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Lie triple systems and warped products

M. BERTOLA – D. GOUTHIER

Riassunto: *Si investigano condizioni che garantiscano che un un prodotto deformato $M = B \times_\omega F$ sia localmente simmetrico. In particolare si mostra che è necessario che esista un sistema triplo di Lie di codimensione uno; tali strutture vengono descritte e studiate, portando alla classificazione dei prodotti deformati localmente simmetrici.*

Abstract: *We investigate the conditions ensuring that a warped product $M = B \times_\omega F$ is locally symmetric. In particular, we show that it is necessary that there exists a codimension-one Lie Triple System; such structures are described and studied, leading to classification of locally symmetric warped products*

1 – Introduction

A *warped product* (M, g) is topologically a product manifold $M = B \times F$: its Riemannian metric g is obtained as a deformation of the product metric $g_B + g_F$ via a map $\omega : B \to \mathbb{R}^+$:

$$g = g_B + \omega^2 g_F \,.$$

In the literature warped products were first used to construct examples of manifolds with negative curvature; of locally symmetric spaces (whose factors are space forms); of curvature homogeneous spaces which are not locally homogeneous and so on. Therefore it is natural to study and

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classify warped products with special Riemannian curvature. This motivated [2], where we described and classified the warped products with constant curvature and the Einstein warped products with both factors Einstein.

The present paper extends those results to the case of locally symmetric spaces.

The classification of symmetric warped products is achieved via the system

$$\begin{cases} \varkappa\omega^2 + \|\Omega\|^2 = \mathfrak{h} \\ \nabla\Big(\dfrac{H^\omega}{\omega}\Big) = 0\,. \end{cases}$$

The second equation generalizes the condition $H^\omega \propto \omega g$ which allowed our classification, in [2], of the warped products with constant sectional curvature. Such an equation, in the form $H^\omega + \omega g = 0$ has been used in order to give conditions on the curvature. In particular GALLOT in [4] proves that if a complete Riemannian manifold (M^n, g) admits a function ω such that $\triangle\omega = n\omega$, then ω satisfies $H^\omega + \omega g = 0$. Furthermore OBATA's theorem [7] assures that — under the hypotheses of compactness — such a function exists iff M^n is the canonical sphere.

It is possible to prove directly that necessary conditions for $(M = B\times F, g_\omega)$ to be (locally) symmetric are that F has constant curvature and B is locally symmetric (Proposition 4.2). Further conditions involving the function ω imply the existence of a codimension one *Lie Triple System* σ orthogonal to $\Omega := \mathrm{grad}(\omega)$ (Lemma 4.5).

We are thus led to study which conditions must be fulfilled in order that a symmetric Lie algebra $(\mathfrak{g}, \theta)$ possesses a codimension one Lie Triple System.

Two different results have been proved here in order to describe such Lie Triple Systems and consequently the Riemannian manifold (B, g_B). First of all we prove that if Ω is a vector orthogonal to the Lie Triple System σ then Ω belongs to an irreducible factor or to the Euclidean one (Theorem 4.9): therefore we study codimension one Lie Triple Systems embedded into an irreducible factor.

The second result assures that an irreducible symmetric space with a codimension one Lie Triple System has constant curvature (Theorem 3.5).

Thus, a locally symmetric warped product is (locally) the direct product (i.e. a warped product with $\omega \equiv 1$) of some irreducible symmetric spaces (and an Euclidean one, if the case) and of a warped product with constant curvature. Moreover, the warping function of this factor is the same ω thought on a unique irreducible factor (Corollary 4.8).

The paper is divided into four sections. In Section 2, we recall the basics of symmetric spaces in order to introduce the notations: we essentially adopt the notations of [5] and the relevant facts are taken ibidem.

Section 3 describes the Lie Triple Systems of codimension one: it contains the original results in the classification of such objects.

In Section 4 we recall symbols and known results on warped products (obtained from O'Neill [8]). Then we obtain necessary and sufficient conditions for local symmetricity (Proposition 4.2); hence, we show the existence of locally symmetric codimension one submanifolds. Such submanifolds define a Lie Triple System of the same kind studied in Section 3. In such a way we can achieve a characterization of locally symmetric warped products, which is the main goal of this paper.

2 – Preliminaries on symmetric spaces

In this section we quickly recall some facts about (locally) symmetric spaces and fix the notations; as they are quite well known we will recall only the facts we are going to use referring the reader to [5] for more extensive coverage of the subject.

A globally [locally] symmetric space is a Riemannian n-dimensional manifold M with metric g such that for any $x \in M$ there exists a geodesic involutive global [local] isometry θ_x fixing the point x. Necessary and sufficient condition for the local symmetry of M is that $\nabla R = 0$.

If G is the largest connected group of isometries of M and K is the isotropy subgroup of a certain arbitrarily fixed point $x_0 \in M$ then G acts transitively on M and hence $M = G/K$.

The isometry θ_{x_0} can be uniquely lifted to an involutive automorphism of G (which we denote again with θ) for which the set of fixed points is K.

Since we are dealing with Riemannian symmetric spaces, K is a compact group (it is a subgroup of $SO(n)$).

Let $\mathfrak{g}$ and $\mathfrak{k}$ be respectively the Lie algebras of the groups G and K: denote by ϑ the action of the automorphism θ on the Lie algebra of G.

Let us denote with $\mathfrak{k}$ and $\mathfrak{m}$ the subspaces realizing the Cartan decomposition of $\mathfrak{g}$, namely the eigenspaces of ϑ with eigenvalues $+1$ and -1 respectively. Clearly $\mathfrak{k}$ is the subalgebra generating K and the orthogonal complement $\mathfrak{m}$ is naturally identified with $T_{x_0}M$.

We remind that the connection of M does not depend upon the choice of the G-invariant metric on $\mathfrak{m}$. Since we are interested in the curvature, a specific choice of the invariant metric does not affect the generality of the result. Hereafter we will work under the assumption that $\mathfrak{g}$ is semisimple:

a) if $\mathfrak{g}$ is semisimple and compact we can choose minus the Killing form as the metric;

b) if it is semisimple, non-compact, without compact ideals then the Killing form on $\mathfrak{m}$ is positive definite and therefore we may take that as our invariant Riemannian metric. This also implies that in the non-compact case $\mathfrak{k}$ must be a maximal compactly embedded subalgebra if we want to obtain a Riemannian symmetric space.

Recall that the curvature tensor at the base point x_0 satisfies the equation:

$$R_{XY}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z = [[X,Y],Z] \quad X, Y, Z \in \mathfrak{m}\,.$$

Let $\mathfrak{h} \subseteq \mathfrak{m}$ be a maximal Abelian subalgebra (**Cartan subalgebra**) in the tangent space $T_{x_0}M$; let $E_\alpha \equiv \{X \in \mathfrak{g}\ s.t.\ \forall\, H \in \mathfrak{h}\ ad_H(X) = \alpha(H)X\}$ be the corresponding root spaces with dimensions m_α.

The set $\overline{R} = \{(\alpha, m_\alpha) \in \mathfrak{h}^* \times \mathbb{N}\ s.t.\ m_\alpha > 0\}$ is called the **system of restricted roots** and we have the decomposition $\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha\in\overline{R}} E_\alpha$ where g_0 is the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. The **rank** of M is the dimension of $\mathfrak{h}$ or equivalently the dimension of a maximal totally geodesic flat submanifold of M. We introduce the following spaces

$$\mathfrak{k}_\alpha := \pi^+ E_\alpha := \frac{1}{2}\Big(id + \vartheta\Big)E_\alpha \subset \mathfrak{k}; \qquad \mathfrak{m}_\alpha := \pi^- E_\alpha := \frac{1}{2}\Big(id - \vartheta\Big)(E_\alpha) \subseteq \mathfrak{m}$$
$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha\in\overline{R}_+} \mathfrak{k}_\alpha; \qquad \mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k}; \qquad \mathfrak{m} = \mathfrak{h} + \sum_{\alpha\in\overline{R}_+} \mathfrak{m}_\alpha\,.$$

Notice that the splitting of $\mathfrak{m}$ is orthogonal w.r.t. the Killing metric.

We complete $\mathfrak{h}$ to a Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{g}$ and consider the similar Cartan decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{H} + \sum_{\alpha \in R} g_\alpha$$

where R denotes the set of roots of the semisimple Lie algebra $\mathfrak{g}$: then the restricted roots contain (as a set) the root system $R \subset \mathfrak{H}^*$ of $\mathfrak{g}$ (restricted to $\mathfrak{h}^*$). More precisely, if $\alpha \in R$, then $\alpha_{|_\mathfrak{h}}$, *if not zero*, belongs to $\overline{R}$. This in turn implies that $\overline{R}$ spans the dual $\mathfrak{h}^*$.

Conversely for any restricted root $\alpha \in \overline{R} \subset \mathfrak{h}^* \subset \mathfrak{H}^*$ there exists a root $\tilde{\alpha} \in R$ such that $\tilde{\alpha}_{|_\mathfrak{h}} = \alpha$.

3 – Codimension one Lie Triple Systems

In this paragraph we will denote with σ the tangent space $T_{x_0}\Sigma$, subspace of $\mathfrak{m} \simeq T_{x_0}M$. We have the following definition and proposition, both to be found in [5], which work for a generic subspace $\sigma \in \mathfrak{m}$, M being a locally symmetric space.

Definition 3.1. A **Lie Triple System** σ is a subspace of $\mathfrak{g}$ with the following property

$$[[\sigma, \sigma], \sigma] \subseteq \sigma\,.$$

It can be proved the following

Proposition 3.2. *Let $\Sigma \subset M$ be a totally geodesic submanifold of the (locally) symmetric manifold M; then Σ is a (locally) symmetric manifold and $T_{x_0}\Sigma \simeq \sigma \subset \mathfrak{m}$ is a Lie Triple System.*

Conversely if $\sigma \subset \mathfrak{m}$ is a Lie Triple System, then $\Sigma = \mathrm{Exp}(\sigma)$ is a totally geodesic submanifold.

We now begin the study of the codimension one Lie Triple Systems: the following lemmas and propositions are not to be found in the literature.

Suppose that σ is of codimension one in $\mathfrak{m}$. We take an orthogonal vector W to σ in $\mathfrak{m}$; we can always assume that $\mathfrak{h}$ contains W by completing the one dimensional algebra spanned by W to a maximal Abelian subalgebra of $\mathfrak{m}$ and hence we can give the orthogonal decomposition of $\mathfrak{m}$ as

$$\mathfrak{m} = \underbrace{\mathbb{R}[W] \overset{\perp}{\oplus} \sigma_0}_{:=\mathfrak{h}} \overset{\perp}{\oplus} \sum_{\alpha\in\overline{R}} \mathfrak{m}_\alpha$$
$$\sigma := \sigma_0 \overset{\perp}{\oplus} \sum_{\alpha\in\overline{R}} \mathfrak{m}_\alpha \,.$$

Let $\{H_i,\ i = 1 \dots \mathrm{rank}(M)\}$ be a basis of $\mathfrak{h}$ corresponding to simple roots $\{\alpha_1 \dots \alpha_r\}$ normalized to $\|H_i\| = 2$ for long roots.

Thus we have that

LEMMA 3.3. *If $\sigma \subset \mathfrak{m}$ is a codimension one Lie Triple System, then the Dynkin diagram of the restricted roots has at least a rank one connected component.*

PROOF. First of all notice that if $X \in E_\alpha$, then $[X, \vartheta X]$ is an element of $\mathfrak{h}$ because it belongs to the centralizer of $\mathfrak{h}$ and it is skew w.r.t. ϑ; moreover $\forall\, H \in \mathfrak{h}$

$$\begin{aligned}\langle H, [X, \vartheta X]\rangle &= \langle [H, X], \vartheta X\rangle = \alpha(H)\langle X, \vartheta X\rangle = \\ &= -\frac{\alpha(H)}{2}\langle \pi^- X, \pi^- X\rangle = -\frac{1}{2}\alpha(H)\|\pi^- X\|^2 \,,\end{aligned}$$

and hence $[X, \vartheta X] = -\frac{\|\pi^- X\|^2}{2} H_\alpha$. There follows that for any fixed simple root α_i, and any $X \in E_{\alpha_i}$ we have (as in the above equation)

$$[[H, \pi^- X], \pi^- X] = \frac{\|\pi^- X\|^2}{4}\alpha_i(H) H_i \quad \forall\, H \in \sigma_0 \,.$$

Imposing that this is orthogonal to W for all $i = 1 \dots r$ (remember that all $\mathfrak{m}_\alpha$ are included into σ) we have that, $\forall\, i = 1 \dots r$, $\forall\, H \in \sigma_0$,

$$\alpha_i(H)\langle W, H_i\rangle = 0 \,,$$

but, since H_i and α_i are linearly independent and the dimension of σ_0 is $r - 1$ it follows that there exist a unique i (which we assume from now on to be 1) such that $\langle W, H_1\rangle \neq 0$, and $\langle W, H_j\rangle = 0$, $\forall\, j = 2\dots r$. Hence $W \in \operatorname{span}(H_1)$ and $\langle H_1, H_j\rangle = 0$, $j = 2\dots r$.

This proves that the Dynkin diagram is disconnected and it has (at least) one rank-one component. ∎

The second lemma characterizes the irreducible component of the set of restricted roots to which α_1 belongs.

LEMMA 3.4. *In the hypotheses of Lemma* 3.3 (*setting* $\alpha := \alpha_1$) *we have that* 2α *is not a root, i.e.* $\mathfrak{m}_{2\alpha}$ *is trivial.*

PROOF. Suppose $X, Y \in E_\alpha$ and $Z \in E_{2\alpha}$, then

$$\begin{aligned}[][\pi^- Z, \pi^- X], \pi^- Y] &= -\frac{1}{2}[\pi^+[\vartheta X, Z], \pi^- Y] = \\ &= -\frac{1}{4}\langle \pi^-[\vartheta X, Z], \pi^- Y\rangle H_\alpha - \frac{1}{4}\underbrace{\pi^-[[\vartheta X, Z], Y]}_{\in \mathfrak{m}_{2\alpha}}\,.\end{aligned}$$

Now the vector $\pi^-[\vartheta X, Z]$ cannot vanish for all $X \in E_\alpha$, $Z \in E_{2\alpha}$ since $\vartheta X \in E_{-\alpha}$ and $-\alpha, 2\alpha$ belong to the α–string through 0. Thus $\langle \pi^-[\vartheta X, Z], \pi^- Y\rangle$ does not vanish identically and this leads to a contradiction with the fact that $W^\perp$ is a Lie Triple System. ∎

As a consequence we get

THEOREM 3.5. *An irreducible Riemannian symmetric space* M *of rank one with reduced diagram of restricted roots* (*i.e.* $\alpha \in \overline{R} \Rightarrow 2\alpha \notin \overline{R}$) *has constant sectional curvature.*

PROOF. We compute the curvature tensor directly. Let $X, Y, Z \in E_\alpha$ (here α is the only positive restricted root) and notice that $[X, Y] = [X, Z] = [Y, Z] = 0$; let $\xi = \pi^- X$, $\eta = \pi^- Y$, $\zeta = \pi^- Z$ and set $H \in \mathfrak{h}$ such that $\alpha(H) = 2$. A straightforward but lengthy computation — which we spare to the readers —, gives the following equations:

$$\begin{aligned} R_{H\xi}\eta &= [[H, \xi], \eta] = 2[\pi^+ X, \pi^- Y] = \langle \xi, \eta\rangle H \\ R_{H\xi}H &= [[H, \xi], H] = 4\xi = \langle H, H\rangle \xi \\ R_{\xi\eta}H &= [[\xi, \eta], H] = \frac{1}{2}[\pi^+[\vartheta X, Y], H] = 0 \\ R_{\xi\eta}\zeta &= \langle \eta, \zeta\rangle \xi - \langle \xi, \zeta\rangle \eta\,. \end{aligned}$$

This proves that $R = -R^0$ where $R^0_{\eta\xi}\zeta := \langle \xi, \zeta\rangle\eta - \langle \eta, \zeta\rangle\xi$ and ends the proof. $\square$

Finally let us consider the more general case in which $\mathfrak{g}$ is semisimple; as it is proved in [5], we can split it into the direct sum of irreducible ideals $\mathfrak{g}^{(i)}$, $i = 1 \ldots N$, (where irreducible means that the algebra $\mathfrak{k}^{(i)} \subset \mathfrak{g}^{(i)}$ is simple compact). Now the picture is the following

$$\mathfrak{g} = \bigoplus_{i=1}^{N} \Big(\overbrace{\mathfrak{m}^{(i)} + \mathfrak{k}^{(i)}}^{:=\mathfrak{g}^{(i)}} \Big)$$

and the orthogonal of our Lie Triple System of codimension one is spanned by the single vector $W \in \bigoplus \mathfrak{m}^{(i)}$. Let us introduce the canonical projections $p^{(i)} : \mathfrak{g} \to \mathfrak{g}^{(i)}$ and the following objects

$$W^i := p^{(i)} W \ ; \qquad \sigma^i := \sigma \cap \mathfrak{m}^{(i)} = \langle W^i \rangle^{\perp}_{\mathfrak{m}^{(i)}} \,.$$

Notice that $\sigma \supset \bigoplus \sigma^i$ and that (from the fact that $\mathfrak{g}^{(i)}$ are ideals) each σ^i is a Lie Triple System inside $\mathfrak{m}^{(i)}$ and coincides with the orthogonal of W^i in $\mathfrak{m}^{(i)}$. Therefore σ^i coincides with $\mathfrak{m}^{(i)}$ if $W^i = 0$, and has codimension one otherwise. It follows from the previous discussion that

COROLLARY 3.6. *If the symmetric pair* $(\mathfrak{g}, \vartheta)$ *is semisimple and* $\mathfrak{g} = \bigoplus \mathfrak{g}^{(i)}$ *is the corresponding splitting into irreducibles, then for all* j *such that* $\mathfrak{m}^{(j)} \ni W^j \neq 0$ *the corresponding irreducible symmetric pair* $(\mathfrak{g}^{(j)}, \vartheta^{(j)})$ *is of rank one with reduced diagram, namely it defines a factor with constant curvature.*

Summarizing, each irreducible factor on which W^i does not vanish has constant curvature, while there are no requirements on the others.

We conclude the section with the

THEOREM 3.7. *If* $\mathfrak{g}$ *is semisimple and has a codimension one Lie Triple System* σ *in* $\mathfrak{m}$, *then the orthogonal to* σ *is contained in one irreducible component* (*which has therefore constant curvature*).

PROOF. Let $\mathfrak{g} = \bigoplus_1^N g^{(i)}$ be the decomposition into irreducibles as above; let us recall that $\sigma^i := \sigma \cap \mathfrak{m}^{(i)} \subset \sigma$ are Lie Triple Systems into $\mathfrak{g}^{(i)}$. Now the generic vector in σ has the form

$$X = \sum_1^N x_i W^i + \pi^- X^{(i)} ,$$

$$X^{(i)} \in \mathfrak{m}^{(i)}_{\alpha_i} \equiv \sigma^i, \qquad \sum_1^N x_i \|W^i\|^2 = 0$$

where the condition on the x_i comes from the requirement of orthogonality, while the $\pi^- X^{(i)}$ are already orthogonal. Now take a further vector of the form $Y = \sum \eta_i = \sum \pi^- Y^{(i)}$ (and noticing that $[W^i, \pi^- X^{(j)}] = \alpha_i(W^i) X^{(i)} \delta_{ij} := \lambda_i X^{(i)} \delta_{ij}$) and compute

$$\begin{aligned}[][X,Y],Y] &= \Big[\Big[\sum_i x_i W^i, \sum_j \eta^j\Big], \sum_k \eta^k\Big] = \\ &= \sum_i x_i \lambda_i [\pi^+ Y^{(i)}, \pi^- Y^{(i)}] = \sum_i x_i \lambda_i^2 \|\eta^i\|^2 W^i ;\end{aligned}$$

imposing the orthogonality we get

$$\sum_i x_i \lambda_i^2 \|\eta^i\|^2 \|W^i\|^2 = 0$$

which is a contradiction since Y is generic, unless there is only one component W^i different from 0. ∎

4 – Symmetric warped products

Let now (B, g_B) and (F, g_F) be two Riemannian manifolds whose dimensions are b and f, respectively. We consider the smooth manifold $M := B \times F$ with the canonical projections $\pi_B : M \to B$ and $\pi_F : M \to F$.

Given a smooth map $\omega : B \to \mathbb{R}^+$, we can define a Riemannian metric $g = g_\omega$ on M (called *warped metric*)

$$g_\omega := \pi_B^* g_B + (\omega \circ \pi)^2 \pi_F^* g_F .$$

The pair (M, g) is also denoted by $M = B \times_\omega F$ and it is called a *warped product*. We shall denote the scalar product $g_\omega(X, Y)$ as $\langle X, Y\rangle$, while g_B and g_F will be explicitly written.

The *fibers* $\pi_B^{-1}(p) = \{p\} \times F$ and the *leaves* $\pi_F^{-1}(q) = B \times \{q\}$ are Riemannian submanifolds of M.

We recall the formulae of the Levi-Civita connection ∇ and the Riemannian tensor R (see [8]). Let X, Y, Z be sections in $\Gamma(\pi^* TB)$ and U, V, W in $\Gamma(\sigma^* TF)$. The Levi-Civita connection is given by

$$\begin{aligned}
\nabla_X Y &= \nabla^B_X Y\\
\nabla_X V &= \nabla_V X = \frac{\langle X, \Omega\rangle}{\omega} V\\
\nabla_V W &= \nabla^F_V W - \frac{\langle V, W\rangle}{\omega}\Omega\,.
\end{aligned}$$

Via a direct computation, we have the Riemannian curvature tensor R,

$$R_{XY}Z = R^B_{XY}Z; \qquad R_{VX}Y = \frac{H^\omega(X,Y)}{\omega}V; \qquad R_{XY}V = R_{VW}X = 0$$

$$R_{XV}W = \frac{\langle V, W\rangle}{\omega}\nabla_X(\Omega); \quad R_{VW}U = R^F_{VW}U - \frac{\|\Omega\|^2}{\omega^2}\{\langle V, U\rangle W - \langle W, U\rangle V\}\,.$$

This section contains our main results on the classification of warped products which are symmetric spaces (briefly *symmetric warped products*): this relies on the previous study of codimension one Lie Triple Systems as it will appear in due course.

The first step is to compute the tensor ∇R on M: since the computations are wearisome but straightforward, we just list the results in the following proposition.

Beside the standard tensors we use the notation R^0 to denote the tensor $R^0_{XY}Z = g(X, Z)Y - g(Y, Z)X$, and obviously R^{F0} is the analogous constructed with g_F.

Proposition 4.1. *Let X, Y, Z, T be in TB, and U, V, W, S in TF.*

Then the expressions of ∇R *are given by*

(1) $(\nabla_X R)_{YZ}T = (\nabla^B_X R^B)_{YZ}T$

(2) $(\nabla_X R)_{VY}Z = \nabla_X\Big(\dfrac{H}{\omega}\Big)(Y,Z)V$

(3) $(\nabla_X R)_{YZ}V = (\nabla_X R)_{VW}Y = 0$

(4) $(\nabla_X R)_{YU}W = \dfrac{\langle V,W\rangle}{\omega}\Big(\nabla_X\nabla_Y\Omega - \nabla_{\nabla_X Y}\Omega - \dfrac{X(\omega)}{\omega}\nabla_Y\Omega\Big)$

(5) $(\nabla_X R)_{VW}U = -2\dfrac{X(\omega)}{\omega}R_{VW}U - 2H(X,\Omega)R^{F0}_{VW}U$

(6) $(\nabla_V R)_{XY}Z = \Big(\dfrac{R_{XYZ\Omega}}{\omega} - \dfrac{X(\omega)H(Y,Z) - Y(\omega)H(X,Z)}{\omega^2}\Big)V$

(7) $(\nabla_V R)_{UY}Z = \dfrac{\langle U,V\rangle}{\omega}\Big\{R^B_{\Omega Y}Z + \dfrac{Z(\omega)}{\omega}\nabla_Y\Omega - \dfrac{H(Y,Z)}{\omega}\Omega\Big\}$

(8) $(\nabla_V R)_{YZ}U = \dfrac{\langle U,V\rangle}{\omega}\Big\{R^B_{YZ}\Omega + \dfrac{Y(\omega)}{\omega}\nabla_Z\Omega - \dfrac{Z(\omega)}{\omega}\nabla_Y\Omega\Big\}$

(9) $(\nabla_V R)_{UW}Y = \dfrac{H(\Omega,Y)}{\omega^2}R^0_{WU}V + \dfrac{Y(\omega)}{\omega}R_{UW}V$

(10) $(\nabla_V R)_{YU}W = \dfrac{\langle U,W\rangle H(Y,\Omega)}{\omega^2}V - \dfrac{\langle V,W\rangle H(Y,\Omega)}{\omega^2}U - \dfrac{Y(\omega)}{\omega}R_{VU}W$

(11) $(\nabla_V R)_{UW}S = (\nabla^F_V R^F)_{UW}S - \dfrac{R_{UWSV}}{\omega}\Omega - \dfrac{R^0_{UWSV}}{\omega^2}\nabla_\Omega\Omega$

Imposing the symmetry, namely setting $\nabla R = 0$, the above equations are redundant since — e.g. — (4) implies (6) which is equivalent to (8):

1. $\nabla^B R^B = 0$;
2. $\nabla_X\Big(\dfrac{H}{\omega}\Big) = 0$;
3. $\nabla_X\nabla_Y\Omega = \nabla_{\nabla_X Y}\Omega + \dfrac{X(\omega)}{\omega}\nabla_Y\Omega$;
4. $\dfrac{\langle X,\Omega\rangle}{\omega}R_{VW}U + \dfrac{H(X,\Omega)}{\omega^2}R^0_{VW}U = 0$;
5. $R_{\Omega Y}Z + \dfrac{Z(\omega)}{\omega}\nabla_Y\Omega = \dfrac{H(Y,Z)}{\omega}\Omega$;
6. $\nabla^F R^F = 0$.

All the above equations are equivalent to the following

PROPOSITION 4.2. *The warped product M is locally symmetric if and only if*

1. *B is locally symmetric* ($: \nabla^B R^B = 0$)*;*
2. *The fiber F has constant sectional curvature $K^F = \mathfrak{h}$;*
3. *We have*

$$\nabla_X \Big(\frac{H}{\omega} \Big) = 0\,; \tag{12}$$

4. *There exists a constant $\varkappa \in \mathbb{R}$ such that*

$$\varkappa \omega^2 + \|\Omega\|^2 = \mathfrak{h}\,. \tag{13}$$

We must investigate if and under which assumptions there exists a suitable warping function ω satisfying the above overdetermined system (12)+(13).

Let $\mathfrak{q}$ be a regular value for ω (it suffices that $\mathfrak{h} - \varkappa \mathfrak{q}^2 \neq 0$) and let $\Sigma = \Sigma_{\mathfrak{q}} := \omega^{-1}(\mathfrak{q})$. Then it follows from equation (13) that ω is an appropriate (hyperbolic) trigonometric function of the geodesic distance from Σ according to the signs of $\mathfrak{h}$ and $\varkappa$.

Moreover we derive the necessary and sufficient properties for such a Σ to be a level surface of ω using equation (12); this way we will have a classification of all solutions of the joint system (12)+(13).

Specifically we will prove as a consequence of the tensor equation (12), that Σ must be locally symmetric and hence it will be enough to classify all codimension one embeddings of a locally symmetric space into another one.

In the sequel we will need the second fundamental form S of Σ; as a notational remark, since we are dealing only with the local geometry of B, we will omit the superscript ^{B} in this paragraph.

Now, for $X, Y \in T\Sigma$ we find (notice that $(T\Sigma)^\perp = \mathbb{R}\Omega$)

$$\begin{aligned} S_{XY} :&= \nabla_X Y^\perp = \langle \nabla_X Y, \Omega \rangle \frac{\Omega}{\|\Omega\|^2} = (X\langle Y, \Omega\rangle - H^\omega(X,Y)) \frac{\Omega}{\|\Omega\|^2} = \\ &= -H^\omega(X,Y) \frac{\Omega}{\|\Omega\|^2}\,. \end{aligned}$$

hence we compute the curvature of Σ by means of Gauss–Codazzi formula in correspondence with generic vectors X, Y, Z, W of $T\Sigma$

$$-R_{XYZW} = -R^{\Sigma}_{XYZW} + \langle S_{XZ}, S_{YW}\rangle - \langle S_{YZ}, S_{XW}\rangle\,.$$

Proposition 4.3. *Let (B, g_B) be a (locally) symmetric space and $\omega : B \to \mathbb{R}$ a smooth function satisfying the system*

$$\nabla\Big(\frac{H}{\omega}\Big) = 0; \qquad \varkappa\omega^2 + \|\Omega\|^2 = \mathfrak{h} \qquad \text{for some } \mathfrak{h}, \varkappa \in \mathbb{R}\,.$$

Let $\mathfrak{q}$ be a fixed regular value of ω and $\Sigma := \omega^{-1}(\mathfrak{q})$: then the intrinsic curvature R^{Σ} of Σ is parallel, namely Σ is a locally symmetric space.

Proof. Let, in this proof, X, Y, Z, T, W be vectors in $T\Sigma$ (hence $\langle\Omega, X\rangle = \langle\Omega, Y\rangle = \ldots = 0$). We first notice that

$$0 = \nabla_T\Big(\frac{H}{\omega}\Big) = \frac{\nabla_T H}{\omega} - \overbrace{\langle\Omega, T\rangle}^{=0}\frac{H}{\omega^2} = \frac{\nabla_T H}{\omega}$$

then $\nabla_T H = 0$.

From Gauss-Codazzi formula we have

$$\begin{aligned}
S_{XY} &= -\frac{H(X,Y)}{\|\Omega\|^2}\Omega\,,\\
\nabla^{\Sigma}_X Y &= \nabla_X Y - S_{XY} = \nabla_X Y + \frac{H(X,Y)}{\|\Omega\|^2}\Omega\,,\\
R^{\Sigma}_{XYZW} &= R_{XYZW} + \langle S_{XZ}, S_{YW}\rangle - \langle S_{YZ}, S_{XW}\rangle = \\
&= R_{XYZW} + \frac{1}{2\|\Omega\|^2} H \owedge H_{XYZW}\,,
\end{aligned}$$

where we have used the *Kulkarni-Nomizu* symbol $\owedge$ (see, e.g., [1]). A quite long but straightforward computation now shows that the derivatives of R^{Σ} are all proportional to $R_{\Omega YZW}$: recalling formula (7) we have

$$R_{\Omega YZW} = \langle R_{\Omega Y}Z, W\rangle = \Big\langle\frac{H(Y,Z)}{\omega}\Omega + \frac{\langle\Omega, Z\rangle}{\omega}\nabla_Y\Omega, W\Big\rangle = 0\,. \qquad \square$$

REMARK 4.4. It is not really essential to assume the explicit form of $\varkappa\omega^2 + \|\Omega\|^2 = \mathfrak{h}$ in the previous proposition. It is easy to realize that the proposition would hold true and the proof would go through in exactly the same way if we had to substitute to $\varkappa\omega^2 + \|\Omega\|^2 = \mathfrak{h}$ any other functional relation between the norm of the gradient and the value of ω, e.g. any smooth functional relation like $f(\omega, \|\Omega\|) = 0$.

The problem arises now as to whether a locally symmetric space B allows the existence of an embedded locally symmetric space Σ of codimension one; in order to answer to this problem, let us fix a point $p_0 \in B = G/K$ and identify as usual the tangent space at that point with the Lie algebra $\mathfrak{g}$ of G; we have

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}\,.$$

Let W denote the vector $\Omega(p_0) \in T_{p_0}B \simeq \mathfrak{m}$, and $\langle|\rangle$ the metric at the point p_0.

From the general theory of the symmetric spaces it follows that $\langle[X,A]|B\rangle = \langle A|[B,X]\rangle$ for any two vectors $A, B \in \mathfrak{m}$ and $X \in \mathfrak{k}$. We first prove the

LEMMA 4.5. *The group G^Σ of isometries of Σ lifts uniquely to local isometries of B in a neighborhood of Σ, under the request of preserving the level surfaces of ω.*

PROOF. We give only a sketch of the proof leaving the details to the reader.

Let $U = \omega^{-1}((q-\epsilon, q+\epsilon)) \subset M$ and let φ be an isometry of Σ. Now, all level surfaces of ω in U are diffeomorphic to Σ by means of a retraction $\mathfrak{r}$ of U onto Σ defined as follows: for any point $p \in U$ the point $\mathfrak{r}(p) \in \Sigma$ is the intersection with Σ of the geodesic through p generated by the gradient Ω. It is not difficult to show that using this diffeomorphism we can extend φ to all level surfaces of ω in U and that the result is a local isometry. One can also check that this lift is unique and that defines a group homomorphism $G^\Sigma \to G$. ∎

It follows from the proof that an alternative and equivalent definition of G^Σ is the following

$$G^\Sigma := \{g \in G \ s.t. \ \omega \circ \mathfrak{g} = \omega\}\,.$$

It is also clear that the maximal open neighborhood of Σ in the statement of Lemma 4.5 is $\omega^{-1}(I)$ where I is the (open) interval of regular values of ω.

The lift defined in Lemma 4.5 allows us to inject the isotropy subgroup $K^\Sigma \subset G^\Sigma$ of $p_0 \in \Sigma$ into the isotropy subgroup $K \subset G$ of $p_0 \in B$. It follows that K^Σ (when injected in K) must preserve $\Omega(p_0)$; in terms of the Lie algebra picture this means that

$$\mathfrak{k}^\Sigma \subseteq \{X \in K \ s.t. \ [X, \Omega] = 0\}$$

namely it is a subalgebra of the centralizer of Ω in $\mathfrak{k}$.

Now the tangent space $T_{p_0}\Sigma \subset T_{p_0}B$ injects naturally in $\mathfrak{m}$ as the orthogonal to Ω. We thus have the following relations

$$\begin{aligned}
\sigma &:= \{A \in \mathfrak{m} \ s.t. \ <\Omega|A>= 0\} = \{\Omega\}^\perp\,;\\
\mathfrak{k}^\Sigma &\subseteq Z_{\mathfrak{k}}(\Omega)\,;\\
[\sigma, \sigma] &\subset \mathfrak{k}^\Sigma\,;\\
[[\sigma, \sigma], \sigma] &\subset [\mathfrak{k}^\Sigma, \sigma] \subset \sigma
\end{aligned}$$

where actually the relation $[\mathfrak{k}^\Sigma, \sigma] \subset \sigma$ is trivially satisfied because it would have held true even if we had substituted the centralizer $Z_{\mathfrak{k}}(\Omega)$: in fact $\forall\, Z \in Z_{\mathfrak{k}}(\Omega)$, $\forall\, A \in \sigma$

$$\langle [Z, A]|\Omega\rangle = \langle A|[\Omega, Z]\rangle = \langle A|0\rangle = 0\,.$$

It follows that σ is a Lie Triple System of codimension one in $\mathfrak{m}$, therefore a necessary condition for our function ω to exist is that $\exists\, \Omega \in \mathfrak{m}$ s.t. $\sigma := \{\Omega\}^\perp \subset \mathfrak{m}$ is a Lie Triple System.

As it was proven in Lemmas 3.3 and 3.4, a necessary and sufficient condition for the existence of a codimension one Lie Triple System is that B has an irreducible component with constant sectional curvature, namely $B = B_1 \times B_2 \times \ldots$, and $K^{B_1} = \varkappa =$ constant.

Since Ω must belong to TB_1, then ω is actually a function only of B_1 and constant on the other factors.

A direct computation shows that the Hessian of ω is proportional to the metric as we now prove.

LEMMA 4.6. *For any $\xi, \eta \in \sigma \simeq T_{p_0}\Sigma$ we have $H^\omega(\xi,\eta) \propto g(\xi,\eta)$. The constant of proportionality depends only on the level surface, namely on the value of ω.*

PROOF. Let $\xi = \pi^- X$, $\eta = \pi^- Y$; then, from $R_{\Omega\xi}\eta + \frac{\eta(\omega)}{\omega}\nabla_\xi\Omega = \frac{H(\xi,\eta)}{\omega}\Omega$, $H^\omega(\xi,\eta) \propto \langle \Omega, R_{\xi\Omega}\eta\rangle$ and hence

$$H^\omega(\xi,\eta) \propto \langle \Omega, [[\xi,\Omega],\eta]\rangle \propto \langle \Omega, [\pi^+ X, \eta]\rangle = \langle [\Omega, \pi^+ X], \eta\rangle \propto \langle \xi, \eta\rangle\,,$$

where all constants of proportionality depend only on the point p_o via a function $\rho(p_0)$; but now equation (12) implies that $H^\omega_{|T\Sigma} = \rho(p_0)g_\Sigma$ is covariantly constant on Σ and hence ρ is actually a constant depending only on the level surface $\Sigma = \omega^{-1}(q)$ and hence on the value of ω. $\square$

To find the explicit dependence we must use equation (13); in fact we have

$$H^\omega = \rho(\omega) g_{B_1}\,,$$

and taking one covariant derivative along Ω we find (recall that $0 = \nabla(\frac{H^\omega}{\omega}) = \frac{1}{\omega}\nabla H^\omega - \frac{1}{\omega^2} d\omega \otimes H^\omega$)

$$\nabla_\Omega H^\omega = \frac{\|\Omega\|^2}{\omega} H^\omega = \frac{\rho(\omega)\|\Omega\|^2}{\omega} g_{B_1} = \rho'(\omega)\|\Omega\|^2 g_{B_1}\,.$$

Thus the function ρ must satisfy

$$\frac{\rho'}{\rho} = \frac{1}{\omega}$$

and hence $\rho = C\omega$. To fix the constant we use again equation (13) in the derivative form, namely

$$C\omega\langle \Omega, X\rangle = H^\omega(\Omega, X) = -\varkappa\omega\langle \Omega, X\rangle \ \Rightarrow C = -\varkappa\,.$$

In conclusion we summarize the results for the case in which the manifold B is irreducible in the

THEOREM 4.7. *Let $M = B \times_\omega F$ be a locally symmetric space such that ω is non-constant and B is irreducible. Then*

i) *M has constant curvature $K^M = \varkappa$;*

ii) *(B, g) is locally isometric to a warped product $I \times_\alpha \Sigma_{\mathfrak{q}}$ where $\Sigma_{\mathfrak{q}} := \omega^{-1}(\mathfrak{q})$ for a regular value $\mathfrak{q}$, $\alpha(t)$ satisfies*

$$\begin{cases} (\alpha'(t))^2 = \dfrac{\mathfrak{h}\varkappa}{\mathfrak{h} - \varkappa\mathfrak{q}^2} - \varkappa\alpha^2(t) \\ \alpha(0) = 1 \end{cases}$$

and $0 \in I \subseteq \mathbb{R}$;

iii) *$\Sigma_{\mathfrak{q}}$ has constant curvature $K^{\Sigma_{\mathfrak{q}}} = \frac{\mathfrak{h}\varkappa}{\mathfrak{h} - \varkappa\mathfrak{q}^2}$.*

COROLLARY 4.8. *A generic symmetric warped product $M = B \times_\omega F$ is (locally) the product of a symmetric space $B_2 \times \ldots \times B_N$ and of a warped product with constant curvature $M_1 := B_1 \times_\omega F$.*

In conclusion we investigate also the case in which $\mathrm{Isom}(B)$ has an Euclidean part: in the above discussion we always assumed that the group of isometries of B was semisimple, thus excluding the Euclidean case; we now consider the general case in which the Lie algebra of isometries is a generic *effective orthogonal symmetric Lie algebra* $(\mathfrak{g}, \vartheta)$ (see [5], Ch. V). We can always split it into three ideals $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_+ + \mathfrak{g}_-$ of the type, respectively Euclidean, compact and non-compact.

Let us denote by π_0, π_+, π_- the respective projections (which are Lie algebra homomorphisms). We can state the

PROPOSITION 4.9. *Let Ω be the gradient of ω at some point, seen in the algebra and let $\{\Omega\}^\perp =: \sigma \subset \mathfrak{m}_0 \overset{\perp}{\oplus} \mathfrak{m}_+ \overset{\perp}{\oplus} \mathfrak{m}_-$ be a Lie Triple System. Then Ω belongs either to the Euclidean part $\mathfrak{m}_0$ or to one irreducible component in the semisimple part $\mathfrak{m}_+ \oplus \mathfrak{m}_-$.*

PROOF. Reasoning as in Theorem 3.7 we can conclude that Ω belongs to $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_1$ is an irreducible component; this implies that the function ω depends only on the points in the (locally) direct product $B_0 \times B_1$ where B_0 is an Euclidean affine space (flat). In order to conclude that ω depends only on the point on B_0 or B_1 we use the tensor equation $\nabla(H^\omega/\omega) =: \nabla(L) = 0$. Take a global flat coordinate system on B_0 and denote by ∂_i the corresponding coordinates in the tangent space to $B_0 \times B_1$; then $\nabla_{\partial_i}(L) = 0 = \partial_{\partial_i} L$ says that the tensor L does not depend on the coordinates of B_0; on the other hand, for a vector X tangent to B_1 the identity

$$0 = (\nabla_X L)(\partial_i, \partial_j) = X(L(\partial_i, \partial_j))$$

implies that the matrix $K_{ij} := L(\partial_i, \partial_j) = K_{ji}$ is constant. This means that

$$\partial_i \partial_j \omega = \omega\, K_{ij}\,.$$

Assuming that the matrix K is non-vanishing, the compatibility of this equation imposes $\partial_k \omega\, K_{ij} = \partial_i \omega\, K_{kj}$ which shows that the solution is of the form

$$\omega(x_0, x_1) := C(x_1)\, \cos(v(x_1) \cdot x_0 + \Phi(x_1))\,, \tag{14}$$

where x_1 stands for the point in B_1 while x_0 stands for the Euclidean coordinates of B_0 (and $\cdot$ is the Euclidean scalar product); moreover $K = v \circ v$; since now we saw that K does not depend on B_1 then $v = v^i \partial_i$ is actually a constant vector.

By now $\omega(x_0, x_1) = C(x_1) \cos(v \cdot x_0 + \Phi(x_1))$. To complete the proof, we compute

$$L(X, \partial_i) = \frac{X(C)}{C} \tan(v \cdot x_0 + \Phi) v^i + X(\Phi) v^i$$

that is covariantly constant iff C is constant on B_1 as well as Φ: this proves that ω depends only on the point on B_0. Clearly if at the beginning $K_{ij} \equiv 0$ one similarly proves that ω can depend only on the point of B_1. □

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Indirizzo degli Autori:

Marco Bertola – e-mail: bertola@sissa.it

Daniele Gouthier – e-mail: gouthier@sissa.it.

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