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Intersection results and fixed point theorems in H-spaces

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RIASSUNTO: *In questo lavoro si ottiene un teorema di raccordamento del tipo Fan negli H-spazi. Sono date applicazioni riguardanti il punto fisso e disuguaglianze minimax.*

ABSTRACT: *In this paper we obtain a Fan's matching theorem in H-spaces. Applications concerning intersection results, fixed point theorems, minimax inequalities are given.*

1 – Introduction

In [12] FAN obtained a matching theorem as a consequence of the Fan-Glicksberg-Kakutani fixed point theorem. Fan's result is a versatile toole in nonlinear functional analysis and was used in [1], [2], [5], [12], [24] in order to prove fixed point theorems, selection theorems and minimax results.

Motivated by HORVATH's papers [13], [14], BARDARO and CEPPITELLI [3], [4] introduced the notion of H-space and proved some $H-KKM$ theorems. In this framework of H-spaces we extend, in the next

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section, Fan's matching theorem. The remaining sections are devoted to applications of this H-matching theorem.

A set-valued map (simply called map) will be denoted by capital letters $T : X \to Y$ according to Berge's notation, while small letters will designate univalued functions.

As defined by Bardaro and Ceppitelli [3] an *H-space* is a pair $(X, \{\Gamma_A\})$ consisting of a topological space X and a family $\{\Gamma_A\}$ of nonempty contractible subsets of X, indexed by the finite subsets of X, such that $A \subset B$ implies $\Gamma_A \subset \Gamma_B$. A set $Y \subset X$ is called *H-convex* if $\Gamma_A \subset Y$ for every nonempty finite set $A \subset Y$. Observe that an arbitrary intersection of H-convex sets is H-convex.

If $(X, \{\Gamma_A\})$ is an H-space and D a nonempty subset of X, then:

(i) a family $\{Y_x : x \in D\}$ of subsets of X is called *$H - KKM$ family* if $\Gamma_A \subset \cup\{Y_x : x \in A\}$ for each nonempty finite subset A of D;
(ii) a map $S : D \to X$ is said to be $H - KKM$ if the family of subsets of X $\{S(x) : x \in D\}$ is an $H - KKM$ family.

2 – Intersection results

The "closed" variant of the following $H - KKM$ theorem is Proposition 1 in [14] as well as Lemma 1 in [8].

Theorem 2.1. *Let $(X, \{\Gamma_A\})$ be an H-space, D a nonempty finite subset of X and $\{Y_x : x \in D\}$ an $H - KKM$ family of subsets of X. Assume that the sets Y_x are all open or all closed. Then*

$$\cap\{Y_x : x \in D\} \neq \emptyset\,.$$

The statement remains valid if X is compact, D infinite and all sets Y_x are closed.

Proof. Let $D = \{x_1, x_2, \ldots, x_n\}$ and let $\Delta = \mathrm{conv}\{e_1, e_2, \ldots, e_n\}$ be the standard simplex of dimension $n-1$ ($\{e_1, e_2, \ldots, e_n\}$ being the canonical basis of $\mathbb{R}^n$). By Theorem 1 in [14] there exists a continuous function $f : \Delta \to X$ such that for every nonempty set $A \subset D$ $f(\mathrm{conv}\{e_i :$

$x_i \in A\}) \subset \Gamma_A$. Then the subsets of Δ, $f^{-1}(Y_x)$, $x \in D$, are all open in Δ or all closed in Δ and for every nonempty set $A \subset D$

$$\mathrm{conv}\{e_i : x_i \in A\} \subset f^{-1}(\Gamma_A) \subset \cup\{f^{-1}(Y_{x_i}) : x_i \in A\}\,.$$

Using the "open" variant, respectively the "closed" variant of the KKM principle (see [20] or [25]) we obtain $\cap\{f^{-1}(Y_x) : x \in D\} \neq \emptyset$ and consequently $\cap\{Y_x : x \in D\} \neq \emptyset$.

The final part of the theorem follows immediately using a standard topological argument. □

Theorem 2.1 may be restated in its contraposition form and in terms of the complement Z_x of Y_x, obtaining in this way a generalization of FAN's matching theorem [12, Th 2].

THEOREM 2.2. *Let $(X, \{\Gamma_A\})$ be an H-space, D be a nonempty finite subset of X and $\{Z_x : x \in D\}$ be a family of subsets of X, all closed or all open. If $\cup\{Z_x : x \in D\} = X$, then there exists a nonempty (finite) set $A \subset D$ such that*

$$\Gamma_A \cap \cap\{Z_x : x \in A\} \neq \emptyset\,.$$

The statement remains valid if X is compact, D is infinite and all sets Z_x are open.

THEOREM 2.3. *In an H-space $(X, \{\Gamma_A\})$, let $\{Y_i : 1 \leq i \leq n\}$ be a family of H-convex sets, every $n-1$ having a common point and let $\{Z_i : 1 \leq i \leq n\}$ be a covering of X having all members closed or all open. Then there exists a nonempty set $I \subset \{1, 2, \dots, n\}$ such that*

$$\cap\{Y_j : j \in \bar{I}\} \cap \cap\{Z_i : i \in I\} \neq \emptyset\,,$$

where $\bar{I}$ denotes the complement of I in $\{1, 2, \dots, n\}$.

PROOF. For each $i \in \{1, 2, \dots, n\}$ select a single point x_i in the intersection $\cap_{j \neq i} Y_j$ and let D be the set of all selected points. By Theorem 2.2 there exists a nonempty set $A \subset D$ such that $\Gamma_A \cap \cap\{Z_i : x_i \in A\} \neq \emptyset$. Denote by I the set $\{i \in \{1, 2, \dots, n\} : x_i \in A\}$. Since $x_i \in \cap\{Y_j : j \in \bar{I}\}$ for each $i \in I$ and the set $\cap\{Y_j : j \in \bar{I}\}$ is H-convex it follows $\Gamma_A \subset \cap\{Y_j : j \in \bar{I}\}$ and thereby

$$\cap\{Y_j : j \in \bar{I}\} \cap \cap\{Z_i : i \in I\} \neq \emptyset\,. \qquad \square$$

The above theorem generalizes a classical result of KLEE [19] known as BERGE's intersection theorem [6]. This corresponds to the particular case when X is a convex subset of a Hausdorff topological vector space and $Y_i = Z_i$ for each $i \in \{1, 2, \dots, n\}$. Topological versions of Berge and Klee type theorems, under conditions weaker than contractibility, as well as short topological proof of Fan's matching theorem can be found in HORVATH and LASSONDE [16].

3 – Fan-Browder's fixed point theorem in H-spaces

Based on Theorem 2.2 we shall extend to H-spaces a fixed point theorem of FAN-BROWDER [9], [7] and other close results obtained by BALAJ [1], KIM [18], PARK [25].

THEOREM 3.1. *Let $(X, \{\Gamma_A\})$ be an H-space and $T : X \to X$ be a map. Suppose that:*

(a) *$T(x)$ is H-convex for each $x \in X$.*
(b) *There exists a finite set $D \subset X$ such that:*

  (b_1) *$T(x) \cap D \neq \emptyset$ for each $x \in X$;*
  (b_2) *the fibers $T^{-1}(y)$ are either all closed or all open, for $y \in D$.*

Then there exists an element $x_0 \in X$ such that $x_0 \in T(x_0)$.

Proof. From (b_1), $X = \cup\{T^{-1}(y) : y \in D\}$. By Theorem 2.2, there are a nonempty set $A \subset D$ and a point x_0 such that

$$x_0 \in \Gamma_A \cap \cap\{T^{-1}(y) : y \in A\}\,.$$

From $x_0 \in \cap\{T^{-1}(y) : y \in A\}$ it follows that $A \subset T(x_0)$ and by (a) $x_0 \in \Gamma_A \subset T(x_0)$. ∎

The proof of the following corollary and of the others which will appear in the paper use the same argument. For this reason the next corollary will be the only one accompanied by proof.

Corollary 3.1. *Let $(X, \{\Gamma_A\})$ be a compact H-space and $T : X \to X$ be a map satisfying the following conditions:*

(a) *For each $x \in X$, $T(x)$ is nonempty and H-convex.*
(b) *For each $y \in X$, $T^{-1}(y)$ is open.*

Then T has a fixed point.

Proof. Observe that $X = \cup\{T^{-1}(y) : y \in X\}$ so, by compactness there exist $y_1, y_2, \dots, y_n \in X$ such that $X = \cup_{i=1}^{n} T^{-1}(y_i)$. It suffices to set $D = \{y_1, y_2, \dots, y_n\}$. ∎

Using the infinite version of the Knaster-Kuratowski-Mazurkiewicz theorem, Fan proved in [9] a section theorem leading to a proof of Tychonoff's fixed point theorem. An open variant of Fan's section theorem appears in [1]. Both will be extended below to H-spaces.

Theorem 3.2. *Let $(X, \{\Gamma_A\})$ be an H-space and let E be a subset of $X \times X$, having the following properties:*

(a) *$(x, x) \in E$ for all $x \in X$.*
(b) *For each $x \in X$ the set $\{y \in X : (x, y) \notin E\}$ is H-convex (possibly empty).*
(c) *The sets $\{x \in X : (x, y) \in E\}$ are either (c_1) all closed or (c_2) all open, for $y \in X$.*

Then for every nonempty finite set $D \subset X$ there exists an element $x_D \in X$ such that $\{x_D\} \times D \subset E$.

PROOF. Suppose that the assertion of the theorem is false. Then there exists a nonempty finite set $D \subset X$ such that

$$\{x\} \times D \not\subset E \text{ for every } x \in X\,.$$

Define a map $T : X \to X$ by $T(x) = \{y \in X : (x, y) \notin E\}$. Then for each $x \in X$, $T(x)$ is H-convex, $T(x) \cap D \neq \emptyset$, and the fibers $T^{-1}(y) = X \setminus \{x \in X : (x, y) \in E\}$ are either all open (in case (c_1)), or all closed (in case (c_2)).

By Theorem 3.1, there exists an element $x_0 \in X$ such that $x_0 \in T(x_0)$, hence $(x_0, x_0) \notin E$, which contradicts (a). ∎

COROLLARY 3.2. *Let $(X, \{\Gamma_A\})$ be a compact H-space and let E be a subset of $X \times X$ satisfying the conditions* (a), (b) *and* (c_1) *in Theorem* 3.3. *Then there exists an element $x_0 \in X$ such that $\{x_0\} \times X \subset E$.*

In [10] FAN obtained an intersection theorem concerning finite families of compact sets with convex sections. Fan's result was generalized in two directions. On the one hand it was extended by MA [21] to arbitrary families (finite or infinite) of compact sets and on the other, by FAN [12] relaxing the compactness condition. A unified generalization has been given by SHIH and TAN in [26]. A close result was obtained by the author in [1]. Fan's result will be further extended to H-spaces.

Let I be an index set; in the case when I is finite we assume that I contains at least two indices. Let $\{(X_i, \{\Gamma^i_A\})\}_{i\in I}$ be a family of H-spaces and $X = \prod_{i\in I} X_i$. For each $i \in I$ set

$$X^i = \prod_{j\neq i} X_j \text{ (so that } X = X_i \times X^i)$$

and let $p_i : X \to X_i$ and $p^i : X \to X^i$ be the projections. For each $x \in X$ we write $p_i(x) = x_i$ and $p^i(x) = x^i$, hence $x = (x_i, x^i)$ for every $i \in I$. For any subset Y of X let $Y_i = p_i(Y)$. With these notations we have:

THEOREM 3.3. *Let $\{E_i\}_{i\in I}$ be a family of subsets of X satisfying the following conditions:*

(a) *For each $i \in I$ and any $x^i \in X^i$ the section*

$$E_i(x^i) = \{x_i \in X_i : (x_i, x^i) \in E_i\}$$

is H-convex and nonempty.

(b) *There exists a finite set $D \subset X$ such that:*

(b$_1$) $\prod_{i\in I} E_i(x^i) \cap D \neq \emptyset$, *for each $x \in X$;*

(b$_2$) *Either the sections*

$$E_i(y_i) = \{x^i \in X^i : (y_i, x^i) \in E_i\}$$

are all closed or I is finite and the sections $E_i(y_i)$ are all open for each $i \in I$ and $y_i \in D_i$.

Then the intersection $\cap_{i\in I} E_i$ is nonempty.

PROOF. For every nonempty finite subset A of X let $\Gamma_A = \prod_{i\in I} \Gamma^i_{A_i}$; then it's clear that $(X, \{\Gamma_A\})$ is an H-space. Consider the map $T : X \to X$ given by $T(x) = \prod_{i\in I} E_i(x^i)$ for all $x \in X$, where $x^i = p^i(x)$. By (b$_1$), $T(x) \cap D \neq \emptyset$ for each $x \in D$.

Let x be an element of X and A be a nonempty finite subset of $T(x)$. Then, for each $i \in I$ $A_i \subset E_i(x_i)$ and by (a), $\Gamma^i_{A_i} \subset E_i(x^i)$. It follows $\Gamma_A \subset T(x)$, hence $T(x)$ is H-convex.

For a given $y \in D$ consider $T^{-1}(y) = \{x \in X : y \in T(x)\}$. Then x lies in $T^{-1}(y)$ iff $y_i \in E_i(x^i)$, that is $x^i \in E_i(y_i)$ for all $i \in I$. So $T^{-1}(y) = \cap_{i\in I}(X_i \times E_i(y_i))$, and by (b$_2$) it follows that either $T^{-1}(y)$ is closed for each $y \in D$ or $T^{-1}(y)$ is open for each $y \in D$.

By Theorem 3.1 there is an element $z \in X$ such that $z \in T(z)$. By the definition of T, this means that $z_i \in E_i(z^i)$ i.e. $z = (z_i, z^i) \in E_i$ for all $i \in I$. Hence $z \in \cap_{i\in I} E_i$ and the proof is complete. ∎

COROLLARY 3.3. *The conclusion of Theorem* 3.3 *remains valid if I is finite, each X_i is compact and conditions* (b$_1$), (b$_2$) *are replaced by the following one:*

(b′) *For each $i \in I$ and any $x_i \in X_i$ the section $E_i(x_i)$ is open.*

Given an H-space $(X, \{\Gamma_A\})$, a function $g : X \to \mathbb{R}$ will be called *H-quasiconcave* if for each $\lambda \in \mathbb{R}$ the set $\{x \in X : g(x) > \lambda\}$ is H-convex, and *H-quasiconvex* if $-g$ is H-quasiconcave. Similarly to [12], [21], [26]. Theorem 3.3 and Corollary 3.3 can receive an analytic formulation. For instance Corollary 3.3 can be restated as follows:

Corollary 3.4. *Let $\{(X_i, \{\Gamma^i_A\})\}_{i\in I}$ be a finite family of compact H-spaces and $X = \prod_{i\in I} X_i$. Let $\{f_i\}_{i\in I}$ be a family of real-valued functions defined on X and $\{t_i\}_{i\in I}$ be a family of real numbers. Suppose that the following conditions are satisfied:*

(a) *For each $i \in I$ and any $x^i \in X^i$, $f_i(\cdot, x^i)$ is an H-quasiconcave function on X_i and the set $\{x_i \in X_i : f_i(x_i, x^i) > t_i\}$ is nonempty.*
(b) *For each $i \in I$ and any $x_i \in X_i$, $f_i(x_i, \cdot)$ is a lower semicontinuous function on X^i.*

Then there exists a point $z \in X$ such that $f_i(z) > t_i$ for each $i \in I$.

As a consequence of Corollary 3.3 we get Nash's equilibrium theorem [23] in H-spaces.

Corollary 3.5. *Let $\{(X_i, \{\Gamma^i_A\})\}_{i\in I}$ be a finite family of compact H-spaces and for each $i \in I$ let $f_i : X = \prod_{i\in I} X_i \to \mathbb{R}$ be a continuous function such that for any $x^i \in X^i$ $f_i(\cdot, x^i)$ is an H-quasiconcave function on X_i. Then there exists a point $z \in X$ such taht*

$$f_i(z) = \max_{y_i\in X_i} f_i(y_i, z^i) \text{ for each } i \in I$$

Proof. For any $\varepsilon > 0$ and $i \in I$ let

$$E_{\varepsilon,i} = \left\{ x = (x_i, x^i) : f_i(x) > \max_{y_i\in X_i} f_i(y_i, x^i) - \varepsilon \right\}.$$

From the uniform continuity of f_i on X it is easily seen that the function $\max_{y_i\in X_i} f_i(y_i, \cdot)$ is continous on X^i, hence the sections $E_{\varepsilon,i}(y_i) = \{x^i \in X^i : (y_i, x^i) \in E_{\varepsilon,i}\}$ are open for all $y_i \in X_i$. Since f_i is an H-quasiconcave function of x_i on X_i, the sections $E_i(x^i)$ are H-convex and clearly nonempty for all $x^i \in X^i$. By Corollary 3.3 we have $\cap_{i\in I} E_{\varepsilon,i} \neq \emptyset$, for every

$\varepsilon > 0$. Then, in view of compactness of X, there is a point z which is in $\bigcap_{i\in I} \mathrm{cl}E_{\varepsilon,i}$, for every $\varepsilon > 0$. This point satisfies $f_i(z) = \max_{y_i\in X_i} f_i(y_i, z^i)$ for each $i \in I$. □

4 – A fixed point theorem for composed maps and applications

The following result extends Theorem 1 in [2].

THEOREM 4.1. *Let $(X, \{\Gamma_A\})$ be an H-space, $S : X \to X$ be an $H - KKM$ map and $T : X \to X$ be a map. Suppose that there exists a finite set $D \subset X$ such that:*

(a) *$T(x) \cap D \neq \emptyset$ for all $x \in X$.*
(b) *The fibers $T^{-1}(y)$ are either all closed or all open, for $y \in D$.*

Then $S \circ T$ has a fixed point.

PROOF. From (a) $X = \cup\{T^{-1}(y) : y \in D\}$. By Theorem 2.2 applied to the closed (open respectively) covering $\{T^{-1}(y) : y \in D\}$ there exist a nonempty set $A \subset D$ and an element $x_0 \in \Gamma_A \cap \cap\{T^{-1}(y) : y \in A\}$.

Since S is an $H - KKM$ map, $x_0 \in \Gamma_A \subset \cup\{S(y) : y \in A\}$ hence for at least one $y_0 \in A$, $x_0 \in S(y_0)$. By $x_0 \in T^{-1}(y_0)$ it follows $y_0 \in T(x_0)$. Therefore $x_0 \in S(y_0) \subset S(T(x_0))$ and the proof is complete. □

COROLLARY 4.1. *Let $(X, \{\Gamma_A\})$ be a compact H-space, $S : X \to X$ be an $H - KKM$ map, $T : X \to X$ be a map. Suppose that for each $x \in X$, $T(x)$ is a nonempty subset of X and $T^{-1}(x)$ is open. Then $S \circ T$ has a fixed point.*

From Corollary 4.1 we get the following generalization of Theorem 3 in [2] which is in turn a generalization of the known FAN's minimax inequality [11].

THEOREM 4.2. *Let $(X, \{\Gamma_A\})$ be a compact H-space and f be a real-valued function defined on $X \times X$ such that for each fixed $x \in X$, $f(x, \cdot)$ is a lower semicontinuous function on X. If there exists a real-valued function g on $X \times X$ satisfying the following conditions:*

(a) *$f(x, y) \leq g(x, y)$ for all $(x, y) \in X \times X$,*

(b) $g(x,x) \leq 0$ *for all* $x \in X$,

(c) *for each fixed* $y \in X$, *the set* $\{x \in X : g(x,y) > 0\}$ *is* H*-convex* (*or empty*),

then there exists a point $y_0 \in X$ *such that* $f(x,y_0) \leq 0$ *for all* $x \in X$.

Proof. Suppose that the conclusion of the theorem is false, i.e. for each $y \in X$ the set $\{x \in X : f(x,y) > 0\}$ is nonempty. Define the maps $S, T : X \to X$ by

$$S(x) = \{y \in X : g(x,y) \leq 0\} \text{ and } T(y) = \{x \in X : f(x,y) > 0\}, \quad x, y \in X\,.$$

We shall show that S is an $H-KKM$ map. Suppose that there exists a nonempty finite set $D \subset X$ and a point $y \in \Gamma_D \setminus \cup\{S(x) : x \in D\}$. Then $g(x,y) > 0$ for each $x \in D$ and by (c), $\Gamma_D \subset \{x \in X : g(x,y) > 0\}$. Thus $g(y,y) > 0$, in contradiction with (b). By hypothesis, for each $x \in X$, $T^{-1}(x) = \{y \in X : f(x,y) > 0\}$ is open.

Thus, by Corollary 4.1, $S \circ T$ has a fixed point, that is there exist $x_0, y_0 \in X$ such that $x_0 \in T(y_0)$ and $y_0 \in S(x_0)$. These relations and (a) lead to the following contradiction

$$0 < f(x_0,y_0) \leq g(x_0,y_0) \leq 0\,. \qquad \square$$

Remark 4.1. Note that the condition (c) in Theorem 4.2 is implied by the following condition:

(c′) for each fixed $y \in X$, $g(\cdot, y)$ is an H-quasiconcave function on X.

If we put aside the condition (b) and replace the condition (c) by (c′) then the conclusion of Theorem 4.2 can be given by the following minimax inequality:

$$\inf_{y\in X} \sup_{x\in X} f(x,y) \leq \sup_{x\in X} g(x,x)\,.$$

Let X be an arbitrary set and $\mathbb{R}^X$ be the set of all real-valued functions defined on X. A map $S : X \to \mathbb{R}^X$ will be called *monotone* if for all $x, y \in X$, each $u \in S(x)$ and each $v \in S(y)$, $(v-u)(y) \geq (v-u)(x)$. For any H-space $(X, \{\Gamma_A\})$ we denote by $\widetilde{X}$ the set $\{u : X \to \mathbb{R} | u \text{ continuous}$ and $H\text{-quasiconvex}\}$. The next result generalizes under many aspects Theorem 6 in [7].

THEOREM 4.3. *Let $(X, \{\Gamma_A\})$ be an H-space and $S : X \to \mathbb{R}^X$ be a monotone map such that for each $x \in X$, $S(x)$ is a nonempty subset of $\widetilde{X}$. Then there exists an element $y_0 \in X$ such that*

$$\sup_{u \in S(x)} (u(y_0) - u(x)) \leq 0 \textit{ for all } x \in X .$$

PROOF. Let $f, g : X \to \mathbb{R}$ be the functions defined by

$$f(x,y) = \sup_{u \in S(x)} (u(y) - u(x)),\ g(x,y) = \inf_{v \in S(y)} (v(y) - v(x)) .$$

Then, for each fixed $x \in X$ the function $f(\cdot, y)$ is lower semicontinuous as the upper envelope of a family of continuous functions. By monotonicity of S, for each $x, y \in X$, $u \in S(x)$ and $v \in S(y)$ we have $u(y) - u(x) \leq v(y) - v(x)$, whence $f(x,y) \leq g(x,y)$. Clearly $g(x,x) = 0$ for all $x \in X$.
Theorem 4.2 is applicable as soon as we prove that $\{x \in X : g(x,y) > 0\}$ is H-convex for each $y \in X$.

The intersection of a family of H-convex sets as well as the union of a family of H-convex sets totally ordered by inclusion are H-convex (cf. [22], p. 282). Thus the above statement follows from

$$\{x \in X : g(x,y) > 0\} = \bigcup_{\varepsilon > 0} \bigcap_{v \in S(y)} \{x \in X : v(x) < v(y) - \varepsilon\} .$$

By Theorem 4.2 there is a point $y_0 \in X$ such that $f(x, y_0) \leq 0$ i.e.

$$\sup_{u \in S(x)} (u(y_0) - u(x)) \leq 0 \text{ for all } x \in X .$$

The last result is a variant of Theorem 4.2 and it admits a similar proof.

THEOREM 4.4. *Let $(X, \{\Gamma_A\})$ be an H-space and f be a real-valued function defined on $X \times X$ such that for each fixed $x \in X$, $f(x, \cdot)$ is an upper semicontinuous function on X. If there exists a real-valued function g on $X \times X$ satisfying the following conditions:*

(a) *$f(x,y) \leq g(x,y)$ for all $(x,y) \in X \times X$;*

(b) $g(x,x) < 0$ *for all* $x \in X$;
(c) *for each fixed* $y \in X$, *the set* $\{x \in X : g(x,y) \geq 0\}$ *is* H*-convex,*

then for every nonempty finite set $D \subset X$ *there exists a point* $y_D \in X$ *such that* $f(x, y_D) < 0$ *for all* $x \in D$.

5 – Applications in hyperconvex metric spaces

There are a lot of examples of H-spaces (see [13]) and in each of them the results of this paper could be interpreted in specific terms. The referee emphasized the connection between our intersection results and the KKM type theorems in hyperconvex metric spaces obtained by Khamsi in [17]. For this reason we shall make a little discussion of our results in this framework of hyperconvex metric spaces. The basic definitions and terminology are those of Khamsi [17].

Let M be a metric space. For a nonempty bounded subset A of M put

$$\text{co}A = \bigcap \{B : B \text{ is a closed ball in } M \text{ containing } A\}.$$

Let $\mathcal{A}(M) = \{A \subset M : A = \text{co}A\}$; that is $A \in \mathcal{A}(M)$ if and only if A is an intersection of closed balls. In this case we will say A is an *admissible* subset of M.

A subset Y of a metric space M is called *finitely closed* (resp. *open*) if for every finite set $A \subset M$ the set $\text{co}A \cap Y$ is closed (resp. open).

A metric space (H,d) is called *hyperconvex* if for any family of points $\{x_i\}_{i \in I}$ in H and for each family $\{r_i\}_{i \in I}$ of nonnegative reals such that $d(x_i, x_j) \leq r_i + r_j$, we have

$$\bigcap_{i \in I} B(x_i, r_i) \neq \emptyset.$$

Here $B(x,r)$ denotes the closed ball with center $x \in H$ and radius $r > 0$.

It is kown that any hyperconvex space (H,d) is an H-space $(H, \{\Gamma_A\})$ with $\Gamma_A = \text{co}A$ for each nonempty finite set $A \subset H$ (see Horvath [15]). If D is a nonempty set in H, an $H - KKM$ map $S : D \to H$, regarding H as an H-space will be called *KKM map.*
The "closed" version of the following result is Theorem 3 in Khamsi [17].

Theorem 5.1. *Let H be a hyperconvex metric space, X an arbitrary subset of H and $S : X \to H$ a KKM map such that the values $S(x)$ are either all closed or all open for $x \in X$. Then the family $\{S(x) : x \in X\}$ has the finite intersection property.*

Proof. Let D be a nonempty finite subset of X. Then $\{S(x) \cap \mathrm{co} D : x \in D\}$ is in the H-space $(H, \{\Gamma_A\})$ an $H - KKM$ family of subsets of H having all members closed or all open. By Theorem 2.1 we obtain $\bigcap_{x \in D}(S(x) \cap \mathrm{co} D) \neq \emptyset$, hence $\bigcap_{x \in D} S(x) \neq \emptyset$. ∎

Using Theorem 4.2 we shall prove the following fixed point theorem which is a slight improvement of Theorem 6 in [17].

Theorem 5.2. *Let (H, d) be a hyperconvex metric space and $X \in \mathcal{A}(H)$ compact. Let $h : X \to H$ be a continuous function such that, for every $x \in X$ with $x \neq h(x)$ there exists $\alpha \in (0, 1)$ satisfying*

$$X \cap B(h(x), \alpha d(x, h(x))) \neq \emptyset\,.$$

Then h has a fixed point.

Proof. For each nonempty finite set $A \subset X$ we have $\mathrm{co} A \subset \mathrm{co} X = X$, hence $(X, \{\Gamma_A\})$ is an H-space if we take $\Gamma_A = \mathrm{co} A$, for each nonempty finite set $A \subset X$.

We intend to apply Theorem 4.2 in the case $f = g : X \times X \to \mathbb{R}$, $f(x, y) = d(y, h(y)) - d(x, h(y))$. Since h is continuous, for each $x \in X$ and $\lambda \in \mathbb{R}$ the set $\{y \in X : f(x, y) \leq \lambda\}$ is closed, hence $f(x, \cdot)$ is a lower semicontinuous function on X. Clearly condition (ii) of Theorem 4.2 holds. We shall prove that for each fixed $y \in X$ the set

$$Z = \{x \in X : d(y, h(y)) - d(x, h(y)) > 0\}$$

is H-convex (that is $Z \in \mathcal{A}(X)$). Let $\{x_1, x_2, \ldots, x_n\} \subset Z$ and $\varepsilon > 0$ such that

$$d(x_i, h(y)) \leq d(y, h(y)) - \varepsilon, \text{ for } i \in \{1, 2, \ldots, n\}\,.$$

Hence $x_i \in B(h(y), d(y, h(y)) - \varepsilon)$ for each $i \in \{1, 2, \ldots, n\}$. Therefore we have

$$\mathrm{co}\{x_1, x_2, \ldots, x_n\} \subset B(h(y), d(y, h(y)) - \varepsilon) \subset Z$$

whence Z is H-convex.

By Theorem 4.2 there exists $y_0 \in X$ such that $f(x, y_0) \leq 0$, that is $d(y_0, h(y_0)) \leq d(x, h(y_0))$ for each $x \in X$.

We claim that such an element y_0 is a fixed point of h. Indeed, assume not, that is $y_0 \neq h(y_0)$. By hypothesis, there exist $\alpha \in (0,1)$ and $x_0 \in X \cap B(y_0, \alpha d(y_0, h(y_0)))$. Since $d(x_0, y_0) \leq \alpha d(y_0, h(y_0)) < d(y_0, h(y_0))$, we clearly get a contradiction. This completes our proof. □

Each of our results can be effortlessly formulated in hyperconvex metric spaces. For instance from Theorem 2.3 we obtain

THEOREM 5.3. *In a hyperconvex metric space X, let $\{Y_i : 1 \leq i \leq n\}$ be a family of admissible sets, every $n-1$ having a common point and let $\{Z_i : 1 \leq i \leq n\}$ be a covering of X having all members closed or all open. Then there exists a nonempty set $I \subset \{1, 2, \ldots, n\}$ such that*

$$\bigcap\{Y_j : j \in \bar{I}\} \cap \bigcap\{Z_i : i \in I\} \neq \emptyset\,.$$

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