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Theorems of Perron type for evolution operators

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RIASSUNTO: *Lo scopo di questo articolo è di dare condizioni necessarie e sufficienti per la stabilità esponenziale uniforme degli operatori di evoluzione in spazi di Banach. Si ottengono cosi generalizzationi di alcuni teoremi dovuti a Datko, Neerven, Clark, Latushkin, Montgomery-Smith, Randolph, van Minh, Räbiger e Schnaubelt.*

ABSTRACT: *The aim of this paper is to give necessary and sufficient conditions for uniform exponential stability of evolution operators in Banach spaces. Thus, there are obtained generalizations of some theorems due to Datko, Neerven, Clark, Latushkin, Montgomery-Smith, Randolph, van Minh, Räbiger and Schnaubelt.*

1 – Introduction

In Perron's classical papers on stability a central concern is to establish connections between the asymptotic properties of the solutions of the equation

$$\text{(E)} \qquad \frac{d}{dt}u(t) = A(t)u(t), \quad t \in \mathbb{R}_+ ,$$

on a Banach space X and the specific properties of the operator defined by

$$\mathbb{P}\varphi(t) = \frac{d}{dt}\varphi(t) - A(t)\varphi(t)$$

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on a space of X-valued functions. The cases when the operators $A(t)$ are bounded for every t or X is finite dimensional were treated in outstanding works (see [3], [5]).

Important attempts to the case of infinite dimensional Banach spaces, have been made by employing an evolution operator technique ([11]). Thus the solutions of (E) have been characterized in terms of asymptotic properties of an evolution operator $\mathbf{\Phi} = \{\Phi(t,s)\}_{t\geq s\geq 0}$ and instead of the operator $\mathbb{P}$ one started to investigate the operator $\mathcal{P}u := P_u$, where

$$P_u(t) = \int_0^t \Phi(t,s)u(s)\,ds$$

for all $t \geq 0$ and all $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ (see [1], [2], [4], [8]).

Two of the most important results of Perron type concerning exponential stability of evolution operators in Banach spaces are:

THEOREM 1.1. *Let $\mathbf{\Phi} = \{\Phi(t,s)\}_{t\geq s\geq 0}$ be an evolution operator on the Banach space X and $p, q \in [1,\infty)$. If $\mathbf{\Phi}$ is $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$-stable then $\mathbf{\Phi}$ is uniformly exponentially stable.*

THEOREM 1.2. *Let $\mathbf{\Phi} = \{\Phi(t,s)\}_{t\geq s\geq 0}$ be an evolution operator on the Banach space X and $p \in [1,\infty)$. Then the following assertions are equivalent:*

(i) *$\mathbf{\Phi}$ is uniformly exponentially stable;*
(ii) *$\mathbf{\Phi}$ is $(C_0(\mathbb{R}_+, X), C_0(\mathbb{R}_+, X))$-stable;*
(iii) *$\mathbf{\Phi}$ is $(C_0(\mathbb{R}_+, X), C_b(\mathbb{R}_+, X))$-stable;*
(iv) *$\mathbf{\Phi}$ is $(L^p(\mathbb{R}_+, X), L^p(\mathbb{R}_+, X))$-stable.*

Theorem 1.1 has been obtained by DATKO (see [4]), using one of his outstanding results contained in the same paper. Its generalization was given by MEGAN in [6].

In the last few years there were given remarkable methods of proving Theorem 1.2. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) were proved by NEERVEN for the special case of C_0-semigroups of linear operators (see [9], [10]). Using Neerven's result for the evolution semigroup associated to an evolution operator CLARK, LATUSHKIN, MONTGOMERY-SMITH and RANDOLPH have proved in [2] the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv), but with the additional hypothesis that the mapping $(t,s) \mapsto \Phi(t,s)x$ is continuous, for every $x \in X$. The equivalence (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) has been also proved by van Minh, Räbiger and Schnaubelt ([8]), using an evolution semigroup technique and the same additional condition on the continuity of $\mathbf{\Phi}$. The equivalence (i) $\Leftrightarrow$ (ii) has been also treated by Buşe in [1].

Other variants for these theorems, based on the use of some Banach sequence spaces, are proved in [7], the integral operator being replaced by a sequence-valued operator.

The purpose of the present paper is to give an unified approach for the results from above. We not only obtain new proofs for these results — without using theorems already known — but also show that the proofs can be done in the same manner for $C_0(\mathbb{R}_+, X)$ and $L^p(\mathbb{R}_+, X)$. It is pointed out that for $p \leq q$ the uniform exponential stability of an evolution operator on the half-line is equivalent to its $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$-stability. An example shows that this last result fails for $p > q$. It is also proved that the uniform exponential stability of an evolution operator can be expressed using boundedly locally dense subsets of $C_b(\mathbb{R}_+, X)$ and $L^p(\mathbb{R}_+, X)$, respectively.

2 – Preliminaries

Let X be a real or complex Banach space. The norm on X and on the space $\mathcal{B}(X)$ of all bounded linear operators on X will be denoted by $\|\cdot\|$.

Definition 2.1. A family $\mathbf{\Phi} = \{\Phi(t,s)\}_{t \geq s \geq 0}$ of bounded linear operators on X is called an *evolution operator* if the following properties are satisfied:

(e_1) $\Phi(t,t) = I$, the identity operator on X;
(e_2) $\Phi(t,s)\Phi(s,t_0) = \Phi(t,t_0)$, for all $t \geq s \geq t_0 \geq 0$;
(e_3) for all $x \in X$ and all $t, t_0 \geq 0$ the function $\Phi(t,\cdot)x$ is continuous on $[0,t]$ and the function $\Phi(\cdot,t_0)x$ is continuous on $[t_0,\infty)$;
(e_4) there exist $M \geq 1, \omega > 0$ such that

$$\|\Phi(t,t_0)\| \leq M e^{\omega(t-t_0)}, \quad \text{for all } t \geq t_0 \geq 0\,. \tag{2.1}$$

Definition 2.2. An evolution operator $\mathbf{\Phi} = \{\Phi(t,s)\}_{t\geq s\geq 0}$ is said to be *uniformly exponentially stable* if there are $N \geq 1$ and $\nu > 0$ such that

$$\|\Phi(t,t_0)\| \leq Ne^{-\nu(t-t_0)}, \quad \text{for all } t \geq t_0 \geq 0\,.$$

Let $C_b(\mathbb{R}_+, X)$ be the linear space of all bounded continuous functions $u : \mathbb{R}_+ \to X$ and

$$C_0(\mathbb{R}_+, X) = \Big\{u \in C_b(\mathbb{R}_+, X) : u(0) = \lim_{t\to\infty} u(t) = 0\Big\}\,.$$

Endowed with the coresponding sup-norm

$$|||u||| := \sup_{t\geq 0} \|u(t)\|$$

$C_b(\mathbb{R}_+, X)$ and $C_0(\mathbb{R}_+, X)$ are Banach spaces.

We denote by $\mathcal{F}(\mathbb{R}_+, X)$ the linear space of all Lebesgue measurable functions $u : \mathbb{R}_+ \to X$ identifying the functions which are equal almost everywhere. For every $p \in [1,\infty)$ the linear space

$$L^p(\mathbb{R}_+, X) = \Big\{u \in \mathcal{F}(\mathbb{R}_+, X) : \int_{\mathbb{R}_+} \|u(t)\|^p\, dt < \infty\Big\}$$

is a Banach space with respect to the norm

$$\|u\|_p := \Big(\int_{\mathbb{R}_+} \|u(t)\|^p\, dt\Big)^{1/p}\,.$$

Definition 2.3. A subset E of $C_b(\mathbb{R}_+, X)$ is said to be *boundedly locally dense* in $C_b(\mathbb{R}_+, X)$ if there exists $c > 0$ such that

(i) for every $T > 0$ and every $u \in C_b(\mathbb{R}_+, X)$ there exists a sequence $(u_n) \subset E$ with $u_n \to u$ almost everywhere on $[0,T]$;

(ii) $|||u_n||| \leq c\,|||u|||$, for all $n \in \mathbb{N}$.

Remark 2.1. (i) It is easy to see that $C_c(\mathbb{R}_+, X)$ — the space of all X-valued, continuous functions on $\mathbb{R}_+$ with compact support — is an example of boundedly locally dense subspace of $C_b(\mathbb{R}_+, X)$.

(ii) Let $BUC(\mathbb{R}_+, X)$ be the space of all X-valued, bounded, uniformly continuous functions on $\mathbb{R}_+$ and $AP(\mathbb{R}_+, X)$ — the closure in $BUC(\mathbb{R}_+, X)$ of the linear span of the functions $\{e^{i\lambda(\cdot)}x : \lambda \in \mathbb{R}, x \in X\}$ (see [10]). Then $BUC(\mathbb{R}_+, X)$ and $AP(\mathbb{R}_+, X)$ are two remarkable examples of boundedly locally dense subspaces of $C_b(\mathbb{R}_+, X)$.

Definition 2.4. Let $p \in [1, \infty)$. A subset E of $L^p(\mathbb{R}_+, X)$ is said to be *boundedly locally dense* in $L^p(\mathbb{R}_+, X)$ if there exists $c > 0$ such that

(i) for every $T > 0$ and every $u \in L^p(\mathbb{R}_+, X)$ there exists a sequence $(u_n) \subset E$ with $u_n \to u$ in $L^p([0, T], X)$;
(ii) $\|u_n\|_p \leq c\, \|u\|_p$, for all $n \in \mathbb{N}$.

Remark 2.2. $C_c(\mathbb{R}_+, X)$ and $\mathcal{S}(\mathbb{R}_+, X)$ —the space of all Lebesgue measurable simple functions $s : \mathbb{R}_+ \to X$ — are boundedly locally dense subspaces of $L^p(\mathbb{R}_+, X)$, for every $p \in [1, \infty)$.

For every $u : \mathbb{R}_+ \to X$ locally integrable and every evolution operator $\mathbf{\Phi} = \{\Phi(t, s)\}_{t \geq s \geq 0}$ we define the function

$$P_u : \mathbb{R}_+ \to X, \quad P_u(t) := \int_0^t \Phi(t, s)u(s)\, ds\,.$$

Definition 2.5. Let

$$U, Y \in \{C_0(\mathbb{R}_+, X), C_b(\mathbb{R}_+, X)\} \cup \{L^p(\mathbb{R}_+, X) : p \in [1, \infty)\}.$$

We say that the evolution operator $\mathbf{\Phi} = \{\Phi(t, s)\}_{t \geq s \geq 0}$ is (U, Y)-*stable* if for every $u \in U$ we have that $P_u \in Y$.

Lemma 2.1. *Let $\mathbf{\Phi} = \{\Phi(t, s)\}_{t \geq s \geq 0}$ be an evolution operator on the Banach space X. If there exist $\delta > 0$ and $c \in (0, 1)$ such that:*

$$\|\Phi(t_0 + \delta, t_0)\| < c, \qquad \textit{for all } t_0 \geq 0$$

then $\mathbf{\Phi}$ is uniformly exponentially stable.

PROOF. Let $\nu > 0$ such that $c = e^{-\nu\delta}$. For $t \geq t_0 \geq 0$ there exist $n \in \mathbb{N}$ and $r \in [0, \delta)$ such that $t = t_0 + n\delta + r$. Then we have

$$\|\Phi(t,t_0)\| \leq \|\Phi(t,t_0+n\delta)\| \, \|\Phi(t_0+n\delta,t_0)\| \leq \\ \leq Me^{\omega\delta}e^{-\nu n\delta} \leq Me^{(\omega+\nu)\delta}e^{-\nu(t-t_0)}\,,$$

where M, ω are given by (2.1). It follows that $\mathbf{\Phi}$ is uniformly exponentially stable.

PROPOSITION 2.1. *If the evolution operator $\mathbf{\Phi} = \{\Phi(t,s)\}_{t\geq s\geq 0}$ is uniformly exponentially stable and $p, q \in [1, \infty)$ with $p \leq q$ then $\mathbf{\Phi}$ is $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$-stable.*

PROOF. It is a simple exercise, using Hölder's inequality.

3 – The main results

THEOREM 3.1. *Let $\mathbf{\Phi} = \{\Phi(t,s)\}_{t\geq s\geq 0}$ be an evolution operator on the Banach space X. Then the following assertions are equivalent:*

(i) *$\mathbf{\Phi}$ is uniformly exponentially stable;*

(ii) *$\mathbf{\Phi}$ is $(C_0(\mathbb{R}_+, X), C_0(\mathbb{R}_+, X))$-stable;*

(iii) *$\mathbf{\Phi}$ is $(C_0(\mathbb{R}_+, X), C_b(\mathbb{R}_+, X))$-stable.*

PROOF. The implication (i) $\Rightarrow$ (ii) is trivial and (ii) $\Rightarrow$ (iii) is obvious. To prove (iii) $\Rightarrow$ (i) we consider the linear operator

$$T : C_0(\mathbb{R}_+, X) \to C_b(\mathbb{R}_+, X), \quad Tu := P_u\,.$$

It is easy to see that T is closed and hence by the closed graph principle we obtain that T is bounded. Setting $K = \|T\|$ we have that

$$|||P_u||| \leq K\,|||u|||, \quad \text{for all } u \in C_0(\mathbb{R}_+, X)\,. \tag{3.1}$$

Consider $t_0 \geq 0$ and $x \in X$. Let $\alpha : \mathbb{R}_+ \to [0, 2]$ be a continuous function with the support contained in $(t_0, t_0 + 1)$ and

$$\int_{t_0}^{t_0+1} \alpha(s)\,ds = 1\,.$$

We consider the function

$$u : \mathbb{R}_+ \to X, \quad u(t) = \begin{cases} 0, & t \in [0, t_0] \\ \alpha(t)\Phi(t, t_0)x, & t \geq t_0 \,. \end{cases}$$

Then $u \in C_0(\mathbb{R}_+, X)$ and

$$|||u||| = \sup_{t \in [t_0, t_0+1]} \|u(t)\| \leq 2Me^{\omega}\|x\|$$

where M, ω are given by (2.1). We observe that for $t \geq t_0 + 1$

$$P_u(t) = \int_{t_0}^{t} \alpha(s)\, ds\, \Phi(t, t_0)x = \Phi(t, t_0)x \,.$$

Then using (3.1) we have that

$$\|\Phi(t, t_0)x\| \leq |||P_u||| \leq 2KMe^{\omega}\|x\| \,. \tag{3.2}$$

Since for $t \in [t_0, t_0 + 1]$

$$\|\Phi(t, t_0)\| \leq Me^{\omega} \,, \tag{3.3}$$

denoting by $L = (2K + 1)Me^{\omega}$ and using relations (3.2) and (3.3), it follows that

$$\|\Phi(t, t_0)\| \leq L, \quad \text{for all } t \geq t_0 \geq 0 \,.$$

Consider $\nu = e/4LK$ and

$$\varphi : \mathbb{R}_+ \to \mathbb{R}_+, \quad \varphi(t) = \int_0^t se^{-\nu s} ds \,.$$

The function φ is strictly increasing on $\mathbb{R}_+$ with

$$\lim_{t \to \infty} \varphi(t) = \frac{1}{\nu^2} \,.$$

We choose $\delta > 0$ such that $\varphi(\delta) > 1/2\nu^2$.

Let $t_0 \geq 0$ and $x \in X$. We consider the function

$$v : \mathbb{R}_+ \to X, \quad v(t) = \begin{cases} 0, & t \in [0, t_0] \\ (t - t_0)e^{-\nu(t-t_0)}\Phi(t, t_0)x, & t \geq t_0 \,. \end{cases}$$

Then $v \in C_0(\mathbb{R}_+, X)$ with

$$|||v||| = \sup_{t \geq t_0} \|v(t)\| \leq L\|x\| \sup_{r \geq 0} re^{-\nu r} = \frac{L}{\nu e} \|x\| .$$

By observing that $P_v(t_0 + \delta) = \varphi(\delta)\Phi(t_0 + \delta, t_0)x$, it follows that:

$$\|\Phi(t_0 + \delta, t_0)x\| < 2\nu^2 \varphi(\delta) \|\Phi(t_0 + \delta, t_0)x\| \leq \\ \leq 2\nu^2 \, |||P_v||| \leq \frac{2\nu LK}{e} \|x\| .$$

Taking in account the way ν was chosen, we obtain

$$\|\Phi(t_0 + \delta, t_0)x\| \leq \frac{1}{2} \|x\|, \quad \text{for all } t_0 \geq 0 \text{ and all } x \in X .$$

Applying Lemma 2.1. we obtain that $\mathbf{\Phi}$ is uniformly exponentially stable.

Corollary 3.1. *Let* $\mathbf{\Phi} = \{\Phi(t,s)\}_{t \geq s \geq 0}$ *be an evolution operator on the Banach space* X *and let* E *be a boundedly locally dense subset of* $C_b(\mathbb{R}_+, X)$*. If* $P_u \in C_b(\mathbb{R}_+, X)$ *for all* $u \in E$ *and*

(i) *there exists* $K > 0$ *such that* $|||P_u||| \leq K|||u|||$*, for all* $u \in E$*,*
or

(ii) E *is a closed linear subspace of* $C_b(\mathbb{R}_+, X)$*,*

then $\mathbf{\Phi}$ *is uniformly exponentially stable.*

Proof. (i) Let $u \in C_0(\mathbb{R}_+, X)$, $T > 0$ and $(u_n) \subset E$ a sequence with $u_n \to u$ almost everywhere on $[0, T]$ and

$$|||u_n||| \leq c\, |||u|||, \quad \text{for all } n \in \mathbb{N} ,$$

where $c > 0$ is given by Definition 2.3. Using Lebesgue's dominated convergence theorem we obtain that

$$P_{u_n}(T) \to P_u(T), \quad \text{as } n \to \infty .$$

Using the fact that

$$\|P_{u_n}(T)\| \leq |||P_{u_n}||| \leq K|||u_n||| \leq cK|||u||| ,$$

as $n \to \infty$, we have that

$$\|P_u(T)\| \leq cK|||u|||\,.$$

Because T was arbitrary chosen, it follows that $P_u \in C_b(\mathbb{R}_+, X)$. Applying Theorem 3.1 we obtain that $\mathbf{\Phi}$ is uniformly exponentially stable.

(ii) It follows from (i) and the closed graph principle.

REMARK 3.1. Neerven proved that a C_0-semigroup $\mathbb{T} = \{T(t)\}_{t\geq 0}$ is uniformly exponentially stable if and only if convolution with $\mathbb{T}$ maps certain subspaces of $BUC(\mathbb{R}_+, X)$ into $C_b(\mathbb{R}_+, X)$ and hence he obtained characterizations for uniform exponential stablity of C_0-semigroups in terms of almost periodic functions (see [10], pp. 90-94). Thus Corollary 3.1. is a generalization of Neerven's result for evolution operators.

THEOREM 3.2. *Let $\mathbf{\Phi} = \{\Phi(t,s)\}_{t\geq s\geq 0}$ be an evolution operator on the Banach space X and $p, q \in [1,\infty)$. If $\mathbf{\Phi}$ is $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$-stable then $\mathbf{\Phi}$ is uniformly exponentially stable.*

PROOF. We consider the linear operator

$$T : L^p(\mathbb{R}_+, X) \to L^q(\mathbb{R}_+, X), \quad Tu := P_u\,.$$

It is easy to see that T is closed. By the closed graph principle we obtain that T is bounded and let $K = \|T\|$.

Let $t_0 \geq 0$, $x \in X$ and

$$u : \mathbb{R}_+ \to X, \quad u(t) = \begin{cases} \Phi(t,t_0)x, & t \in [t_0, t_0+1] \\ 0, & t \in [0,t_0] \cup [t_0+1, \infty)\,. \end{cases}$$

If M, ω are given by (2.1) then $\|u\|_p \leq Me^{\omega}\|x\|$ and

$$P_u(t) = \Phi(t,t_0)x, \quad \text{for } t \geq t_0 + 1\,.$$

For $t \geq t_0 + 2$ we have that

$$\begin{aligned}\|\Phi(t,t_0)x\| &\leq Me^{\omega}\Big(\int_{t-1}^{t} \|\Phi(\tau,t_0)x\|^q \, d\tau\Big)^{\frac{1}{q}} \leq \\ &\leq Me^{\omega}\Big(\int_{t_0+1}^{\infty} \|\Phi(\tau,t_0)x\|^q \, d\tau\Big)^{\frac{1}{q}} = \\ &= Me^{\omega}\Big(\int_{t_0+1}^{\infty} \|P_u(\tau)\|^q \, d\tau\Big)^{\frac{1}{q}} \leq \\ &\leq Me^{\omega}\|P_u\|_q \leq KMe^{\omega}\|u\|_p \leq KM^2e^{2\omega}\|x\| \,.\end{aligned}$$

Setting $L = (KM+1)Me^{2\omega}$ we obtain that

$$\|\Phi(t,t_0)\| \leq L, \quad \text{for all } t \geq t_0 \geq 0 \,.$$

Let

$$\varphi : \mathbb{R}_+ \to \mathbb{R}_+, \quad \varphi(t) = \int_0^t se^{-s} \, ds \,.$$

Then φ is a strictly increasing function with $\lim\limits_{t\to\infty} \varphi(t) = 1$. Let $c > 0$ such that

$$\varphi(t) > \frac{1}{2}, \quad \text{for all } t \geq c \,. \tag{3.4}$$

For $t_0 \geq 0$ and $x \in X$ we consider the function

$$v : \mathbb{R}_+ \to X, \quad v(t) = \begin{cases} 0, & t \in [0,t_0] \\ (t-t_0)e^{-(t-t_0)}\Phi(t,t_0)x, & t \geq t_0 \,. \end{cases}$$

Then $v \in L^p(\mathbb{R}_+, X)$ and

$$\|v\|_p^p = \int_{t_0}^{\infty} \|v(t)\|^p \, dt \leq L^p\|x\|^p \int_0^{\infty} r^p e^{-pr} \, dr = L_1^p \|x\|^p$$

where $L_1 = L(\int_0^{\infty} r^p e^{-pr} \, dr)^{1/p}$.

We observe that for every $t \geq 0$

$$P_v(t_0 + t) = \varphi(t)\, \Phi(t_0 + t, t_0)x \,.$$

For $t > c$ and $\tau \in [c, t]$ using (3.4), we obtain that

$$\frac{1}{2}\,\|\Phi(t_0+t,t_0)x\| \le L\varphi(\tau)\|\Phi(t_0+\tau,t_0)x\|\,.$$

It follows that

$$\frac{(t-c)^{\frac{1}{q}}}{2}\|\Phi(t_0+t,t_0)x\| \le L\Big(\int_c^t \|P_v(t_0+\tau)\|^q\,d\tau\Big)^{\frac{1}{q}} \le \\ \le L\,\|P_v\|_q \le KL\,\|v\|_p \le KLL_1\,\|x\|\,.$$

Let $\delta > 0$ with $(\delta - c)^{1/q} > 4KLL_1$. Then

$$\|\Phi(t_0+\delta,t_0)x\| \le \frac{1}{2}\,\|x\|\,.$$

Since $t_0 \ge 0$ and $x \in X$ were arbitrary chosen and δ does not depend on t_0 or x, it follows that

$$\|\Phi(t_0+\delta,t_0)\| < \frac{1}{2}, \qquad \text{for all } t \ge t_0 \ge 0\,.$$

Using Lemma 2.1, we conclude that $\mathbf{\Phi}$ is uniformly exponentially stable.

Corollary 3.2. *Let $\mathbf{\Phi} = \{\Phi(t,s)\}_{t\ge s\ge 0}$ be an evolution operator on the Banach space X and $p, q \in [1,\infty)$ with $p \le q$. Then $\mathbf{\Phi}$ is uniformly exponentially stable if and only if $\mathbf{\Phi}$ is $(L^p(\mathbb{R}_+,X), L^q(\mathbb{R}_+,X))$-stable.*

Proof. It follows from Proposition 2.1 and Theorem 3.2.

Remark 3.2. Generally, if the evolution operator $\mathbf{\Phi} = \{\Phi(t,s)\}_{t\ge s\ge 0}$ is uniformly exponentially stable and $p, q \in [1,\infty)$ with $p > q$ then it does not result that it is $(L^p(\mathbb{R}_+,X), L^q(\mathbb{R}_+,X))$-stable. This fact is illustrated by the following example.

Example. Let $X = \mathbb{R}$ and

$$\Phi(t,s)x = e^{-(t-s)}x, \qquad \text{for all } t \ge s \ge 0 \text{ and all } x \in X\,.$$

Then $\mathbf{\Phi} = \{\Phi(t,s)\}_{t\ge s\ge 0}$ is an uniformly exponentially stable evolution operator on $\mathbb{R}$.

If $p, q \in [1, \infty)$ with $p > q$ let $\delta \in (q, p)$. We consider the function

$$u : \mathbb{R}_+ \to \mathbb{R}, \quad u(t) = \frac{1}{(t+1)^{1/\delta}}\,.$$

We have that $u \in L^p(\mathbb{R}_+, \mathbb{R}) \setminus L^q(\mathbb{R}_+, \mathbb{R})$ and

$$P_u(t) = e^{-t} \int_0^t e^s u(s)\, ds, \qquad \text{for all } t \geq 0\,.$$

Because

$$\lim_{t\to\infty} \frac{P_u(t)}{u(t)} = \lim_{t\to\infty} \frac{\displaystyle\int_0^t e^s u(s)\, ds}{e^t u(t)} = \\ = \lim_{t\to\infty} \frac{e^t u(t)}{e^t u(t) - \dfrac{1}{\delta(t+1)} e^t u(t)} = \lim_{t\to\infty} \frac{1}{1 - \dfrac{1}{\delta(t+1)}} = 1$$

and using the fact that $u \notin L^q(\mathbb{R}_+, \mathbb{R})$ we obtain that $P_u \notin L^q(\mathbb{R}_+, \mathbb{R})$ and hence $\mathbf{\Phi}$ is not $(L^p(\mathbb{R}_+, \mathbb{R}), L^q(\mathbb{R}_+, \mathbb{R}))$-stable.

Corollary 3.3. *Let $\mathbf{\Phi} = \{\Phi(t, s)\}_{t\geq s\geq 0}$ be an evolution operator on the Banach space X, let $p, q \in [1, \infty)$ and let E be a boundedly locally dense subset of $L^p(\mathbb{R}_+, X)$. If $P_u \in L^q(\mathbb{R}_+, X)$ for all $u \in E$ and*

(i) *there exists $K > 0$ such that $\|P_u\|_q \leq K\|u\|_p$, for all $u \in E$,*

or

(ii) *E is a closed linear subspace of $L^p(\mathbb{R}_+, X)$,*

then $\mathbf{\Phi}$ is uniformly exponentially stable.

Proof. (i) Let $M \geq 1$ and $\omega \geq 0$ given by (2.1).

Let $u \in L^p(\mathbb{R}_+, X)$ and $T > 0$. By Definition 2.4 there exist $c > 0$ and a sequence $(u_n) \subset E$ such that $u_n \to u$ in $L^p([0, T], X)$ and

$$\|u_n\|_p \leq c\|u\|_p, \qquad \text{for all } n \in \mathbb{N}\,.$$

For $t \in [0, T]$ we have that

$$\begin{aligned} \|P_{u_n}(t) - P_u(t)\| &\leq M e^{\omega T} \int_0^T \|u_n(s) - u(s)\|\, ds \leq \\ &\leq M e^{\omega T} \delta \Big(\int_0^T \|u_n(s) - u(s)\|^p\, ds \Big)^{\frac{1}{p}} \end{aligned} \tag{3.5}$$

where

$$\delta = \begin{cases} 1, & p = 1 \\ T^{1/q}, & p \in (1,\infty) \text{ and } q = \frac{p}{p-1} \end{cases}$$

so $P_{u_n}(t) \to P_u(t)$, as $n \to \infty$. Because

$$\|P_{u_n}(t)\| \le Me^{\omega T} \int_0^T \|u_n(s)\|\, ds \le Me^{\omega T} \delta \|u_n\|_p \le$$
$$\le Me^{\omega T} \delta c \|u\|_p, \quad \text{for all } t \in [0,T] \text{ and all } n \in \mathbb{N},$$

by Lebesgue's dominated convergence theorem we obtain that

$$\int_0^T \|P_{u_n}(t)\|^q\, dt \to \int_0^T \|P_u(t)\|^q\, dt \quad \text{as } n \to \infty. \tag{3.6}$$

But, for every $n \in \mathbb{N}$

$$\int_0^T \|P_{u_n}(t)\|^q\, dt \le \|P_{u_n}\|_q^q \le K^q \|u_n\|_p^q \le c^q K^q \|u\|_p^q. \tag{3.7}$$

So for $n \to \infty$ in (3.7) and using (3.6), we obtain that

$$\int_0^T \|P_u(t)\|^q\, dt \le c^q K^q \|u\|_p^q.$$

Since $T > 0$ was arbitrary chosen it follows that $P_u \in L^q(\mathbb{R}_+, X)$. Using Theorem 3.2 we obtain that $\mathbf{\Phi}$ is uniformly exponentially stable.

(ii) It follows from (i) and the closed graph principle.

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