{f_I<\lambda\}} f^{-\frac{1}{p-1}} dx + \int_{\{f_I\ge\lambda\}} f^{-\frac{1}{p-1}} dx \le \\ &\le \lambda^{-\frac{1}{p-1}}\Big(\frac{p-1}{p-c}\Big)|\{f_I<\lambda\}| + \int_{\{f_I\ge\lambda\}} f^{-\frac{1}{p-1}} dx \le \\ &\le \lambda^{-\frac{1}{p-1}}\Big(\frac{p-1}{p-c}\Big)|\{f_I<\lambda\}| + \lambda^{-\frac{1}{p-1}}|\{f_I\ge\lambda\}| \le \\ &\le \lambda^{-\frac{1}{p-1}}\Big(\frac{p-1}{p-c}\Big)|\{f_I<\lambda\}| + \\ &\quad + \lambda^{-\frac{1}{p-1}}\Big(\frac{p-1}{p-c}\Big)|\{f_I\ge\lambda\}| = \lambda^{-\frac{1}{p-1}}\Big(\frac{p-1}{p-c}\Big)|I|\,.\end{aligned}$$

By choosing $\lambda = c⨍_I f(x)dx$ we obtain

$$\int_I f^{-\frac{1}{p-1}} dx \le \Big(c⨍_I f(x)dx\Big)^{-\frac{1}{p-1}} \frac{p-1}{p-c}|I|$$

that is

$$\Big(⨍_I f(x)dx\Big)\Big(⨍_I f(x)^{-\frac{1}{p-1}} dx\Big)^{p-1} \le \frac{1}{c}\Big(\frac{p-1}{p-c}\Big)^{p-1}. \tag{3.6}$$

The constant $\frac{1}{c}(\frac{p-1}{p-c})^{p-1}$ and the lower bound $c$ for the exponent $p$ are the best possible. Indeed the function

$$f : x = (x_1, x_2, \cdots, x_n) \in (0,1)^n \to x_1^{c-1}, \qquad c > 1\,,$$

is in $G_\infty(I_0; c)$ and verifies (3.7) as equalities, for every $p > c$; furthermore $f$ lies in $L^{-\frac{1}{p-1}}$ if, and only if, $p > c$. ∎

## REFERENCES

[1] L. Basile – L. D'Apuzzo – M. Squillante: *Relations between Gehring classes and Muckenoupt classes via the Hardy operator*, Ricerche di Matematica, **45** (1996), 157-177.

[2] L. Basile – L. D'Apuzzo – M. Squillante: *The limit class of Gehring type*, Bollettino dell'Unione Matematica Italiana, (7) **11-B** (1997), 871-884.

[3] B. Bojarski – C. Sbordone – I. Wik: *The Muckenhoupt class $A_1(R)$*, Studia Mathematica, **101** (1992), 155-163.

[4] C. Coifmann – C. Fefferman: *Weighted norm inequalities for maximal functions and singular integrals*, Studia Mathematica, **51** (1974), 241-250.

[5] D. Cruz-Uribe: *Piecewise monotonic doubling measures*, Rocky Mtn. J. Math., **26** (1996), 1-39.

[6] L. D'Apuzzo – C. Sbordone: *Reverse Hölder inequalities. A sharp result*, Rendiconti di Matematica, serie VII, **10** (1990), 357-366.

[7] F.W. Gehring: *The $L^p$-integrability of the partial derivatives of a quasiconformal mapping*, Acta Mathematica, **130** (1973), 145-152.

[8] J. Garcia-Cuerva – J. Rubio de Francia: *Weighted norm inequalities and related topics*, North Holland, 1985.

[9] G.H. Hardy – J.E. Littlewood – G. Polya: *Inequalities*, Cambridge University Press, Cambridge, 1934.

[10] J. Kinnunen: *Sharp results on reverse Hölder inequalities*, Annales Accademiae Sientiarum Fennicae, Series A, I. Mathematica, **95** (1994).

[11] J. Kinnunen: *A stability result on Muckenhoupt's Weights*, Publicacions Matematiques, **42** (1998), 153-163.

[12] A.A. Korenovskii: *The exact continuation of a reverse Hölder inequality and Muckenhoupt conditions*, Math. Notes, **52** (1993) 1192-1201.

[13] G. Moscariello – C. Sbordone: *$A_\infty$ as a limit case of reverse-Hölder inequalities when the esponent tends to 1*, Ricerche Mat., **44** (1995), 131-144.

[14] B. Muckenhoupt: *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., **165** (1972), 207-226.

[15] W. Rudin: *Real and complex analysis*, McGraw-Hill Interntional editions, Mathematics Series.

[16] C. Sbordone: *Weighted norm inequalities and their limit cases*, "Proc. of Variational Methods Nonlinear Analysis and Differential Equations", Genova Nervi September 1993, pp. 164-171.

[17] C. Sbordone – I. Wik: *Maximal functions and related weight classes*, Publicacions Matemàtiques, **38** (1994), 127-155.

[18] J.O. Strömberg – A. Torchinsky: *Weighted Hardy spaces*, Lectures Notes in Math., **1381** (1989).

*Lavoro pervenuto alla redazione il 21 ottobre 1999*
*modificato il 20 ottobre 2000*
*ed accettato per la pubblicazione il 29 gennaio 2001.*
*Bozze licenziate il 13 giugno 2001*

Indirizzo degli Autori:

L. Basile – L. D'Apuzzo – Dipartimento di Costruzioni e Metodi Matematici in Architettura – Facoltà di Architettura – Università degli Studi di Napoli "Federico II" – Napoli – Italy

M. Squillante – Facoltà di Economia – Università del Sannio – Benevento – Italy