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A generalization of Vesentini's theorem to locally m-convex Q-algebras

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RIASSUNTO: *Lo scopo di questo lavoro è la generalizzazione del teorema di Vesentini sulle algebre m-convesse. Si considera il caso delle Q-algebre m-convesse e sequenzialmente complete. Si ottiene un teorema del tipo di quello di Vesentini per le algebre m-convesse e sequenzialmente complete che non sono necessariamente Q-algebre. Il risultato finale è un teorema del tipo di quello di Kleinecke-Shirokov per algebre m-convesse e sequenzialmente complete.*

ABSTRACT: *The aim of this paper is to generalize Vesentini's theorem to m-convex algebras. We obtain this result in case of sequentially complete m-convex Q-algebras. As an auxiliary result a Vesentini type theorem for sequentially complete m-convex algebras which are not necessarily Q-algebras is previously proved. As a final result a Kleinecke-Shirokov type theorem for sequentially complete m-convex algebras is obtained.*

1 – Preliminaries

We now briefly recall some facts needed in the sequel.

Let A be a locally m-convex algebra with identity, whose topology is defined by the separating family $(p_\alpha)_{\alpha\in I}$ of submultiplicative seminorms such that $p_\alpha(1) = 1$ for each $\alpha \in I$. Moreover, A is a Q-algebra if the set of all invertible elements of A is open.

KEY WORDS AND PHRASES: *m-convex algebra – Spectral radius – Subharmonic function.*
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We denote by $G(A)$ the set of all invertible elements of A, by $\sigma(x)$ the set

$$\sigma(x) = \{\lambda \in \mathbb{C}/\lambda \cdot 1 - x \notin G(A)\}$$

called the spectrum of the element x of A and by $\rho(x)$ the spectral radius of x,

$$\rho(x) = \sup_{\lambda \in \sigma(x)} |\lambda| .$$

The element $x \in A$ is called quasi-nilpotent if $\rho(x) = 0$.

It is well known that

$$\lim_{n\to\infty} (p_\alpha(x^n))^{1/n} = \inf_n (p_\alpha(x^n))^{1/n}$$

for any $x \in A$ and $\alpha \in I$; therefore, if we set

$$\rho_\alpha(x) = \lim_{n\to\infty} (p_\alpha(x^n))^{1/n}$$

for $x \in A$ and $\alpha \in I$, then in accordance with the generalization to sequentially complete m-convex algebras of Beurling's formula (see [5], [7]) we have

$$\rho(x) = \sup_{\alpha\in I} \lim_{n\to\infty} (p_\alpha(x^n))^{1/n}$$

for all $x \in A$ and $\alpha \in I$, and so

$$\rho(x) = \sup_{\alpha \in I} \rho_\alpha(x)$$

for any $x \in A$ and $\alpha \in I$.

We can consider $(\rho_\alpha)_{\alpha\in I}$ as a family of spectral radiuses and it is easy to prove that each of these have the following properties:

(i) $\rho_\alpha(xy) = \rho_\alpha(yx)$, for any x, y of A

(ii) $\rho_\alpha(\lambda x) = |\lambda|\rho_\alpha(x)$, for any $\lambda \in \mathbb{C}$ and $x \in A$.

These properties are used at the end of this paper when we give an application of the generalization of Vesentini's theorem.

Let $D \subset \mathbb{C}$ be an open connected set. Then $u : D \to [-\infty, \infty)$ is a subharmonic function if u is upper semicontinuous and

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta$$

whenever $\overline{D_r(z_0)} \subset D$.

As it is known, subharmonicity is preserved by finite sums, products with positive scalars, finite supremums, limits of decreasing sequences. Moreover, $\varphi \circ u$ is subharmonic whenever u is subharmonic and φ is convex and increasing.

We recall that for u a subharmonic function on the domain $D \subset \mathbb{C}$ and for $z_0 \in D$ one has

$$u(z_0) = \overline{\lim_{\substack{z \to z_0 \\ z \neq z_0}}} u(z)\,.$$

2 – A generalization of Vesentini's theorem

As an auxiliary result, we prove the following theorem.

THEOREM 2.1. *Let A be a sequentially complete unital m-convex algebra and $D \subset \mathbb{C}$ an open connected set. If f is an analytic function defined on D and A valued, then for each $\alpha \in I$, $p_\alpha(f(\cdot))$ as well as $\ln p_\alpha(f(\cdot))$ are subharmonic functions.*

PROOF. Let $\alpha \in I$. First we prove that $p_\alpha(f(\cdot))$ is a subharmonic function. To this end, let λ be an arbitrary real number and

$$M = \{z \in D / p_\alpha(f(z)) < \lambda\}\,.$$

We claim that M is an open set. Indeed, if we suppose that M is not an open set, then there exists $z_0 \in M$ and the sequence $z_n \underset{n}{\longrightarrow} z_0$ such that $p_\alpha(f(z_n)) \geq \lambda$. Since $z_0 \in M$, we have $p_\alpha(f(z_0)) < \lambda$. Further f being an analytic function, f is continuous which implies $f(z_n) \underset{n}{\longrightarrow} f(z_0)$. From this and from the inequalities

$$-p_\alpha(f(z_n) - f(z_0)) \leq p_\alpha(f(z_n)) - p_\alpha(f(z_0)) \leq p_\alpha(f(z_n) - f(z_0))$$

we get $p_\alpha(f(z_n)) \underset{n}{\longrightarrow} p_\alpha((f(z_0))$. Hence, $p_\alpha(f(z_0)) \geq \lambda$, contradicting the fact that $z_0 \in M$. We now prove that whenever $\overline{D_r(z_0)} \subset D$,

$$p_\alpha(f(z_0)) \leq \frac{1}{2\pi} \int_0^{2\pi} p_\alpha(f(z_0 + re^{i\theta}))d\theta\,.$$

From Cauchy's integral representation theorem it follows that

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(\zeta)}{\zeta - z_0} d\zeta\,.$$

From the Hahn-Banach theorem (see [3]) we get the existence of a linear and continuous form F_α on A such that $p_\alpha(f(z_0)) = F_\alpha(f(z_0))$ and $|F_\alpha(y)| \leq p_\alpha(y)$ for any $y \in A$.

Now we have

$$\begin{aligned} p_\alpha(f(z_0)) = F_\alpha(f(z_0)) &= F_\alpha\Big(\frac{1}{2\pi i}\int_{\partial D_r(z_0)} \frac{f(\zeta)}{\zeta - z_0} d\zeta\Big) = \\ &= \frac{1}{2\pi i}\int_{\partial D_r(z_0)} \frac{F_\alpha(f(\zeta))}{\zeta - z_0} d\zeta = \\ &= \frac{1}{2\pi}\int_0^{2\pi} F_\alpha(f(z_0 + re^{i\theta}))d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} p_\alpha(f(z_0 + re^{i\theta}))d\theta\,. \end{aligned}$$

Hence, $p_\alpha(f(\cdot))$ is a subharmonic function.

Now we show that $\ln p_\alpha(f(\cdot))$ is upper semicontinuous. To this end we are going to prove that for λ an arbitrary real number, the set

$$M = \{z \in D / \ln p_\alpha(f(z)) < \lambda\}$$

is an open set. Let $z_0 \in M$. Then $\ln p_\alpha(f(z_0)) < \lambda$ and hence $p_\alpha(f(z_0)) < e^\lambda$. Because $p_\alpha(f(\cdot))$ is upper semicontinuous there exists $r > 0$ with the property that for every z for which $|z - z_0| < r$ we have $p_\alpha(f(z)) < e^\lambda$. It follows that there exists $r > 0$ such that for every z for which $|z - z_0| < r$ we have $\ln p_\alpha(f(z)) < \lambda$.

From the fact that $D_r(z_0) \subset M$ it follows that M is an open set and hence $\ln p_\alpha(f(\cdot))$ is upper semicontinuous.

As it is known, if u is a complex analytic function then $\ln|u|$ is a subharmonic function.

Now, using Hahn-Banach theorem we get the existence of a linear and continuous form F_α on A such that $p_\alpha(f(z_0)) = F_\alpha(f(z_0))$ and $|F_\alpha(y)| \leq p_\alpha(y)$ for any $y \in A$. Also we use the fact that $F_\alpha(f(\cdot))$ is an analytic function. So we have

$$\begin{aligned} \ln p_\alpha(f(z_0)) = \ln F_\alpha(f(z_0)) &\leq \frac{1}{2\pi}\int_0^{2\pi} \ln F_\alpha(f(z_0 + re^{i\theta}))d\theta \leq \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \ln p_\alpha(f(z_0 + re^{i\theta}))d\theta\,, \end{aligned}$$

whenever $\overline{D_r(z_0)} \subset D$.

So, $\ln p_\alpha(f(\cdot))$ is a subharmonic function. $\square$

THEOREM 2.2. *Let A be a sequentially complete m-convex algebra, $D \subset C$ be open connected and $f : D \to A$ be an analytic function. Then $\rho_\alpha(f(\cdot))$ is subharmonic for any $\alpha \in I$.*

PROOF. Let $\alpha \in I$. For $z \in D$

$$\rho_\alpha(f(z)) = \lim_{n\to\infty} p_\alpha(f(z)^n)^{1/n} = \inf_n p_\alpha(f(z)^n)^{1/n}\,.$$

Because f is an analytic function, f^n is also analytic for every n positive integer. By Theorem 2.1. the mapping $z \to \ln p_\alpha(f(z)^n)$ is subharmonic which implies that the mappings

$$\begin{aligned} &z \to \frac{1}{n}\ln p_\alpha(f(z)^n)\\ &z \to \ln p_\alpha(f(z)^n)^{1/n}\\ &z \to p_\alpha(f(z)^n)^{1/n}\\ &z \to \inf_n p_\alpha(f(z)^n)^{1/n}\end{aligned}$$

are subharmonics.

Thus, the mapping $z \to \rho_\alpha(f(z))$ is a subharmonic function. $\square$

THEOREM 2.3. *Let A be a sequentially complete unital m-convex Q-algebra, $D \subset \mathbb{C}$ be an open connected set and $f : D \to A$ be an analytic function. Then $\rho(f(\cdot))$ is a subharmonic function.*

PROOF. Let $z_0 \in D$, $r > 0$ such that $\overline{D_r(z_0)} \subset D$ and $\alpha \in I$. From Theorem 2.2. it follows that

$$\rho_\alpha(f(z_0)) \le \frac{1}{2\pi}\int_0^{2\pi} \rho_\alpha(f(z_0 + re^{i\theta}))d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \rho(f(z_0 + re^{i\theta}))d\theta\,.$$

Therefore

$$\rho(f(z_0)) = \sup_{\alpha\in I} \rho_\alpha(f(z_0)) \le \frac{1}{2\pi}\int_0^{2\pi} \rho(f(z_0 + re^{i\theta}))d\theta\,.$$

Now we have to prove the upper semicontinuity of $\rho(f(\cdot))$.

For this, we first prove that if $M \subset A$ is a bounded set then $\sup_{x\in M} \rho(x)$ is finite.

We suppose that $\sup_{x\in M} \rho(x)$ is not finite. Then $\sup_{x\in M} \rho(x) > n$ for any n positive integer.

For n positive integer there exists $x_n \in M$ such that $\rho(x_n) > n$.

Let $\lambda_n \in \sigma(x_n)$ satisfying $\rho(x_n) > |\lambda_n| > n$.

From the fact that $0 \neq \lambda_n \in \sigma(x_n)$ it follows that $1 \in \sigma(\frac{1}{\lambda_n} x_n)$ and because $|\lambda_n| \underset{n}{\longrightarrow} \infty$ we have $\frac{1}{\lambda_n} x_n \underset{n}{\longrightarrow} 0$ and so $1 - \frac{1}{\lambda_n} x_n \underset{n}{\longrightarrow} 1$.

Since A is a Q-algebra and 1 is an invertible element we get $1 - \frac{1}{\lambda_n} x_n$ is an invertible element, in contradiction with $1 \in \sigma(\frac{1}{\lambda_n} x_n)$.

Therefore, we must have that $\sup_{x\in M} \rho(x)$ is finite.

To prove the upper semicontinuity of $\rho(f(\cdot))$ we consider r one real number and the set $M = \{z \in \mathbb{C}/\rho(f(z)) < r\}$.

In order to show that M is an open set, we take $z_0 \in M$, $r_0 = \rho(f(z_0)) < r$ and $r_1 > 0$ such that $r_0 < r_1 < r$.

Suppose M is not open. Then there exists the sequence $z_n \underset{n}{\longrightarrow} z_0$ and $\rho(f(z_n)) \geq r > r_1$. Let $\lambda_n \in \sigma(f(z_n))$ such that $|\lambda_n| \geq r_1$. Since for each n positive integer $|\lambda_n| \leq \rho(f(z_n))$ we have $\sup_n |\lambda_n| \leq \sup_n \rho(f(z_n))$.

Now because $\sup_n |\lambda_n|$ is finite we can choose a convergent subsequence of $(\lambda_n)_n$, $\lambda_n \underset{n}{\longrightarrow} \lambda_0$, and $|\lambda_0| \geq r_1$.

From the facts $1 - \frac{1}{\lambda_n} f(z_n) \underset{n}{\longrightarrow} 1 - \frac{1}{\lambda_0} f(z_0)$ and $1 - \frac{1}{\lambda_n} f(z_n) \notin G(A)$ it follows that $1 - \frac{1}{\lambda_0} f(z_0) \notin G(A)$.

Hence $\lambda_0 \in \sigma(f(z_0))$ and $\rho(f(z_0)) \geq |\lambda_0| \geq r_1$. We have $r_0 \geq r_1$, contradicting the choice of r_0, r_1 such that $r_0 < r_1$. Thus we get that $\rho(f(\cdot))$ is a subharmonic function. □

Corollary 2.4 (Vesentini's theorem, see [1], [6]). *If f is an analytic Banach algebra valued mapping on D then $\rho(f(\cdot))$ is subharmonic.*

3 – An application

Theorem 3.1. *Let A be a sequentially complete unital m-convex algebra and x, $y \in A$ such that x commutes with $xy - yx$. Then $xy - yx$ is quasi-nilpotent.*

Proof. For x, $y \in A$ we denote $[x, y] = xy - yx$.

The mapping $\lambda \to \exp(\lambda x) y \exp(-\lambda x)$ is an analytic function and $f'(\lambda) = [x, \exp(\lambda x) y \exp(-\lambda x)]$.

By induction is easy to verify that its n-th derivative is

$$f^{(n)}(\lambda) = \underbrace{[x, [x, \dots [}_{n} x, \exp(\lambda x) y \exp(-\lambda x)] \dots]] .$$

Because f is an analytic function, we can use Taylor's formula

$$f(\lambda) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \lambda^n .$$

So, $f(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} [x, [x, \dots [x, y] \dots]] = y + \lambda [x, y]$.

Consequently, we have for any $z \neq 0$ complex number

$$zy + [x, y] = \exp\Big(\frac{1}{z} x\Big)(zy) \exp\Big(-\frac{1}{z} x\Big) .$$

Let $\alpha \in I$. We have

$$\rho_\alpha(zy + [x, y]) = \rho_\alpha\Big(\exp\Big(\frac{1}{z} x\Big)(zy) \exp\Big(-\frac{1}{z} x\Big)\Big) = |z| \rho_\alpha(y) .$$

By the subharmonicity of $z \to \rho_\alpha(zy + [x, y])$ and by the above property of subharmonic functions we get

$$\rho_\alpha(xy - yx) = \varlimsup_{\substack{z \to 0 \\ z \neq 0}} \rho_\alpha(zy + [x, y]) = \varlimsup_{\substack{z \to 0 \\ z \neq 0}} |z| \rho_\alpha(y) = 0 .$$

So $\rho_\alpha(xy - yx) = 0$ for any $\alpha \in I$. Hence $\rho(xy - yx) = 0$ and $xy - yx$ is a quasi-nilpotent element of A. $\square$

Corollary 3.2 (Kleinecke-Shirokov). *If x, y belong to a Banach algebra and x commutes with $xy - yx$ then $xy - yx$ is quasi-nilpotent.*

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