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On the radial weak solutions of a conservative system modeling the isentropic flow

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RIASSUNTO: *Si studia l'esistenza globale ed il comportamento asintotico delle soluzioni deboli del sistema iperbolico quasilineare e singolare* (1). *Con i metodi già usati nei lavori* [3] *e* [6] *si dimostra l'esistenza di soluzioni entropiche deboli per il problema di Cauchy ed applicando il metodo di* CHEN *e* FRID [1] *se ne dimostra una proprietà di decadimento. Nell'ultima parte si costruiscono alcune soluzioni particolari che sono regolari a tratti ed autosimilari.*

ABSTRACT: *In this paper we study the existence and large time behavior of the weak solutions for the quasilinear hyperbolic singular system*

$$(1) \qquad \begin{cases} a_t + (au)_x + \dfrac{2au}{x} = 0 \\ u_t + \dfrac{1}{2}\,(a^2 + u^2)_x = 0 \end{cases} \qquad x > 0, \quad t \geq 0$$

First, in the framework of [3] *and* [6], *we prove the existence of entropy weak solutions for the Cauchy problem and then, by applying the method of* CHEN *and* FRID *in* [1], *we derive a decay (in time) result for those solutions. In the last part of the paper we analyse the existence of special solutions for this system namely we construct some global piecewise smooth weak solutions that are self-similar.*

KEY WORDS AND PHRASES: *Radial weak solutions – Self-similar solutions – Conservative system – Isentropic flow.*
A.M.S. CLASSIFICATION: 35L65 – 35Q35

1 – Introduction and main results

In this paper we consider the existence of global weak solutions for the quasilinear hyperbolic system with a singular source term

$$\text{(1.1)}\qquad \begin{cases} a_t + (au)_x + \dfrac{2au}{x} = 0 \\ u_t + \dfrac{1}{2}\,(a^2+u^2)_x = 0 \end{cases} \qquad x>0, \quad t\geq 0$$

with the initial data

$$\text{(1.2)}\qquad (a(x,0), u(x,0)) = (a_0(x),\, u_0(x)), \qquad x>0\,.$$

The system (1.1) appears in the study of the radial symetric solutions in $\mathbf{R}^3\times\mathbf{R}$ for a conservative system modelling the isentropic flow introduced by G.B. Whitham in [9, Chap. 9] where a is the sound speed and u is the radial velocity. If $f:\mathbf{R}^2 \longrightarrow \mathbf{R}^2$ is defined by $f(a,u) = (au, \frac{1}{2}(a^2+u^2))$, then two eingenvalues of ∇f are

$$\text{(1.3)}\qquad \lambda_1 = u-a, \quad \lambda_2 = u+a$$

and so the strict hyperbolicity fails if $a=0$, but the system is genuinely nonlinear with Riemann invariants

$$\text{(1.4)}\qquad l = -u+a, \quad r = u+a\,.$$

If $a_0, u_0 \in H_0^1(\mathbf{R}_+)$, we will say that $v=(a,u)\in (L^\infty_{\text{loc}}(\mathbf{R}_+\times[0,+\infty[))^2$ is a weak solution for the Cauchy problem (1.1), (1.2) in $\mathbf{R}_+\times[0,+\infty[$ if, for each pair $\varphi\in C_0^\infty(\mathbf{R}_+\times[0,+\infty[)$, $\psi\in C_0^\infty([0,+\infty[\times[0,+\infty[)$

$$\text{(1.5)}\qquad \begin{aligned} &\int_{\mathbf{R}_+\times[0,+\infty[} \Big(a\varphi_t + au\varphi_x - \frac{2au}{x}\,\varphi\Big)\,dx\,dt + \\ &+ \int_{\mathbf{R}_+\times[0,+\infty[} \Big(u\psi_t + \frac{1}{2}(a^2+u^2)\psi_x\Big)\,dx\,dt + \\ &+ \int_{\mathbf{R}_+} a_0(x)\varphi(x,0)\,dx + \int_{\mathbf{R}_+} u_0(x)\psi(x,0)\,dx = 0\,. \end{aligned}$$

A weak notion of null boundary condition for v (at $x=0$) is contained in (1.5). Moreover, we will say that $v=(a,u)$ verifying (1.5) is an entropy weak solution if, for every pair of smooth functions $\eta, q : \mathbf{R}^2 \longrightarrow \mathbf{R}$, η convex (entropy/entropy flux pair) such that $\nabla\eta \cdot \nabla f = \nabla q$ in $\mathbf{R}^2$, we have

$$(1.6)\qquad \begin{aligned} &\int_{\mathbf{R}_+\times[0,+\infty[} \Big(\eta(v)\varphi_t + q(v)\varphi_x - \nabla\eta(v)\cdot\Big(\frac{2au}{x},0\Big)\varphi\Big)dx\,dt+ \\ &\quad+\int_{\mathbf{R}_+} \eta(v_0(x))\varphi(x,0)\,dx \geq 0, \quad \forall\,\varphi\in C_0^\infty(\mathbf{R}_+\times[0,+\infty[),\ \varphi\geq 0\,. \end{aligned}$$

By applying the compensated compacteness method of TARTAR, MURAT and DIPERNA (cf. [2], [3] and [6]) and some ideas of M.E. SCHONBEK in [7] we easily prove in §2 the following result:

THEOREM 1. *Assume* $a_0,\ u_0 \in H_0^1(\mathbf{R}_+)$, $u_0(x) \geq a_0(x) \geq 0$, $x \in \mathbf{R}_+$. *Then, there exists* $v = (a,u) \in (L^\infty(\mathbf{R}_+ \times [0,+\infty[))^2$, *with* $u \geq a \geq 0$ *a.e. in* $\mathbf{R}_+ \times [0,+\infty[$, *which is an entropy weak solution for the Cauchy problem* (1.1), (1.2) *in* $\mathbf{R}_+ \times [0,+\infty[$. *Moreover there exists a sequence* $v_\varepsilon = (a_\varepsilon, u_\varepsilon) \in (C([0,+\infty[; H^3 \cap H_0^1) \cap C^1([0,+\infty[; H^1) \cap L^\infty(\mathbf{R}_+^2))^2$ *such that* $v_\varepsilon \longrightarrow v$ *a.e. in* $\mathbf{R}_+ \times [0,+\infty[$ *and in* $(L^\infty(\mathbf{R}_+^2))^2$ *weak* $*$, $v_\varepsilon(.,0) \longrightarrow v(.,0)$ *in* $(H_0^1(\mathbf{R}_+))^2$ *and* v_ε *is the solution of the approximate system* (2.1).

By a similar method to the employed by CHEN and FRID in [1] we also prove the following asymptotic decay for this weak solution:

THEOREM 2. *Let* $v = (a,u)$ *be the entropy weak solution for the Cauchy problem* (1.1), (1.2) *in* $\mathbf{R}_+ \times [0,+\infty[$ *obtained in Theorem* 1 *by the vanishing viscosity method. Then if* $T_n \to \infty$ *we have, a.e. in* $x \in \mathbf{R}_+$,

$$\frac{1}{T_n}\int_0^{T_n} (a(x,t),\ u(x,t))\,dt \underset{n}{\longrightarrow} 0\,.$$

The equilibrium states for the system (1.1) are given by the system

$$\begin{cases} (au)_x + \dfrac{2au}{x} = 0 \\ (a^2+u^2)_x = 0 \end{cases} \qquad x \in \mathbf{R}_+$$

that is in the canonical form, with $\Delta = u^2 - a^2$,

$$\begin{cases} a_x = -\dfrac{2au^2}{\Delta x} \\ u_x = \dfrac{2a^2u}{\Delta x} \end{cases}$$

Let us take $a(1) = a_1 > 0$, $u(1) = u_1 > a_1$. We have $\Delta(1) > 0$ and so there is a unique local C^1 solution with $\Delta > 0$ (and therefore $a > 0$ by the uniqueness theorem). Hence a is a decreasing function and u is an increasing function. We derive, by integration, $a^2 + u^2 = a_1^2 + u_1^2$, $au = e^k/x^2$ where $k = \log(a_1 u_1)$. Therefore $a = u$ for $x = \sqrt{\frac{2a_1u_1}{a_1^2+u_1^2}} \in]0,1[$ and the solution blows-up.

Now we look for solutions of the system (1.1) in the following form: $\bar{v} = (\bar{a}, \bar{u})$ where

$$\bar{a}(x,t) = t^{\beta-1} a\Big(\frac{x}{t^\beta}\Big), \quad \bar{u}(x,t) = t^{\beta-1} u\Big(\frac{x}{t^\beta}\Big) \tag{1.7}$$

(self-similar solutions), with $t > 0$. The system (1.1) becomes, with $y = \frac{x}{t^\beta}$,

$$\begin{cases} (\beta-1)a(y) - \beta\, y\, a'(y) + (au)'(y) + \dfrac{2au(y)}{y} = 0 \\ (\beta-1)u(y) - \beta\, y\, u'(y) + \dfrac{1}{2}(a^2+u^2)'(y) = 0\,. \end{cases} \tag{1.8}$$

Putting $\Delta = (-\beta y + u)^2 - a^2$, the system (1.8) can be reduced to the canonical form (if $\Delta \neq 0$):

$$\begin{cases} a' = \dfrac{1}{\Delta}\Big[(\beta y - u)\dfrac{2au}{y} + (\beta-1)\,(\beta y - u)\,a + (\beta-1)\,au\Big] \\ u' = \dfrac{1}{\Delta}\Big[\dfrac{2a^2u}{y} + (\beta-1)\,a^2 + (\beta-1)\,(\beta y - u)\,u\Big]\,. \end{cases} \tag{1.9}$$

Let us start with the case $\beta = 0$, that is

$$\bar{a}(x,t) = \frac{1}{t}a(x), \quad \bar{u}(x,t) = \frac{1}{t}u(x)\,. \tag{1.10}$$

In this case the system (1.9) becomes, with $\Delta = u^2 - a^2$, $y = x$,

$$\text{(1.11)} \qquad \begin{cases} a' = -\dfrac{1}{\Delta}\dfrac{2au^2}{y} \\ u' = \dfrac{1}{\Delta}\left(\dfrac{2a^2u}{y} - a^2 + u^2\right) = 1 + \dfrac{2a^2u}{\Delta y}. \end{cases}$$

Let us give $a_1 = a(1)$, $u_1 = u(1)$ with $u_1 > a_1 > 0$. We have $\Delta(1) > 0$ and so there is a unique local C^1 solution with $\Delta > 0$ (and therefore $a > 0$ by (1.11) and the uniqueness theorem). Hence, by (1.11), a is a decreasing function and u is an increasing function. We deduce $0 \leq a \leq u \leq u_1$. If there is $y_0 < 1$ such that $u(y_0) < y_0$ it follows that there exists $y_1 < 1$ such that $u(y_1) = a(y_1)$ and the solution blows-up. If $u(y) \geq y$ for $y < 1$, we deduce from the first equation in (1.8) for $\beta = 0$, by multiplying by y and integrating in $[y, 1]$:

$$-\int_y^1 a\xi\, d\xi + (au)(1) - y\,(au)(y) + \int_y^1 au\, d\xi = 0$$

and so $0 < a_1 u_1 = (au)(1) \leq y\,(au)(y) \underset{y\to 0}{\longrightarrow} 0$ which is absurd. Hence the solution blows-up in $]0, 1[$.

Now let us consider the case $\beta = 1$, that is

$$\text{(1.12)} \qquad \bar{a}(x,t) = a\left(\frac{x}{t}\right), \quad \bar{u}(x,t) = u\left(\frac{x}{t}\right).$$

In this case the system (1.9) becomes, with $\Delta = (u-y)^2 - a^2$, $y = x/t$,

$$\text{(1.13)} \qquad \begin{cases} a' = \dfrac{1}{\Delta y}\, 2au(y-u) \\ u' = \dfrac{1}{\Delta y}\, 2a^2u\,. \end{cases}$$

We fix $y_0 > 0$, $a_+ > 0$ and $u_+ = a_+ + y_0$. Hence $\Delta(y_0) = (u_+ - y_0)^2 - a_+^2 = 0$. In the straight line $x = y_0 t$ the RANKINE-HUGONIOT conditions (cf. [5]) for the system (1.1) can be writen as follows

$$\text{(1.14)} \qquad \begin{cases} y_0(a_+ - a_-) = a_+u_+ - a_-u_- \\ y_0(u_+ - u_-) = \dfrac{1}{2}(a_+^2 - a_-^2) + \dfrac{1}{2}(u_+^2 - u_-^2) \end{cases}$$

where

$$(a_-, u_-) = \lim_{y \to y_0^-} (a, u), \quad (a_+, u_+) = \lim_{y \to y_0^+} (a, u).$$

If we take $u_- = 0$, it is easy to check that we must choose

$$a_- = -a_+ \quad \text{and} \quad u_+ = 2a_+ .$$

We will prove the following

THEOREM 3. *Let us choose $y_0 > 0$, $a_+ = y_0$ and $u_+ = a_+ + y_0 = 2a_+$. Then there exists a piecewise smooth weak solution $(\bar{a}, \bar{u})$ of the system (1.1) in $\mathbf{R} \times]0, +\infty[$ of the form (1.12) such that $(\bar{a}, \bar{u}) \equiv (-a_+, 0)$ if $x < y_0 t$ and verifying $(\bar{a}_+, \bar{u}_+) = (a_+, u_+)$ where*

$$\bar{a}_+ = \lim_{\substack{(x,t) \to (y_0 t_1, t_1) \\ x > y_0 t}} \bar{a}(x,t), \qquad \bar{u}_+ = \lim_{\substack{(x,t) \to (y_0 t_1, t_1) \\ x > y_0 t}} \bar{u}(x,t), \quad \forall t_1 > 0 .$$

REMARK. The solution $(\bar{a}, \bar{u})$ in Theorem 3 is not necessarily smooth for $x > y_0 t$.

2 – Proof of Theorems 1 and 2

We start with the proof of Theorem 1. Following [2], [3] and [6] we consider, for each $\varepsilon > 0$, the related parabolic system

$$\text{(2.1)} \qquad \begin{cases} a_t + (au)_x + \dfrac{2au}{x + \varepsilon} = \varepsilon\, a_{xx} \\ u_t + \dfrac{1}{2}(a^2 + u^2)_x = \varepsilon\, u_{xx} \end{cases} \qquad x \geq 0, \quad t \geq 0$$

with initial data

$$\text{(2.2)} \qquad a(x,0) = a_{0\varepsilon}(x), \quad u(x,0) = u_{0\varepsilon}(x), \quad x \geq 0$$

where $a_{0\varepsilon}, u_{0\varepsilon} \in H^3(\mathbf{R}_+) \cap H_0^2(\mathbf{R}_+)$, $u_{0\varepsilon} \geq a_{0\varepsilon} \geq 0$, and

$$(a_{0\varepsilon}, u_{0\varepsilon}) \underset{\varepsilon \to 0}{\longrightarrow} (a_0, u_0) \text{ in } (H_0^1(\mathbf{R}_+))^2 .$$

By the usual fixed point theorem technique and the well known estimates for the heat equation it is not difficult to prove the local (in time) existence and uniqueness of a solution

$$v_\varepsilon = (a_\varepsilon, u_\varepsilon) \in (C([0,T]; H^3 \cap H_0^1) \cap C^1([0,T]; H^1))^2 .$$

For the global existence (for each $\varepsilon > 0$) we need some a priori — L^∞ estimates based on the maximum principle (cf. [4]) and on the invariant regions principle (cf. [8]). First, if we apply the maximum principle for parabolic equations (cf. [4]) to the first eqution in (2.1), it is easy to see that $a_\varepsilon(x,t) \geq 0$, $(x,t) \in \mathbf{R}_+ \times [0,T]$, since $a_{0\varepsilon}(x) \geq 0$, $x \in \mathbf{R}_+$. Assuming that, let $d > \|a_0\|_{L^\infty} + \|u_0\|_{L^\infty}$ (we may assume $d > \|a_{0\varepsilon}\|_{L^\infty} + \|u_{0\varepsilon}\|_{L^\infty}$, $\forall \varepsilon \leq \varepsilon_0$) and consider the functions $G_i : \mathbf{R}^2 \longrightarrow \mathbf{R}$, $i = 1,2,3$ defined by

$$G_1(a,u) = -a, \quad G_2(a,u) = a - u, \quad G_3(a,u) = a + u - d .$$

It is easy to prove (following [7], Chap. 14) that

$$\sum = \Big\{(a,u) \in \mathbf{R}^2 : \max_{1 \leq i \leq 3} G_i(a,u) \leq 0\Big\}$$

is an invariant region for the system (2.1) and so

Proposition 2.1. *We have the following a priori estimate for the solution $v_\varepsilon = (a_\varepsilon, u_\varepsilon)$ of the Cauchy problem* (2.1), (2.2)*:*

(2.3) $\quad 0 \leq a_\varepsilon(x,t) \leq u_\varepsilon(x,t) \leq M,$ *with M not depending on ε.*

The a priori estimate (2.3) implies (see [8] for related results) the global existence for $v_\varepsilon = (a_\varepsilon, u_\varepsilon)$ in $(C\,([0,\infty[; H^3 \cap H_0^1) \cap C^1([0,\infty[; H^1))^2$ verifying (2.3) for $x \geq 0$, $t > 0$ and for every $\varepsilon \leq \varepsilon_0$. The system (1.1) has the strictly convex entropy $\eta_0 = \frac{1}{2}(a^2 + u^2)$ with the corresponding entropy flux $q_0 = a^2 u + \frac{1}{3}u^3$. Since

$$\nabla \eta_0 \cdot \nabla f = \nabla q_0 = (2au, a^2 + u^2)\,,$$

we derive from (2.1) for $v_\varepsilon = (a_\varepsilon, u_\varepsilon)$:

$$\frac{1}{2}(a_\varepsilon^2 + u_\varepsilon^2)_t + \Big(a_\varepsilon^2 u_\varepsilon + \frac{1}{3}u_\varepsilon^3\Big)_x + \frac{2a_\varepsilon^2 u_\varepsilon}{x+\varepsilon} = \varepsilon\,(a_{\varepsilon xx}\, a_\varepsilon + u_{\varepsilon xx}\, u_\varepsilon) \tag{2.4}$$

and so by integration we derive

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbf{R}_+}(a_\varepsilon^2 + u_\varepsilon^2)\,dx + \int_{\mathbf{R}_+}\frac{2a_\varepsilon^2 u_\varepsilon}{x+\varepsilon}\,dx + \varepsilon\int_{\mathbf{R}_+}(a_{\varepsilon x}^2 + u_{\varepsilon x}^2)\,dx = 0$$

and so, since $u_\varepsilon \geq 0$,

$$\frac{1}{2}\int_{\mathbf{R}_+}(a_\varepsilon^2 + u_\varepsilon^2)\,(t)\,dx + \varepsilon\int_0^t\int_{\mathbf{R}_+}(a_{\varepsilon x}^2 + u_{\varepsilon x}^2)\,dx\,d\tau \leq M_1 \tag{2.5}$$

where M_1 is independent of ε and t.

We conclude from (2.5) that, with $\varepsilon \to 0$,

$$\varepsilon^{1/2}a_{\varepsilon x},\ \varepsilon^{1/2}u_{\varepsilon x} \quad \text{are uniformly bounded in } L^2(\mathbf{R}_+^2)\,. \tag{2.6}$$

If (η, q) is a smooth pair entropy/entropy flux, η convex, for the system (1.1) we derive for $v_\varepsilon = (a_\varepsilon, u_\varepsilon)$

$$\eta(v_\varepsilon)_t + q(v_\varepsilon)_x = \varepsilon\,\eta(v_\varepsilon)_{xx} - \varepsilon\, v_{\varepsilon x}^T \nabla^2\eta(v_\varepsilon) v_{\varepsilon x} + \nabla\eta(v_\varepsilon)\cdot\Big(-\frac{2a_\varepsilon u_\varepsilon}{x+\varepsilon}, 0\Big)\,. \tag{2.7}$$

By (2.3) and (2.5) we can, as usually, derive that, for any open set $\Omega \subset [\alpha,\beta]\times]0,T[$, $0 < \alpha < \beta < +\infty$, $0 < T < +\infty$, the second member of (2.7) is in a compact set of $H^{-1}(\Omega)$ (see [7] for a similar result). From (2.3) we conclude that there exists $v = (a, u) \in (L^\infty(\mathbf{R}_+^2))^2$ and a subsequence still denoted by $v_\varepsilon = (a_\varepsilon, u_\varepsilon)$, $\varepsilon \to 0$, such that

$$v_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} v \quad \text{in} \quad \big(L^\infty(\mathbf{R}_+^2)\big)^2 \quad \text{weak} * \,. \tag{2.8}$$

For each open set $\Omega \subset [\alpha,\beta]\times]0,T[$, $0 < \alpha < \beta < +\infty$, $0 < T < +\infty$, we arrive as in [6] to the Tartar relation concerning every smooth pair $(\eta_1, q_1), (\eta_2, q_2)$ where $\eta_i (i = 1, 2)$ is a convex entropy (q_i is the corresponding entropy flux) and the Young measures ν_{xt} associated to the sequence v_ε satisfying (2.8). Reasoning as in [6] (cf. also [3]) and pointing out that in our case the entropy equations are equivalent (in the smooth case) to the equation $\eta_{aa} = \eta_{uu}$ (see (3.2) in [6] in the simpler case $r = 3$), we prove that ν_{xt} reduces to a Dirac measure, a.e. in Ω, and so, there is a subsequence of v_ε, still denoted by v_ε, such that

$$v_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} v \quad \text{a.e. in} \quad \Omega\,. \tag{2.9}$$

By a standard diagonalization procedure we can construct a sub-sequence of the sequence defined in (2.8), still denoted by v_ε, such that

$$v_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} v \quad \text{a.e. in} \quad \mathbf{R}_+ \times [0, +\infty[\,. \tag{2.10}$$

Now by a well known technique, starting from (2.1) and (2.7), it is easy to prove that $v = (a, u)$ is an entropy weak solution of the Cauchy problem (2.1), (2.2), such that

$$0 \le a(x,t) \le u(x,t) \le M, \quad \text{a.e. in} \quad \mathbf{R}_+ \times [0, +\infty[\,.$$

This completes the proof of Theorem 1. □

We can now pass to the proof of Theorem 2:

Let $v_\varepsilon = (a_\varepsilon, u_\varepsilon)$ be the solution of the system (2.1) with initial data (2.2) such that $v_\varepsilon \to v = (a, u)$ in $((L^\infty(\mathbf{R}^2_+))^2$ weak $*$ and a.e. in $\mathbf{R}_+ \times [0, +\infty[$, v the entropy weak solution of the problem (1.1), (1.2). We assume that v_ε verify (2.3). With a technique motivated by the work of CHEN and FRID in [1], we integrate the equations of (2.1) (in $(a_\varepsilon, u_\varepsilon)$) in $[0, x_0] \times [0, T]$ for $x_0 > 0, T > 0$, and take the sum. We derive

$$\begin{aligned}
&\int_0^{x_0} [a_\varepsilon(x,T) + u_\varepsilon(x,T) - a_\varepsilon(x,0) - u_\varepsilon(x,0)]\, dx + \\
&\qquad + \int_0^T \Big[a_\varepsilon u_\varepsilon(x_0,t) + \frac{1}{2}(a_\varepsilon^2 + u_\varepsilon^2)(x_0,t)\Big]\, dt + \int_0^T \int_0^{x_0} \frac{2a_\varepsilon u_\varepsilon}{x+\varepsilon}\, dx\, dt = \\
&= \varepsilon \int_0^T [a_{\varepsilon x}(x_0,t) + u_{\varepsilon x}(x_0,t)]\, dt\,,
\end{aligned}$$

$$\begin{aligned}
&\int_0^T [a_\varepsilon u_\varepsilon(x_0,t) + \frac{1}{2}(a_\varepsilon^2 + u_\varepsilon^2)(x_0,t)]\, dt \le \\
&\quad \le C x_0 + \varepsilon \int_0^T [a_{\varepsilon x}(x_0,t) + u_{\varepsilon x}(x_0,t)]\, dt
\end{aligned}$$

(with C not depending on x_0, T, ε), and so, for each ε and $x > 0$,

$$\begin{aligned} \frac{1}{T}\int_0^T [a_\varepsilon u_\varepsilon(x,t) + \frac{1}{2}(a_\varepsilon^2 + u_\varepsilon^2)(x,t)]\,dt \le \\ \le \frac{C}{T}x + \frac{\varepsilon}{T}\int_0^T [a_{\varepsilon x}(x,t) + u_{\varepsilon x}(x,t)]\,dt\,. \end{aligned} \tag{2.11}$$

Now integrate (2.11) in $x \in [x_1, x_2]$, $0 < x_1 < x_2$. We obtain

$$\begin{aligned} &\frac{1}{T}\int_{x_1}^{x_2}\int_0^T [a_\varepsilon u_\varepsilon(x,t) + \frac{1}{2}(a_\varepsilon^2 + u_\varepsilon^2)(x,t)]\,dt\,dx \le \\ &\quad\le \frac{C}{T}\left(\frac{x_2^2 - x_1^2}{2}\right) + \frac{\varepsilon}{T}\int_0^T [a_\varepsilon(x_2,t) + u_\varepsilon(x_2,t) - a_\varepsilon(x_1,t) - u_\varepsilon(x_1,t)]\,dt\,. \end{aligned}$$

Letting $\varepsilon \to 0$ we derive

$$\frac{1}{T}\int_{x_1}^{x_2}\int_0^T \Big[au(x,t) + \frac{1}{2}(a^2 + u^2)(x,t)\Big]\,dt\,dx \le \frac{C}{T}\Big(\frac{x_2^2 - x_1^2}{2}\Big)\,.$$

Now divide by $x_2 - x_1$ and let $x_2 \to x_1^+$. We obtain

$$\frac{1}{T}\int_0^T \Big[au(x_1,t) + \frac{1}{2}(a^2 + u^2)(x_1,t)\Big]\,dt \le \frac{C}{T}x_1, \quad \text{a.e. in } x_1 \in \mathbf{R}_+\,,$$

and this achieves the proof of Theorem 2. □

3 – Proof of Theorem 3

Let $\delta = \delta_n \searrow 0$, $(a,u) = (a_\delta, u_\delta)$ the (local) C^1 solution of (1.13) with initial data $a(y_0) = a_+$, $u(y_0) = u_+ + \delta : \Delta(y_0) = (u_+ + \delta - y_0)^2 - a_+^2 = (a_+ + \delta)^2 - a_+^2 > 0$ and so $\Delta > 0$ in $[y_0, y_0 + \varepsilon]$ where also $a > 0$ (if $a(y_1) = 0$ with $a(y) \ne 0$ for $y < y_1$ and $\Delta(y) \ne 0$ for $y \le y_1$ we get $a'(y_1) = u'(y_1) = 0$ and so by the uniqueness theorem $a = 0$ in $]y_1 - \varepsilon_1, y_1 + \varepsilon_1[$). Hence, from (1.6) we infer that a is a decreasing function and u is an increasing function. We deduce

$$(a+u)' = \frac{1}{\Delta y}\,2au(a + y - u) = -\frac{2au}{u - y + a} \tag{3.1}$$

and so $a+u$ is also a decreasing function. We derive

$$
\begin{aligned}
&0 \le a \le a_+, \quad u_+ + \delta \le a + u \le a_+ + u_+ + \delta \quad \text{and so} \\
&\qquad\qquad u_+ \le u \le a_+ + u_+ + \delta_0 .
\end{aligned} \tag{3.2}
$$

Since $\Delta(y) = f(y)\,(u-y+a)$, with $f(y) = u - y - a$ ($f(y_0) = \delta$) and

$$
f' = \frac{2au}{yf} - 1 = \frac{2u(u-y)}{yf} - \frac{2u}{y} - 1 , \tag{3.3}
$$

if $\Delta(y_1) = 0$ and $\Delta(y) \neq 0$ for $y \le y_1$ we get $a(y_1) = 0$ ($f(y_1) = 0$, $f(y) > 0$ for $y \in [y_0, y_1[$ and $f'(y_1) = +\infty$ is not possible). Hence, from (3.2) and (3.3) we derive, for $y \in [y_0, y_0+\varepsilon]$

$$
f' \ge C f^{-1} - D, \quad C = \frac{2(y_0+a_+)(a_+ - \varepsilon)}{y_0+\varepsilon}, \ D = 1 + \frac{2(a_+ + u_+ + \delta_0)}{y_0} . \tag{3.4}
$$

In a similar way, we deduce from (3.2), (3.3), for $y \in [y_0, y_0+\varepsilon]$

$$
f' \le E f^{-1}, \quad E = \frac{2(u_+ + a_+ + \delta_0)^2}{y_0} \tag{3.5}
$$

and so, for $y \in [y_0, y_0+\varepsilon]$ we get

$$
f \le \sqrt{\delta_n^2 + 2E\varepsilon} \le \sqrt{\delta_0^2 + 2E\varepsilon} \tag{3.6}
$$

Hence, for certain δ_0 and ε, we derive $C - Df > 0$ in $[y_0, y_0+\varepsilon]$ and so, by (3.4),

$$
ff' \ge C - Df > 0, \quad \frac{ff'}{C - Df} \ge 1 .
$$

Putting $\frac{f}{C-Df} = -\frac{1}{D} + \frac{C}{D}\frac{1}{C-Df}$, we get by integration,

$$
\delta_n - f - \frac{C}{D} \log\left(\frac{C - Df}{C - D\delta_n}\right) \ge D(y - y_0) \ge D\varepsilon_1 \tag{3.7}
$$

for $[y_0 + \varepsilon_1, y_0 + \varepsilon]$, $\varepsilon_1 < \varepsilon$. Hence

$$
\inf_{y \in [y_0+\varepsilon_1, y_0+\varepsilon], n \in \mathbb{N}_1} f \ge C_{\varepsilon_1} > 0 . \tag{3.8}
$$

By (1.13), (3.2) and (3.8) we conclude that, for $\varepsilon_1 < \varepsilon$, the sequence of functions $(a_{\delta_n}, u_{\delta_n})$ is bounded in $(C^1([y_0+\varepsilon_1, y_0+\varepsilon]))^2$ and so by the theorem of Ascoli, by (1.13) and by a suitable diagonalization procedure we can extract a sub-sequence still denoted by $(a_{\delta_n}, u_{\delta_n})$ and a pair $(a, u) \in (C^1(]y_0, y_0+\varepsilon]))^2$ such that $(a_{\delta_n}, u_{\delta_n})$ converge to (a, u) uniformly in each compact interval $[y_0+\varepsilon_1, y_0+\varepsilon]$, $\varepsilon_1 < \varepsilon$.

Moreover, since $a_{\delta_n}(y_0) = a_+$, $u_{\delta_n}(y_0) = u_+ + \delta_n \underset{n}{\longrightarrow} u_+ = a_+ + y_0$ and, by (3.6),

$$0 \le u_{\delta_n}(y) - y - a_{\delta_n}(y) \le \sqrt{\delta_n^2 + 2E\varepsilon} \quad \text{for} \quad y \in [y_0, y_0+\varepsilon]\,,$$

we conclude that (a, u) can be extended to $[y_0, y_0+\varepsilon]$ so that

$$(a, u) \in (C([y_0, y_0+\varepsilon]))^2 \quad \text{and} \quad (a(y_0), u(y_0)) = (a_+, u_+)\,.$$

Furthermore we have (a, u) defined for $y > y_0$ such that $\Delta = (u-y)^2 - a^2 > 0$ and

$$(3.9) \qquad 0 \le a \le a_+, \quad u_+ \le u \le a_+ + u_+$$

Hence, if $f = u - y + a$ there exists $y_1 > y_0$ such that $f(y_1) = 0$ that is $\Delta(y_1) = 0$ and $\Delta(y) \ne 0$ for $y \in]y_0, y_1[$. As remarked in the begining of the proof, we must have $a(y_1) = 0$ ($a(y) \ne 0$ for $y \in [y_0, y_1[$). Hence $u(y_1) = y_1$. We can write (1.13) as follows

$$(3.10) \qquad \begin{cases} a' = -\dfrac{2u}{y}\dfrac{1}{g+1}\dfrac{1}{1-\frac{1}{g}} \\[2ex] u' = \dfrac{2u}{y}\dfrac{1}{g+1}\dfrac{1}{g-1} \end{cases} \qquad y \in]y_0,\ y_1[$$

where $g = \frac{u-y}{a}$.

First assume $a' \underset{y \to y_1^-}{\longrightarrow} A \ne 0, -\infty$. We have $A < 0$. We derive

$$g \underset{y \to y_1^-}{\longrightarrow} A_1, \quad A = -2\frac{1}{A_1+1}\frac{1}{1-\dfrac{1}{A_1}} = 2\frac{A_1}{1-A_1^2} \quad \text{and so} \quad A_1 > 1\,.$$

Fom (3.10) we get $u' \underset{y \to y_1^-}{\longrightarrow} 2\,\frac{1}{A_1^2-1}$. Hence,

$$\lim_{y \to y_1^-} \frac{u'-1}{a'} = \lim_{y \to y_1^-} g = A_1\,,$$

that is, $\frac{2}{A_1^2-1}\,\frac{1}{A} - \frac{1}{A} = A_1$, and so $A_1^3 = -3$, which is absurd.

Now assume $a' \underset{y \to y_1^-}{\longrightarrow} -\infty$. We derive $g \underset{y \to y_1^-}{\longrightarrow} 1$, $u' \underset{y \to y_1^-}{\longrightarrow} +\infty$. Hence

$$\frac{a'}{u'} = \frac{y-u}{a} = -g \underset{y \to y_1^-}{\longrightarrow} -1\,.$$

But we have also

$$\lim_{y \to y_1^-} \frac{a'}{u'} = \lim_{y \to y_1^-} \frac{1-u'}{a'} = 1\,,$$

which is absurd. Hence or $a' \underset{y \to y_1^-}{\longrightarrow} 0$ or a' has no limit. In the first case we derive $g \underset{y \to y_1^-}{\longrightarrow} +\infty$, $u' \underset{y \to y_1^-}{\longrightarrow} 0$. Since $a(y_1) = 0$ and $u(y_1) = y_1$ we can extend (a, u) for $y \geq y_1$ by putting $a(y) = 0$, $u(y) = y_1$. We obtain a C^1 solution. In the second case, since $a(y_1) = 0$ and $u(y_1) = y_1$ we make the same extension and the corresponding functions $(\bar{a}(x,t), \bar{u}(x,t)) = (a(x/t), u(x/t))$ verify for $x = y_1 t$ the Rankine-Hugoniot conditions. Hence we have constructed a global piecewise smooth solution for the system (1.1) in $\mathbf{R}_+ \times]0, +\infty[$. ∎

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