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Transitive groups of quasi-isometries and growth

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RIASSUNTO: *Se G è un gruppo di Lie transitivo di quasi-isometrie di una varietà riemanniana (M, g), il tipo di crescita di G è uguale alla crescita riemanniana di (M, g).*

ABSTRACT: *If G is a transitive Lie group of quasi-isometries of a Riemannian manifold (M, g), the growth-type of G is equal to the Riemannian growth of (M, g).*

– Introduction

In the paper we describe the structure of a Riemannian manifold (M, g) which admits a transitive group G of (uniform) quasi-isometries and prove that the growth of G (in the sense of Y. GUIVARCH [5]) coincides with the growth of the Riemannian ball of (M, g). Some consequences of this result are derived.

The group of (uniform) quasi-isometries of a Riemannian manifold (M, g) is defined as a group G of transformations of M such that

$$\lambda_- g \leq \varphi^* g \leq \lambda_+ g \quad \forall \varphi \in G\,,$$

where $\lambda_\pm$ are some positive constants.

This notion is stronger then the notion of a group of (uniform) quasi-conformal transformations, defined by the condition that the ratio be-

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tween maximal and minimal axis of the ellipsoid, which is the image of the unit sphere $S \subset T_x M$, $x \in M$, under the action of a transformation $\varphi \in G$ is uniformally bounded (see M. Gromov [4]).

In the papers [3] and [11] it is proven that, under some conditions, a transitive group G of quasi-conformal transformations of a domain $D \subset \mathbb{R}^n$ is conjugated to a group of conformal transformations. This result is similar to our Theorem 2. It seems that the problem of characterization of all Riemannian manifolds (or even domains in $\mathbb{R}^n$) which admit transitive group of quasi-conformal transformations is still open.
More precisely, the main result of the paper may be stated as following:

Theorem 1. *Let G be a connected transitive Lie group of (uniform) quasi-isometries of a Riemannian manifold (M, g). Then the growth of the group G is equal to the growth of the Riemannian ball of M.*

See Section 1 for basic concepts.

This theorem generalize the following result [1]:

Theorem (Alekseevsky-Kimelfeld). *Let G be a connected transitive group of isometries of a Riemannian manifold (M, g). Then the growth of G is equal to the growth of the Riemannian ball of M.*

The proof of the Theorem 1 is based on a characterization of linear groups of (uniform) quasi-isometries of an Euclidean vector space as relatively compact groups (Lemma 2) and on the following description of quasi-invariant metrics in a homogeneous manifold $M = G/H$.

Theorem 2. *Let G be a transitive group of (uniform) quasi-isometries of a Riemannian manifold (M, g). Then G preserves a Riemannian metric g^0 which is quasi-isometric to g. In particular, the isotropy representation $j : G_x \to GL(T_x M)$ of the stabilizer G_x of a point x is exact.*

The following corollary shows that we may assume that the group of (uniform) quasi-isometries is a Lie group.

Corollary 1. *A transitive group of (uniform) quasi-isometries of a Riemannian manifold (M, g) is a subgroup of a Lie group $\tilde{G}$ of (uniform) quasi-isometries of (M, g).*

Indeed, it is a subgroup of the group $\tilde{G} = Isom(M, g^0)$ of all isometries of (M, g^0), which is a Lie group, by theorem of Myers-Steenrod [2].

Now we state some corollaries of these results.

Recall that a connected Lie group G is called a group of type R (respectively, $\bar{R}$) if all eigenvalues of the adjoint operators on the Lie algebra $\mathcal{G}$= Lie G have unit modulus (respectively, there exist an adjoint operator Ad_x, $x \in G$ with eigenvalue λ such that $|\lambda| \neq 1$).

Corollary 2. *Let (M, g) be a Riemannian manifold with non negative Ricci curvature (resp., non positive not identically zero sectional curvature). Then a transitive Lie group G of (uniform) quasi-isometries of (M, g) has type R (resp., $\bar{R}$).*

Corollary 3. *Let (M, g_0) be a Riemannian manifold which admits a transitive Lie group G of (uniform) quasi-isometries. If G has type R (resp. $\bar{R}$) then there is no G-quasi-invariant Riemannian metric with non positive not identically zero sectional curvature (resp. with non negative Ricci curvature).*

Corollary 4. *Let (M, g) be a compact Riemannian manifold. Assume that its universal covering $(\tilde{M}, \tilde{g})$ admits a connected transitive group G of (uniform) quasi-isometries. Then the growth of the fundamental group $\pi_1(\tilde{M})$ is finite if and only if G has type R.*

1 – Basic concepts

We refer to the Alekseevsky-Kimelfeld work [1]. We recall that the *growth* of a function $f(t)$ of a real or natural argument is defined as

$$r(f) = r_{(t)}(f(t)) = \overline{\lim_{t \to +\infty}} \frac{\log f(t)}{\log t} ;$$

obviously, $r_{(t)}(t^a) = a$, $r_{(t)}(e^{at}) = \infty \cdot \operatorname{sgn} a)$.

Moreover, the *growth of a Riemannian manifold* (M, g) is defined as $r(M) = r(M, g) = r_{(t)}(\mathrm{vol}B_p(t))$, where $\mathrm{vol}B_p(t)$ is the Riemannian volume of the (closed) geodesic ball $B_p(t)$ of radius t with center at $p \in M$.

It follows readily from the triangle inequality that $r(M)$ is independent of the choice of the point $p \in M$.

Now, let G be a locally compact group with the left invariant Haar measure μ, generated by its relatively compact neighborhood of unity U(i.e. $\bigcup_{n=1}^{\infty} U^n = G$).

Such groups is called *compactly generated* groups. The *growth of the group* G is defined as

$$r_e(G) = r_{(n)}(\mu(U^n)) .$$

It is known (see [5]) that $r_e(G)$ is independent of a choice of the Haar measure μ or of the neighborhood U.

Moreover, we recall that a diffeomorphism $\varphi \in \mathrm{Diff}(M)$ is a *quasi-isometry* of a Riemannian metric g on M if there exist $\lambda_- \leq \lambda_+$ positive constants such that:

$$\lambda_- g \leq g' =: \varphi^* g \leq \lambda_+ g . \tag{1}$$

The metrics g and g' which satisfy (1) are called to be *quasi-isometric*. We will indicate this as $g \sim g'$. Note that $\sim$ is an equivalence relation.

We will write $g' \geq g$ for symmetric bilinear forms g', g on a manifold M if $g' - g$ is non negatively defined form. Remark that if $g' \geq g$, then we have the following inclusion between geodesic balls with respect to these metrics:

$$B_x^{g'}(r) \subset B_x^g(r),\ x \in M,\ r > 0 ,$$

and (for oriented M) the following relation between volume forms:

$$\mathrm{vol}^{g'} \geq \mathrm{vol}^g .$$

This implies the following lemma.

LEMMA 1. *Let (M, g) be a Riemannian manifold. If a metric g' is quasi-isometric to g, then $r(M, g') = r(M, g)$.*

REMARK. As the referee remarks, the inequality $g' \geq g$ doesn't imply any relation between $r(M, g')$ and $r(M, g)$.

Now, we give the following:

DEFINITION 1. A group G of transformations of a Riemannian manifold (M, g) is called *a group of (uniform) quasi-isometries if* there exist $\lambda_\pm > 0$ such that

$$\lambda_- g \leq \varphi^* g \leq \lambda_+ g \quad \forall \varphi \in G . \tag{2}$$

Note that the constants λ_-, λ_+ *do not depend on* φ.

In the future, we will omit the adjective "uniform" and will speak about group of quasi-isometries, for simplicity.

Definition 2. If G is a group of quasi-isometries of (M, g), we will say that the metric g is *quasi-invariant* with respect to G.

Remarks. 1) Any compact group G of transformations of a compact Riemannian manifold (M, g) is a group of quasi-isometries.

2) There exist *non compact* Lie groups of transformations of a *compact* Riemannian manifold (M, g) which are *not* group of quasi-isometries.

The example is the group of conformal transformations of the standard sphere S^2.

Indeed, let $\varphi : S^2 \backslash \{p\} \to \mathbb{R}^2$ be the stereographic projection s.t. $\varphi(q) = 0$ where q is the antipodal point.

Then a homothety λid of $\mathbb{R}^2$ defines a conformal transformation $h_\lambda = \varphi^{-1} \circ \lambda id \circ \varphi$ of S^2 which preserves the antipodal points p, q and has the differential $h_{\lambda_*}|q = \lambda id$.

This shows that the 1-parameter group $\{h_\lambda\}$ of conformal transformations of S^2 is not a group of quasi-isometries.

3) A non compact Lie group can also acts on a compact Riemannian manifold as a group of quasi-isometries. For example, let $M = T^n$ be a torus equiped with any Riemannian metric g. Then any 1-parameter subgroup $G \cong \mathbb{R}$ of T^n acts on (T^n, g) as a group of quasi-isometries.

4) There exist *compact* groups G of transformations of *a non compact* Riemannian manifold (M, g) which are not group of quasi-isometries.

The example is the group $\mathbb{Z}_2 = \{\pm\}$ of transformations of the real line $M = \mathbb{R}$ equiped with the Riemannian metric $g = e^x dx^2$.

Example. A simple construction of a group of quasi-isometries can be given as follows. Let G be a group of isometries of a Riemannian manifold (M, g). Let g_1 a deformation of the metric g which is equal (or, more generally, quasi-isometric) to g out of some compact domain $K \subset M$. Then G is a group of quasi-isometries of the Riemannian manifold (M, g_1).

Now we consider the case of *transitive* groups G of quasi-isometries of (M, g).

Example. Let $(M = G/H, g_0)$ be a homogeneous Riemannian manifold and $f \in C^\infty(M)$ a *positive* function such that $\lambda_- \leq f \leq \lambda_+$ for some positive constants $\lambda_\pm$. Then G is a transitive group of quasi-isometries of $(M = G/H, g = fg_0)$.

2 – Proofs of Theorems 1, 2 and the corollaries

2.1 – Linear groups of quasi-isometries of an Euclidean vector space (V, g_0).

We need the following characterization of linear groups of quasi-isometries of the Euclidean vector space.

LEMMA 2. *A group $G \subset GL(V)$ of linear transformations of the Euclidean vector space (V, g_0) is a group of quasi-isometries if and only if it is relatively compact.*

PROOF. It was remarked by the referee that the result follows from the fact that a subset G of the space $\mathrm{End}(V)$ of endomorphisms of V is relatively compact if and only if it is bounded with respect of the norm

$$\|A\| = \max_{x \in V \setminus \{0\}} \frac{g_0(Ax, Ax)^{1/2}}{g_0(x,x)^{1/2}}, \ A \in \mathrm{End} V\,,$$

that is if

$$\|A\| \leq \lambda_+ \quad \forall A \in G$$

for some $\lambda_+ > 0$.

Indeed, if G is a group the last condition can be rewritten as

$$\lambda_+^{-2} g_0(x,x) \leq g_0(Ax, Ax) = (A^* g_0)(x,x) \leq \lambda_+^2 g_0(x,x)$$

$\forall x \in V \quad \forall A \in G.$

This inequality means that G is a group of quasi-isometries of (V, g_0). □

2.2 – Proof of Theorem 2

To prove Theorem 2 we need the following lemma:

LEMMA 3. *Any two quasi-invariant metrics g, g' on a homogeneous manifold $M = G/H$ are quasi-isometric.*

PROOF. By assumption, there exist positive constants $\lambda_\pm$, $\lambda'_\pm$ such that, for any $\varphi \in G$

$$\lambda_- g \leq \varphi^* g \leq \lambda_+ g \tag{5}$$

$$\lambda'_- g' \leq \varphi^* g' \leq \lambda'_+ g' . \tag{6}$$

We may choose also positive constants $\mu_\pm$ such that

$$\mu_- g_p \leq g'_p \leq \mu_+ g_p . \tag{7}$$

Applying a transformation $\varphi \in G$ to inequality (7) we get

$$\mu_- (\varphi^* g)_{\varphi^{-1}(p)} \leq (\varphi^* g')_{\varphi^{-1}(p)} \leq \mu_+ (\varphi^* g)_{\varphi^{-1}(p)} . \tag{8}$$

Combining (5), (6), (8), we can write

$$\lambda_- \mu_- g_{\varphi^{-1}(p)} \leq \lambda'_+ g'_{\varphi^{-1}(p)}$$
$$\lambda'_- g'_{\varphi^{-1}(p)} \leq \lambda_+ \mu_+ g_{\varphi^{-1}(p)} .$$

This implies

$$\frac{\lambda_- \mu_-}{\lambda'_+} g_{\varphi^{-1}(p)} \leq g'_{\varphi^{-1}(p)} \leq \frac{\lambda_+ \mu_+}{\lambda'_-} g_{\varphi^{-1}(p)} .$$

This shows that g and g' are quasi-isometric. ∎

Now we prove Theorem 2.

The isotropy group $j(G_x)$ is a linear group of quasi-isometries of the Euclidean vector space $(V = T_x M, g_{|x})$. Then, by Lemma 2, it is relatively compact. Hence, there exists a $j(G_x)$-invariant Euclidean metric g^0_x on V. It can be extended to a G-invariant metric g^0 on M. This implies that the isotropy representation is exact. The metrics g, g^0 are quasi-invariant with respect to G.

Hence, by Lemma 3, they are quasi-isometric. ∎

2.3 – Proof of the Theorem 1

The hypothesis of Theorem 1 implies, by the Theorem 2, that there exists a G-invariant metric g^0 which is quasi-isometric to g.

Then, by Lemma 1, $r(M, g) = r(M, g^0)$.

Finally, by theorem of Alekseevsky-Kimelfeld:

$$r_e(G) = r(M, g^0) .$$

Combining these equalities, we have

$$r_e(G) = r(M, g)\,.$$

This finish the proof of the Theorem 1. ∎

2.4 – Proof of the corollaries

The corollaries 2 and 3 follows from our Theorem 1 and the following theorems by Guivarch, Günter and Schwarz-Milnor.

THEOREM (Guivarch [5]). *A connected Lie group G has finite growth if and only if it is group of type R.*

THEOREM (Günter [6]). *Let (M, g) be a Riemannian manifold and $(\tilde{M}, \tilde{g})$ its Riemannian universal covering.*

a) *If the sectional curvature of M is non positive and not identically zero, then $r(\tilde{M}) = \infty$.*
b) *If the Ricci curvature of M is non negative, then $r(M) \le r(\tilde{M}) \le \dim M$.*

Corollary 4 follows from Theorem 1, theorem of Guivarch and the following theorem proved by A.S. Schwarz and independently by Milnor:

THEOREM (Schwarz [10], Milnor [9]). *Let $\tilde{M}$ be a universal covering of the compact Riemannian manifold (M, g) and $\pi_1(M)$ its fundamental group (with discrete topology).*

Then

$$r(\tilde{M}) = r_e(\pi_1(M))\,.$$ ∎

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