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Convergence of some energies for the Dirichlet problem in perforated domains

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RIASSUNTO: *Si studia la convergenza delle energie* $\int_{\Omega_\varepsilon} B\nabla u_\varepsilon \cdot \nabla u_\varepsilon dx$ *per* $\varepsilon \to 0$, *essendo* u_ε *la soluzione del problema omogeneo di Dirichlet in un dominio perforato di* Ω_ε.

ABSTRACT: *We study the convergence of energies,* $\int_{\Omega_\varepsilon} B\nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx$ *as* $\varepsilon \to 0$, *where* u_ε *is the solution of the homogeneous Dirichlet problem in the perforated domain,* Ω_ε.

1 – Introduction

Let Ω be a bounded open set in R^n, $n \geq 2$. Let $\varepsilon > 0$ be a parameter which tends to zero. For ε fixed, one may consider a periodically perforated domain, Ω_ε, obtained as follows. First we cover R^n by cubes of size 2ε periodically arranged with period 2ε. Then it is clear that only finitely many of the cubes, $P_i^\varepsilon, i = 1, 2, ..., k(\varepsilon)$ intersect Ω. Let T_i^ε be closed balls of radius a_ε, $0 < a_\varepsilon \leq \varepsilon$, centred at the centre of the cubes P_i^ε. Then the perforated domain Ω_ε is defined by $\Omega_\varepsilon \equiv \Omega \setminus \cup T_i^\varepsilon$.

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Let $f \in L^2(\Omega)$. The Dirichlet problem in Ω_ε is: find $u_\varepsilon \in H_0^1(\Omega_\varepsilon)$ satisfying,

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{1.1}$$

The existence of u_ε follows from the Lax-Milgram theorem and such a u_ε is unique. The behaviour of u_ε as $\varepsilon \to 0$ was the subject of study in a paper by Cioranescu and Murat [2], [3]. Depending on the size of the holes, a_ε, various types of behaviour are possible. It was shown that there exists a critical size of the holes, c_ε, such that if:

a) $a_\varepsilon = c_\varepsilon$, then there exists a measure μ such that $\widetilde{u_\varepsilon} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and u solves,

$$\begin{cases} -\Delta u + \mu u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

$\widetilde{\ }$ denotes the extension by zero onto the holes. The lower order term is known in the literature as the "strange term".

b) If $a_\varepsilon << c_\varepsilon$, then $\mu = 0$ and $\widetilde{u_\varepsilon} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and u solves the Dirichlet problem in the domain Ω.

c) If $a_\varepsilon >> c_\varepsilon$, then $\widetilde{u_\varepsilon} \longrightarrow 0$ strongly in $H_0^1(\Omega)$.

In addition, in case of a) or b), the following convergence of energies was established.

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx \xrightarrow{\varepsilon \to 0} \int_\Omega |\nabla u|^2 \, dx + \int_\Omega u^2 \, d\mu.$$

Obviously, $\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx \longrightarrow 0$ when $a_\varepsilon >> c_\varepsilon$.

More generally, they (cf. [2], [3]) considered arbitrary perforated domains $\Omega_\varepsilon \equiv \Omega \setminus \cup S_i^\varepsilon$, where $S_i^\varepsilon,\ i = 1, 2, ..., k(\varepsilon)$ are closed subsets of R^n (the holes). Further they assumed the existence of a sequence $w_\varepsilon \in H^1(\Omega)$ and a distribution $\mu \in W^{-1,\infty}(\Omega)$ satisfying,

CM1) $w_\varepsilon = 0$ in $\Omega \setminus \Omega_\varepsilon$,
CM2) $w_\varepsilon \rightharpoonup 1$ weakly in $H^1(\Omega)$ and,

CM3) For any sequence $v_\varepsilon \in H^1(\Omega)$ with $v_\varepsilon = 0$ in $\Omega \setminus \Omega_\varepsilon$ and such that $v_\varepsilon \rightharpoonup v$ weakly in $H^1(\Omega)$ and for any $\phi \in D(\Omega)$, we have,

$$\int_{\Omega_\varepsilon} \nabla w_\varepsilon . \nabla(\phi v_\varepsilon)\, dx \longrightarrow \langle \mu, \phi v \rangle.$$

We will henceforth refer to these conditions jointly as [CM] conditions. Under these assumptions it was shown that

$$\widetilde{u_\varepsilon} \rightharpoonup u \text{ weakly in } H_0^1(\Omega) \tag{1.3}$$

where u_ε, u solve (1.1), (1.2) respectively. Also,

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\, dx \xrightarrow{\varepsilon \to 0} \int_\Omega |\nabla u|^2\, dx + \langle \mu, u^2 \rangle. \qquad \square$$

Remark 1.1. In fact, it is enough to have a sequence $w_\varepsilon \in H^1(\Omega)$ satisfying CM1 and CM2 above. That CM3 is a consequence of these was shown by Casado-Díaz [1]. $\square$

Remark 1.2. From CM3 it follows that (cf. [2] [3])

$$\langle \mu, \phi \rangle = \lim_{\varepsilon \to 0} \int_\Omega |\nabla w_\varepsilon|^2 \phi\, dx.$$

So μ is a positive measure. $\square$

In this paper, we consider the convergence of energies $\int_{\Omega_\varepsilon} B \nabla u_\varepsilon . \nabla u_\varepsilon\, dx$ as $\varepsilon \to 0$ where $B \in L^\infty(\Omega)^{n\times n}$ is a matrix satisfying

$$\|B\|_\infty \leq M \tag{1.4}$$

for some constant M.

If Ω_ε is the periodically perforated domain defined before and holes are above the critical size i.e., $a_\varepsilon >> c_\varepsilon$, then $\int_{\Omega_\varepsilon} B\nabla u_\varepsilon \cdot \nabla u_\varepsilon\, dx \to 0$, since $\widetilde{u_\varepsilon} \longrightarrow 0$ *strongly* in $H_0^1(\Omega)$. When the holes are smaller or equal to critical size, then we remark that there exists a sequence satisfying [CM] conditions (cf. [2] [3]). In the next section, we, therefore, treat the convergence of energies $\int_{\Omega_\varepsilon} B\nabla u_\varepsilon \cdot \nabla u_\varepsilon\, dx$ for the Dirichlet problem on a general perforated domain, Ω_ε but assuming the existence of a sequence,

w_ε satisfying [CM] conditions. In the third section, we apply the results obtained in the second section to the homogenization of optimal control problems whose state equations are Dirichlet problems in perforated domains.

2 – Strange term for the energy

Let Ω_ε be a general perforated domain. Assume that there exists a sequence, w_ε satisfying the [CM] conditions. We show that there exists a subsequence ε' of ε and a distribution μ_B such that given any $f \in L^2(\Omega)$, if $u_{\varepsilon'}$ solves the Dirichlet problem (1.1) in domain $\Omega_{\varepsilon'}$, then

$$\int_{\Omega_{\varepsilon'}} B\nabla u_{\varepsilon'} \cdot \nabla u_{\varepsilon'}\, dx \to \int_\Omega B\nabla u \cdot \nabla u\, dx + \langle \mu_B, u^2\rangle$$

where u solves the Dirichlet problem (1.2). Further we show that

$$B\nabla\widetilde{u_{\varepsilon'}} \cdot \nabla\widetilde{u_{\varepsilon'}} \longrightarrow B\nabla u \cdot \nabla u + u^2\mu_B \text{ in } D'(\Omega).$$

Lemma 2.1. *Let $\psi_\varepsilon \in H_0^1(\Omega_\varepsilon)$ be the solution of the boundary value problem,*

$$\begin{cases} -\Delta\psi_\varepsilon = -\mathrm{div}(B\nabla w_\varepsilon) & \text{in } \Omega_\varepsilon \\ \psi_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{2.1}$$

Then, the sequence, $\widetilde{\psi_\varepsilon}$, is bounded in $H_0^1(\Omega)$.

Proof. Multiplying (2.1) by ψ_ε and integrating by parts we get,

$$\int_{\Omega_\varepsilon} |\nabla\psi_\varepsilon|^2\, dx = \int_{\Omega_\varepsilon} B\nabla w_\varepsilon \cdot \nabla\psi_\varepsilon\, dx \le M|\nabla w_\varepsilon|_{0,\Omega_\varepsilon}|\nabla\psi_\varepsilon|_{0,\Omega_\varepsilon}.$$

Therefore, since w_ε is a bounded sequence in $H^1(\Omega)$,

$$|\nabla\widetilde{\psi_\varepsilon}|_{0,\Omega} = |\nabla\psi_\varepsilon|_{0,\Omega_\varepsilon} \le M|\nabla w_\varepsilon|_{0,\Omega} \le C$$

where C is a generic constant, and this completes the proof. ∎

So, by the lemma, H^1 boundedness of w_ε and the bound for B in L^∞ (cf. (1.4)), we also deduce that the sequence, $\nabla\widetilde{\psi_\varepsilon} - B\nabla w_\varepsilon$ is bounded in $L^2(\Omega)$. Hence, there exists a subsequence ε' of ε and a function $\psi \in H_0^1(\Omega)$ such that,

$$\tag{2.2} \begin{cases} \widetilde{\psi_{\varepsilon'}} \rightharpoonup \psi & \text{weakly in } H_0^1(\Omega), \\ \nabla\widetilde{\psi_{\varepsilon'}} - B\nabla w_{\varepsilon'} \rightharpoonup \nabla\psi & \text{weakly in } L^2(\Omega)^n. \end{cases}$$

Define, $\mu_B \in D'(\Omega)$ by,

$$\tag{2.3} \mu_B \equiv -\Delta\psi + \psi\mu.$$

Note that the definition of μ_B depends only on w_ε and B. Also note that $\mu_B \in H^{-1}(\Omega)$.

PROPOSITION 2.1. *Let μ_B be given by* (2.3). *Let $f \in L^2(\Omega)$ and let $u_{\varepsilon'}$ be the solution of the Dirichlet problem* (1.1) *in $\Omega_{\varepsilon'}$, such that $u_{\varepsilon'} \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Let $p_{\varepsilon'} \in H_0^1(\Omega_{\varepsilon'})$ be the solution of:*

$$\tag{2.4} \begin{cases} -\Delta p_{\varepsilon'} = -\mathrm{div}({}^tB\nabla u_{\varepsilon'}) & \text{in } \Omega_{\varepsilon'} \\ p_{\varepsilon'} = 0 & \text{on } \partial\Omega_{\varepsilon'}. \end{cases}$$

Then, $\widetilde{p_{\varepsilon'}} \rightharpoonup p$ weakly in $H_0^1(\Omega)$ and p is the solution of,

$$\tag{2.5} \begin{cases} -\Delta p + p\mu = -\mathrm{div}({}^tB\nabla u) + u\mu_B & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega. \end{cases}$$

PROOF. It can be shown, as in Lemma 2.1., that $\widetilde{p_{\varepsilon'}}$ is bounded in $H_0^1(\Omega)$. So there is a subsequence ε'' of ε' and $p \in H_0^1(\Omega)$ such that,

$$\widetilde{p_{\varepsilon''}} \rightharpoonup p \text{ weakly in } H_0^1(\Omega),$$
$$\eta_{\varepsilon''} \equiv \nabla\widetilde{p_{\varepsilon''}} -{}^t B\nabla u_{\varepsilon''} \rightharpoonup \eta \equiv \nabla p -{}^t B\nabla u \text{ weakly in } L^2(\Omega)^n.$$

We need to show that

$$-\mathrm{div}\eta + p\mu = u\mu_B.$$

Let $\phi \in D(\Omega)$. By integration by parts and since $-\mathrm{div}\eta_{\varepsilon''} = 0$ in $\Omega_{\varepsilon''}$,

$$\int_{\Omega_{\varepsilon''}} \eta_{\varepsilon''} \cdot \nabla w_{\varepsilon''} \phi \, dx = -\int_{\Omega_{\varepsilon''}} \eta_{\varepsilon''} \cdot \nabla \phi w_{\varepsilon''} \, dx.$$

Therefore, since $w_{\varepsilon''} \to 1$ strongly in $L^2(\Omega)$ and $\eta_{\varepsilon''} \rightharpoonup \eta$ weakly in $L^2(\Omega)^n$,

$$\lim_{\varepsilon'' \to 0} \int_{\Omega_{\varepsilon''}} \eta_{\varepsilon''} \cdot \nabla w_{\varepsilon''} \phi \, dx = -\int_{\Omega} \eta . \nabla \phi \, dx = \langle \mathrm{div}\eta, \phi \rangle. \tag{2.6}$$

Again,

$$\int_{\Omega_{\varepsilon''}} \eta_{\varepsilon''} \cdot \nabla w_{\varepsilon''} \phi \, dx = \int_{\Omega_{\varepsilon''}} \nabla p_{\varepsilon''} \cdot \nabla w_{\varepsilon''} \phi \, dx - \int_{\Omega_{\varepsilon''}} {}^t B \nabla u_{\varepsilon''} \cdot \nabla w_{\varepsilon''} \phi \, dx \equiv$$
$$\equiv I_{\varepsilon''} + J_{\varepsilon''}.$$

Now,

$$\begin{aligned} I_{\varepsilon''} &= \int_{\Omega_{\varepsilon''}} \nabla p_{\varepsilon''} \cdot \nabla w_{\varepsilon''} \phi \, dx = \\ &= \int_{\Omega_{\varepsilon''}} \nabla w_{\varepsilon''} \cdot \nabla (p_{\varepsilon''} \phi) \, dx - \int_{\Omega_{\varepsilon''}} p_{\varepsilon''} \nabla w_{\varepsilon''} \cdot \nabla \phi \, dx \end{aligned}$$

Therefore, using properties CM2 and CM3 of $w_{\varepsilon''}$, the weak convergence of $\widetilde{p_{\varepsilon''}}$ in $H_0^1(\Omega)$ and its strong convergence in $L^2(\Omega)$, we get,

$$\lim_{\varepsilon'' \to 0} I_{\varepsilon''} = \langle \mu, \phi p \rangle. \tag{2.7}$$

Now,

$$\begin{aligned} J_{\varepsilon''} &= -\int_{\Omega_{\varepsilon''}} {}^t B \nabla u_{\varepsilon''} \cdot \nabla w_{\varepsilon''} \phi \, dx = \\ &= -\int_{\Omega_{\varepsilon''}} B \nabla w_{\varepsilon''} \cdot \nabla u_{\varepsilon''} \phi \, dx = \\ &= \int_{\Omega_{\varepsilon''}} (\nabla \psi_{\varepsilon''} - B \nabla w_{\varepsilon''}) \cdot \nabla u_{\varepsilon''} \phi \, dx - \int_{\Omega_{\varepsilon''}} \nabla \psi_{\varepsilon''} \cdot \nabla u_{\varepsilon''} \phi \, dx \equiv \\ &\equiv K_{\varepsilon''} + L_{\varepsilon''}. \end{aligned}$$

We have, by (2.1),

$$
\begin{aligned}
0 &= \langle \operatorname{div}(\nabla\psi_{\varepsilon''} - B\nabla w_{\varepsilon''}), \phi u_{\varepsilon''}\rangle = \\
&= -\int_{\Omega_{\varepsilon''}} (\nabla\psi_{\varepsilon''} - B\nabla w_{\varepsilon''})\cdot\nabla\phi u_{\varepsilon''}\,dx - \int_{\Omega_{\varepsilon''}} (\nabla\psi_{\varepsilon''} - B\nabla w_{\varepsilon''})\cdot\nabla u_{\varepsilon''}\phi\,dx = \\
&= -\int_{\Omega_{\varepsilon''}} (\nabla\psi_{\varepsilon''} - B\nabla w_{\varepsilon''})\cdot\nabla\phi u_{\varepsilon''}\,dx - K_{\varepsilon''}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lim_{\varepsilon''\to 0} K_{\varepsilon''} &= -\lim_{\varepsilon''\to 0}\int_{\Omega} (\nabla\widetilde{\psi_{\varepsilon''}} - B\nabla w_{\varepsilon''})\cdot\nabla\phi\widetilde{u_{\varepsilon''}}\,dx = \\
&= -\int_{\Omega}\nabla\psi\cdot\nabla\phi u\,dx.
\end{aligned}
$$

Therefore,

$$
\lim_{\varepsilon''\to 0} K_{\varepsilon''} = \int_{\Omega}\nabla u\cdot\nabla\psi\phi\,dx + \langle u\Delta\psi, \phi\rangle. \tag{2.8}
$$

$$
\begin{aligned}
L_{\varepsilon''} &= -\int_{\Omega_{\varepsilon''}} \nabla\psi_{\varepsilon''}\cdot\nabla u_{\varepsilon''}\phi\,dx = \\
&= \int_{\Omega_{\varepsilon''}} \nabla u_{\varepsilon''}.\nabla(\psi_{\varepsilon''}\phi)\,dx + \int_{\Omega_{\varepsilon''}} \nabla u_{\varepsilon''}\cdot\nabla\phi\psi_{\varepsilon''}\,dx = \\
&= -\int_{\Omega} f\widetilde{\psi_{\varepsilon''}}\phi\,dx + \int_{\Omega}\nabla\widetilde{u_{\varepsilon''}}\cdot\nabla\phi\widetilde{\psi_{\varepsilon''}}\,dx.
\end{aligned}
$$

Therefore, using (2.2), we have,

$$
\lim_{\varepsilon''\to 0} L_{\varepsilon''} = -\int_{\Omega} f\phi\psi\,dx + \int_{\Omega}\nabla u\cdot\nabla\phi\psi\,dx,
$$

which, using (1.2), gives,

$$
\begin{aligned}
\lim_{\varepsilon''\to 0} L_{\varepsilon''} &= \langle \Delta u - u\mu, \psi\phi\rangle + \int_{\Omega}\nabla u\cdot\nabla\phi\psi\,dx = \\
&= -\int_{\Omega}\nabla u\cdot\nabla\psi\phi\,dx - \langle u\mu, \phi\psi\rangle
\end{aligned}
\tag{2.9}
$$

after an integration by parts. From (2.6)-(2.9), we get,

$$\begin{aligned}\langle \mathrm{div}\eta, \phi\rangle &= \langle p\mu, \phi\rangle + \langle u\Delta\psi, \phi\rangle - \langle u\psi\mu, \phi\rangle \\ &= \langle p\mu, \phi\rangle - \langle u\mu_B, \phi\rangle.\end{aligned}$$

Since the above holds for all $\phi \in D(\Omega)$, we have $-\Delta p + p\mu = -\mathrm{div}({}^tB\nabla u) + u\mu_B$, i.e. p satisfies (2.5). Since μ is a postive measure, the solution to (2.5) is unique, and therefore, it follows that the entire sequence $\widetilde{p_{\varepsilon'}} \rightharpoonup p$ weakly in $H_0^1(\Omega)$. This completes the proof of the proposition. ∎

We now prove our main theorem.

THEOREM 2.1. *Let ε' be the subsequence of ε chosen prior to Proposition* 2.1. *Let $f \in L^2(\Omega)$ and u_ε be the solution of the Dirichlet problem* (1.1). *Let μ_B be given by* (2.3). *Then,*

$$\int_{\Omega_{\varepsilon'}} B\nabla u_{\varepsilon'} \cdot \nabla u_{\varepsilon'}\, dx \to \int_\Omega B\nabla u \cdot \nabla u\, dx + \langle \mu_B, u^2\rangle \tag{2.10}$$

and,

$$B\nabla u_{\varepsilon'} \cdot \nabla u_{\varepsilon'} \to B\nabla u \cdot \nabla u + u^2\mu_B \ \text{in } D'(\Omega). \tag{2.11}$$

PROOF. Define $p_{\varepsilon'} \in H_0^1(\Omega_{\varepsilon'})$ to be the solution of (2.4). We write,

$$\begin{aligned}\int_{\Omega_{\varepsilon'}} B\nabla u_{\varepsilon'} \cdot \nabla u_{\varepsilon'} dx &= \int_{\Omega_{\varepsilon'}} \nabla p_{\varepsilon'} \cdot \nabla u_{\varepsilon'} dx - \int_{\Omega_{\varepsilon'}} (\nabla p_{\varepsilon'} - {}^tB\nabla u_{\varepsilon'}) \cdot \nabla u_{\varepsilon'} dx = \\ &= \int_{\Omega_{\varepsilon'}} \nabla p_{\varepsilon'} \cdot \nabla u_{\varepsilon'}\, dx = \int_\Omega f\widetilde{p_{\varepsilon'}}\, dx\end{aligned}$$

where we have used the fact that $u_{\varepsilon'}$ and $p_{\varepsilon'}$ are solutions of (1.1) and (2.4) respectively. Therefore, by integration by parts and using Proposition 2.1,

$$\begin{aligned}\lim_{\varepsilon'\to 0} \int_{\Omega_{\varepsilon'}} B\nabla u_{\varepsilon'} \cdot \nabla u_{\varepsilon'}\, dx &= \int_\Omega fp\, dx = \langle -\Delta u + u\mu, p\rangle = \\ &= \langle u, -\Delta p + p\mu\rangle = \langle u, -\mathrm{div}({}^tB\nabla u) + u\mu_B\rangle\end{aligned}$$

which proves (2.10).

Let $\phi \in D(\Omega)$. Set $\eta_{\varepsilon'}$ as in Proposition 2.1., then

$$\int_{\Omega_{\varepsilon'}} B\nabla u_{\varepsilon'} \cdot \nabla u_{\varepsilon'} \phi \, dx = \int_{\Omega_{\varepsilon'}} \nabla p_{\varepsilon'} \cdot \nabla u_{\varepsilon'} \phi \, dx - \int_{\Omega_{\varepsilon'}} \eta_{\varepsilon'} \cdot \nabla u_{\varepsilon'} \phi \, dx \equiv$$
$$\equiv I_{\varepsilon'} + J_{\varepsilon'}.$$

On one hand, it can be shown that (cf. arguments for convergence of $L_{\varepsilon''}$ in Proposition 2.1.)

$$\lim_{\varepsilon' \to 0} I_{\varepsilon'} = \lim_{\varepsilon' \to 0} \int_{\Omega_{\varepsilon'}} \nabla p_{\varepsilon'} \cdot \nabla u_{\varepsilon'} \phi \, dx = \langle -\Delta u + u\mu, p\phi \rangle - \int_{\Omega} \nabla u \cdot \nabla \phi p \, dx =$$
$$= \int_{\Omega} \nabla u \cdot \nabla p \phi \, dx + \langle u\mu, p\phi \rangle.$$

On the other hand, using the fact that $p_{\varepsilon'}$ solves (2.4) and using Proposition 2.1.,

$$\lim_{\varepsilon' \to 0} J_{\varepsilon'} = \lim_{\varepsilon' \to 0} - \int_{\Omega_{\varepsilon'}} \eta_{\varepsilon'} \cdot \nabla u_{\varepsilon'} \phi \, dx = \lim_{\varepsilon' \to 0} \int_{\Omega_{\varepsilon'}} \eta_{\varepsilon'} . \nabla \phi u_{\varepsilon'} \, dx =$$
$$= \int_{\Omega} \eta \cdot \nabla \phi u \, dx = - \int_{\Omega} \eta \cdot \nabla u \phi \, dx + \langle -\mathrm{div} \eta, u\phi \rangle =$$
$$= - \int_{\Omega} \eta \cdot \nabla u \phi \, dx + \langle u\mu_B - p\mu, u\phi \rangle.$$

Therefore,

$$\lim_{\varepsilon' \to 0} \int_{\Omega_{\varepsilon'}} B\nabla u_{\varepsilon'} \cdot \nabla u_{\varepsilon'} \phi \, dx = \lim_{\varepsilon' \to 0} (I_{\varepsilon'} + J_{\varepsilon'}) = \int_{\Omega} \nabla u \cdot \nabla p \phi \, dx + \langle u\mu, p\phi \rangle +$$
$$- \int_{\Omega} \eta \cdot \nabla u \phi \, dx + \langle u\mu_B - p\mu, u\phi \rangle =$$
$$= \int_{\Omega} (\nabla p - \eta) \cdot \nabla u \phi \, dx + \langle u^2 \mu_B, \phi \rangle =$$
$$= \int_{\Omega} B\nabla u \cdot \nabla u \phi \, dx + \langle u^2 \mu_B, \phi \rangle.$$

This holds for all $\phi \in D(\Omega)$. This proves (2.11). $\square$

REMARK 2.1. We introduce a parallel notation, Q_ε for the extension operator $\tilde{}$ from $H^1_0(\Omega_\varepsilon)$ to $H^1_0(\Omega)$. Note the following:

a) If $f \in L^2(\Omega)$, the action of the adjoint operator, $Q^*_\varepsilon : H^{-1}(\Omega) \to H^{-1}(\Omega_\varepsilon)$ on f is nothing but the restriction of f to Ω_ε.
b) Proposition 2.1 and Theorem 2.1 continue to be true if the right hand side in the Dirichlet problem (1.1) is $Q^*_\varepsilon f_\varepsilon$ for a sequence $f_\varepsilon \in H^{-1}(\Omega)$ such that $f_\varepsilon \to f$ strongly in $H^{-1}(\Omega)$. An example of such a sequence is $f_\varepsilon \in L^2(\Omega)$ such that $f_\varepsilon \rightharpoonup f$ weakly in $L^2(\Omega)$. ∎

THEOREM 2.2. *Let μ_B be as defined in* (2.3)*. Then,*

$$\langle \mu_B, \phi\rangle = \lim_{\varepsilon'\to 0}\int_{\Omega_{\varepsilon'}} B\nabla w_{\varepsilon'}\cdot\nabla w_{\varepsilon'}\phi\,dx \textit{ for all } \phi\in D(\Omega). \tag{2.12}$$

PROOF. Let $\phi \in D(\Omega)$. We have,

$$\begin{aligned}\int_{\Omega_{\varepsilon'}} B\nabla w_{\varepsilon'}\cdot\nabla w_{\varepsilon'}\phi dx &= \int_{\Omega_{\varepsilon'}}\nabla\psi_{\varepsilon'}\cdot\nabla w_{\varepsilon'}\phi\,dx - \int_{\Omega_{\varepsilon'}}(\nabla\psi_{\varepsilon'} - B\nabla w_{\varepsilon'})\cdot\nabla w_{\varepsilon'}\phi\,dx = \\ &= -\int_{\Omega_{\varepsilon'}}\nabla w_{\varepsilon'}\cdot\nabla\phi\psi_{\varepsilon'}\,dx + \langle -\Delta w_{\varepsilon'}, \widetilde{\psi_{\varepsilon'}}\phi\rangle + \\ &\quad + \int_{\Omega_{\varepsilon'}}(\nabla\psi_{\varepsilon'} - B\nabla w_{\varepsilon'})\cdot\nabla\phi w_{\varepsilon'}\,dx.\end{aligned}$$

Passing to the limit is easy now and we get,

$$\begin{aligned}\lim_{\varepsilon'\to 0}\int_{\Omega_{\varepsilon'}} B\nabla w_{\varepsilon'}\cdot\nabla w_{\varepsilon'}\phi\,dx &= \langle\mu, \psi\phi\rangle + \int_\Omega \nabla\psi\cdot\nabla\phi\,dx = \\ &= \langle\mu,\psi\phi\rangle + \langle -\Delta\psi, \phi\rangle = \langle\mu_B, \phi\rangle.\end{aligned}$$

This completes the proof. ∎

COROLLARY 2.1. *If B is a positive definite matrix, then μ_B is a positive measure.*

We now prove a result on the partial uniqueness of μ_B.

THEOREM 2.3. *Suppose that μ_0 and μ_1 are measures. Let $f \in H^{-1}(\Omega)$ be arbitrary. Let u_ε solve the Dirichlet problem* (1.1) *with right* *hand side* $Q_\varepsilon^* f$ *and* u *solve* (1.2)*, so that* $\widetilde{u_\varepsilon} \rightharpoonup u$ *weakly in* $L^2(\Omega)$ (cf. [3])*. Suppose that,*

$$B\nabla u_\varepsilon \cdot \nabla u_\varepsilon \longrightarrow B\nabla u \cdot \nabla u + u^2\mu_0 \; in D'(\Omega) \; and,$$
$$B\nabla u_\varepsilon \cdot \nabla u_\varepsilon \longrightarrow B\nabla u \cdot \nabla u + u^2\mu_1 \; in \; D'(\Omega).$$

Then $\mu_0 = \mu_1$ *in* $D'(\Omega)$*.*

Proof. Let $v \in H_0^1(\Omega)$ be arbitrary. Then $\Delta v \in H^{-1}(\Omega)$. Since, $\mu \in W^{-1,\infty}(\Omega)$, we also have $v\mu \in H^{-1}(\Omega)$. Thus we are allowed to take $f \equiv -\Delta v + v\mu$ in the hypothesis. Now, let u_ε be the solution of (1.1) with right hand side $Q_\varepsilon^* f$. Then $\widetilde{u_\varepsilon} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ where u solves

$$-\Delta u + u\mu = -\Delta v + v\mu \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega.$$

Since μ is a positive measure, the solution to the above equation is unique and therefore, $u \equiv v$. Now, from the hypothesis of the theorem, we conclude that,

$$v^2\mu_0 = v^2\mu_1 \text{ in } D'(\Omega).$$

But $v \in H_0^1(\Omega)$ was arbitrary. For any $\omega \subset\subset \Omega$, we choose $v \in D(\Omega)$ such that $v \equiv 1$ on ω. Then, $\mu_0(\phi) = \mu_1(\phi)$ for any $\phi \in D(\Omega)$ with supp$\phi \subset \omega$. That is, $\mu_{0|_\omega} = \mu_{1|_\omega}$ or $\mu_0 = \mu_1$ in $D'(\omega)$. As this holds, for all $\omega \subset\subset \Omega$, we have $\mu_0 = \mu_1$ in $D'(\Omega)$. This ends the proof. ∎

Remark 2.2. The proof of Theorem 2.1 becomes simpler, when a strong corrector result of Cioranescu and Murat (cf. [3]) holds. The corrector result is as follows:
Let $f \in L^2(\Omega)$ and let u_ε be the solution of (1.1) and let u be the solution of (1.2). Then, $\widetilde{u_\varepsilon} \longrightarrow u$ weakly in $H_0^1(\Omega)$. Further assume that u is $C_0^1(\Omega)$, then

$$\widetilde{u_\varepsilon} - uw_\varepsilon \to 0 \text{ strongly in } H_0^1(\Omega). \tag{2.13}$$

When this holds we can give a proof of Theorem 2.1 using Theorem 2.2 as follows.

PROOF. Set $r_{\varepsilon'} = \widetilde{u_{\varepsilon'}} - uw_{\varepsilon'}$. By Proposition 2.2, $r_{\varepsilon'} \to 0$ strongly in $H_0^1(\Omega)$. Therefore,

$$I_{\varepsilon'} \equiv \int_{\Omega_{\varepsilon'}} B\nabla u_{\varepsilon'} \cdot \nabla u_{\varepsilon'}\, dx = \int_\Omega B\nabla\widetilde{u_{\varepsilon'}} \cdot \nabla\widetilde{u_{\varepsilon'}}\, dx = $$
$$= \int_\Omega B\nabla(uw_{\varepsilon'}) \cdot \nabla(uw_{\varepsilon'})\, dx + o(1).$$

Now, since $\nabla w_{\varepsilon'} \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and B and u are bounded functions on Ω,

$$\int_\Omega B\nabla(uw_{\varepsilon'})\cdot\nabla(uw_{\varepsilon'})dx = \int_\Omega B\nabla w_{\varepsilon'}\cdot\nabla w_{\varepsilon'} u^2\, dx + \int_\Omega B\nabla u\cdot\nabla u w_{\varepsilon'}^2\, dx + o(1)$$

Therefore, using Theorem 2.2 and the strong convergence of $w_{\varepsilon'}$ to 1 in $L^2(\Omega)$, we conclude that,

$$\lim_{\varepsilon'\to 0} I_{\varepsilon'} = \langle \mu_B, u^2\rangle + \int_\Omega B\nabla u \cdot \nabla u\, dx.$$

This ends the proof. ∎

3 – An application

We apply Theorem 2.1 to obtain the homogenized cost functional corresponding to optimal control problems whose state equations are Dirichlet problems on perforated domains, Ω_ε. The type of optimal control problem which we consider now, has been studied in various situations before; for e.g cf. Kesavan and Vanninathan [6], Kesavan and Saint Jean Paulin [4], [5]. The last of the references deals with the homogenization of an optimal control problem in perforated domains. There, in the state equation, Neumann condition is assumed on the boundary of holes. So the homogenization problem is quite different from what we study now.

For fixed ε, the optimal control problem is as follows: Let $N > 0$ be a constant. Let B be as before and moreover, assumed to be positive. We take the space of *admissible controls* to be $U_{ad}^\varepsilon = L^2(\Omega_\varepsilon)$. For $\varepsilon > 0$ fixed, we define the optimal control problem as follows:

Minimize the *cost functional*,

$$J_\varepsilon(\theta) = \frac{1}{2}\int_{\Omega_\varepsilon} B\nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx + \frac{N}{2}\int_{\Omega_\varepsilon} \theta^2 \, dx \text{ on } U^\varepsilon_{ad}$$

where the *state* $u_\varepsilon \equiv u_\varepsilon(\theta)$ is the solution of the Dirichlet problem,

$$\begin{cases} -\Delta u_\varepsilon = \theta & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \Omega_\varepsilon. \end{cases} \tag{3.1}$$

It can be shown that there exists a unique $\theta^*_\varepsilon \in U^\varepsilon_{ad}$ such that

$$J_\varepsilon(\theta^*_\varepsilon) = \min_{\theta \in U^\varepsilon_{ad}} J_\varepsilon(\theta). \tag{3.2}$$

The homogenization theorem is as follows.

Theorem 3.1. *Let ε' be the subsequence of ε as obtained prior to Proposition* 2.1. *Let μ_B be given by* (2.3). *Then, for a subsequence ε'' of ε', $\widetilde{\theta^*_{\varepsilon''}} \rightharpoonup \theta^*$ weakly in $L^2(\Omega)$ where θ^* is the optimal control of the problem: minimize the cost functional, $J(\theta)$ over $\theta \in L^2(\Omega)$ where,*

$$J(\theta) \equiv \frac{1}{2}\int_{\Omega} B\nabla u \cdot \nabla u \, dx + \frac{1}{2}\langle \mu_B, u^2\rangle + \frac{N}{2}\int_{\Omega} \theta^2 \, dx$$

and where the state $u \equiv u(\theta)$ solves the Dirichlet problem,

$$\begin{cases} -\Delta u + u\mu = \theta & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Proof. Step 1: First we show that $\widetilde{\theta^*_\varepsilon}$ is bounded in $L^2(\Omega)$. Let $\theta \in L^2(\Omega)$ be arbitrary. We will denote its restriction to Ω_ε also by θ. We shall denote the solutions (states) of the Dirichlet problem (3.1) for right hand sides θ^*_ε, θ by u^*_ε and u_ε respectively. It can be shown that $|\nabla\widetilde{u_\varepsilon}|_{0,\Omega} \leq \alpha|\theta|_{0,\Omega}$ where α is the constant which appears in the Poincaré's inequality for Ω. Now, $J_\varepsilon(\theta^*_\varepsilon) \leq J_\varepsilon(\theta)$ implies,

$$\frac{N}{2}|\theta^*_\varepsilon|^2_{0,\Omega_\varepsilon} \leq \frac{1}{2}M\alpha^2|\theta|^2_{0,\Omega} + \frac{N}{2}|\theta|^2_{0,\Omega}.$$

Therefore, $\widetilde{\theta_\varepsilon^*}$ is a bounded sequence in $L^2(\Omega)$.

Step 2: Let ε' be the subsequence of ε mentioned before. From Step 1, we conclude that there is a subsequence ε'' of ε' and θ^* in $L^2(\Omega)$ such that $\widetilde{\theta^*_{\varepsilon''}} \rightharpoonup \theta^*$ weakly in $L^2(\Omega)$. Therefore,

$$\begin{aligned}&\widetilde{u^*_{\varepsilon''}} \rightharpoonup u^*,\\ &\widetilde{u_{\varepsilon''}} \rightharpoonup u \text{ weakly in } H_0^1(\Omega)\end{aligned}$$

where u^*, u are the solutions of (3.3) with right hand sides θ^*, θ respectively. Using a) of Remark 2.1, we conclude that,

$$\int_{\Omega_{\varepsilon''}} B\nabla u^*_{\varepsilon''} \cdot \nabla u^*_{\varepsilon''}\, dx \to \int_\Omega B\nabla u^* \cdot \nabla u^*\, dx + \langle \mu_B, (u^*)^2\rangle$$

and

$$\int_{\Omega_{\varepsilon''}} B\nabla u_{\varepsilon''} \cdot \nabla u_{\varepsilon''}\, dx \to \int_\Omega B\nabla u \cdot \nabla u\, dx + \langle \mu_B, u^2\rangle.$$

Step 3: Let χ_ε be the characteristic function of Ω_ε. Let us assume that $\chi_\varepsilon \rightharpoonup \chi$ weakly * in $L^\infty(\Omega)$. Since $\chi_\varepsilon w_\varepsilon = w_\varepsilon$, passing to the limit we get, $\chi = 1$ identically in Ω. Now we pass to the limit in

$$J_\varepsilon(\theta_\varepsilon^*) \le J_\varepsilon(\theta).$$

From Step 2, it follows that

$$\begin{aligned}\frac{1}{2}\int_\Omega B\nabla u^* \cdot \nabla u^*\, dx + \frac{1}{2}\langle \mu_B, (u^*)^2\rangle + \frac{N}{2}\varlimsup_{\varepsilon''\to 0}\int_{\Omega_{\varepsilon''}} {\theta^*_{\varepsilon''}}^2\, dx \le\\ \le \frac{1}{2}\int_\Omega B\nabla u \cdot \nabla u\, dx + \frac{1}{2}\langle \mu_B, u^2\rangle + \frac{N}{2}\int_\Omega \theta^2\, dx.\end{aligned}$$

Starting with $J_\varepsilon(\theta_\varepsilon^*) \le J_\varepsilon(\theta^*)$ and arguing as before we get,

$$\begin{aligned}\frac{1}{2}\int_\Omega B\nabla u^* \cdot \nabla u^*\, dx + \frac{1}{2}\langle \mu_B, (u^*)^2\rangle + \frac{N}{2}\varlimsup_{\varepsilon''\to 0}\int_{\Omega_{\varepsilon''}} {\theta^*_{\varepsilon''}}^2\, dx \le\\ \le \frac{1}{2}\int_\Omega B\nabla u^* \cdot \nabla u^*\, dx + \frac{1}{2}\langle \mu_B, (u^*)^2\rangle + \frac{N}{2}\int_\Omega (\theta^*)^2\, dx.\end{aligned}$$

Therefore,

$$\overline{\lim_{\varepsilon''\to 0}} \int_{\Omega_{\varepsilon''}} {\theta^*_{\varepsilon''}}^2\, dx \le \int_\Omega (\theta^*)^2\, dx.$$

However, since $\widetilde{\theta^*_{\varepsilon''}} \rightharpoonup \theta^*$ weakly in $L^2(\Omega)$, we have,

$$\underline{\lim}_{\varepsilon''\to 0} \int_{\Omega_{\varepsilon''}} {\theta^*_{\varepsilon''}}^2\, dx \ge \int_\Omega (\theta^*)^2\, dx.$$

Thus,

$$\lim_{\varepsilon''\to 0} \int_{\Omega_{\varepsilon''}} {\theta^*_{\varepsilon''}}^2\, dx = \int_\Omega (\theta^*)^2\, dx. \tag{3.4}$$

Putting the above together we get, $J(\theta^*) \le J(\theta)$. This holds for all $\theta \in L^2(\Omega)$. Thus, θ^* is the optimal control of the problem whose cost functional is J and whose state equation is (3.3). We remark finally that (3.4) implies that we have the strong convergence, $\widetilde{\theta^*_{\varepsilon''}} \to \theta^*$ in $L^2(\Omega)$ of the optimal controls. ∎

We end with a few remarks.

Remark 3.1. We saw in Step 3 of the preceding theorem that if $\chi_\varepsilon \rightharpoonup \chi$ weakly * in $L^\infty(\Omega)$, then $\chi \equiv 1$. We now observe that,

$$\begin{aligned} \chi_\varepsilon - 1 &= \chi_\varepsilon(1-w_\varepsilon) + \chi_\varepsilon w_\varepsilon - 1 \\ &= \chi_\varepsilon(1-w_\varepsilon) + w_\varepsilon - 1 \end{aligned}$$

Therefore, since $w_\varepsilon \to 1$ strongly in $L^2(\Omega)$, we conclude that $\chi_\varepsilon \to 1$ strongly in $L^2(\Omega)$. ∎

Remark 3.2. The optimal control problem can be studied for other choices for U^ε_{ad} like (cf. [4], [5]);

$$\begin{aligned} U^\varepsilon_{ad} &= \{\theta \in L^2(\Omega_\varepsilon | \theta \ge \psi \text{ in } \Omega_\varepsilon\} \\ U^\varepsilon_{ad} &= \{\theta \in L^2(\Omega_\varepsilon | \psi_1 \le \theta \le \psi_2 \text{ in } \Omega_\varepsilon\} \\ U^\varepsilon_{ad} &= \{\theta \in L^2(\Omega_\varepsilon | \int_{\Omega_\varepsilon} \theta^2\, dx \le 1\} \end{aligned}$$

The corresponding space of controls in the homogenized problem can be shown to be

$$U_{ad} = \{\theta \in L^2(\Omega) | \theta \geq \psi \text{ in } \Omega\}$$
$$U_{ad} = \{\theta \in L^2(\Omega) | \psi_1 \leq \theta \leq \psi_2 \text{ in } \Omega\}$$
$$U_{ad} = \{\theta \in L^2(\Omega) | \int_\Omega \theta^2 \, dx \leq 1\}$$

The cost functionals remain the same as in Theorem 3.1. ∎

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