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A note on the monomiality principle and generalized polynomials

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RIASSUNTO: *Si utilizza il principio di monomialità per stabilire forme generalizzate dell'algoritmo di divisione e del teorema del resto per famiglie di polinomi scritte come combinazioni lineari di polinomi di Hermite.*

ABSTRACT: *The monomiality principle is used to state generalized forms of the division algorithm and of the remainder theorem for families of polynomials written as linear combination of Hermite polynomials.*

1 – Introduction

The notion of quasi-monomiality has been exploited within different contexts to deal with isospectral problems [1] and to study the properties of new families of special functions [2]. The concept of quasi-monomiality is fairly straightforward and can be summarized as follows:

a) let $\hat{M}$ and $\hat{P}$ two operators

b) let $f_n(x)$, $(n \in N,\ x \in C)$ a polynomial,

$f_n(x)$ will be said a quasi-monomial, under the action of $\hat{M}$ and $\hat{P}$, if:

$$\hat{M} f_n(x) = f_{n+1}(x)\,, \qquad \hat{P} f_n(x) = n f_{n-1}(x)\,. \tag{1}$$

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The operators $\hat{M}$ and $\hat{P}$ are recognized as multiplicative and derivative operators respectively, furthermore, if $f_0(x) = 1$ from the first of (1) it follows that:

$$\hat{M}^n 1 = f_n(x)\,. \tag{2}$$

The Hermite polynomials have been shown to be quasi-monomials under the action of:

$$\hat{M} = x + 2y\frac{\partial}{\partial x}\,, \qquad \hat{P} = \frac{\partial}{\partial x}\,. \tag{3}$$

In this case we get indeed [3]:

$$\begin{gathered}\hat{M}^n 1 = H_n(x,y)\,, \qquad H_n(x,y) = n!\sum_{r=0}^{[n/2]}\frac{y^r x^{n-2r}}{r!(n-2r)!}\\ \hat{M}^m H_n(x,y) = H_{n+m}(x,y)\,.\end{gathered} \tag{4}$$

It has also been proved that:

$$H_n(x,y) = e^{y\frac{\partial^2}{\partial x^2}}(x^n) \tag{5}$$

and it is also worth noting that, more in general, the following operational identities hold [3]:

$$e^{y\frac{\partial^2}{\partial x^2}} f(x) = \hat{f}\left(x+2y\frac{\partial}{\partial x}\right)\,,\ e^{y\frac{\partial^2}{\partial x^2}}[f(x)g(x)] = \hat{f}\left(x+2y\frac{\partial}{\partial x}\right)e^{y\frac{\partial^2}{\partial x^2}} g(x)\,. \tag{6}$$

In the forthcoming sections we will exploit the previous simple rules, to prove that the quasi-monomial nature of Hermite polynomials is a useful tool to extend to more complicated polynomial structures, the elementary properties of ordinary polynomials.

2 – Generalized polynomials and division algorithm

If we introduce the linear combination of Hermite polynomials

$$\pi(x,y)=\sum_{n=0}^{m} a_n H_n(x,y)\,,\qquad a_n\in C \tag{7}$$

we can conclude that, according to (5):

$$\pi(x,y)=e^{y\frac{\partial^2}{\partial x^2}}P(x)\,,\qquad P(x)=\sum_{n=0}^{m} a_n x^n \tag{8}$$

where $P(x)$ is a polynomial in a field $F[x]$, within the present framework $\pi(x,y)$ will be considered a polynomial in x in the field $G[x]$, and y will be viewed as a parameter.

According to the linearity of the operator $e^{y\frac{\partial^2}{\partial x^2}}$, we can also conclude that:

$$\pi_1(x,y)\pm\pi_2(x,y)=e^{y\frac{\partial^2}{\partial x^2}}[P_1(x)\pm P_2(x)] \tag{9}$$

and according to (6) we can derive the further important operational identity:

$$e^{y\frac{\partial^2}{\partial x^2}}(P_1(x)P_2(x))=\hat{P}_1\Big(x+2y\frac{\partial}{\partial x}\Big)\pi_2(x,y)\,. \tag{10}$$

It is well known that if $P(x)$ and $f(x)=\sum_{n=0}^{p} a_n x^n$ are polynomials in $F[x]$, there exist a unique pair of polynomials in $F[x]q(x)=\sum_{n=0}^{s} b_n x^n$ and $r(x)=\sum_{n=0}^{t} c_n x^n$ called quotient and remainder respectively, such that [4]:

$$P(x)=q(x)f(x)+r(x)\,. \tag{11}$$

We can now state a generalization of the above division algorithm theorem. By applying the operator $e^{y\frac{\partial^2}{\partial x^2}}$ to both sides of (5) we find, as a consequence of equations (8) and (10):

$$\begin{aligned}&\pi(x,y)=\hat{q}\Big(x+2y\frac{\partial}{\partial x}\Big)\varphi(x,y)+\rho(x,y)\,,\\&\varphi(x,y)=e^{y\frac{\partial^2}{\partial x^2}}f(x)\,,\qquad \rho(x,y)=e^{y\frac{\partial^2}{\partial x^2}}r(x)\,.\end{aligned} \tag{12}$$

It is clear that $\varphi(x,y)$ and $\rho(x,y)$ play the role of quotient and remainder polynomials, within the present more general context.

In a less technical language we note that the theorem can be applied as follows

1) Known $\pi(x,y)$, we construct the corresponding $P(x)$ polynomial;
2) We apply the division algorithm theorem for ordinary polynomials;
3) We use the operational rules to identify the quotient and remainder polynomials in the field $G(x)$.

The following example may better clarify the above results. Let

$$\pi(x,y) = 2H_4(x,y) + 3H_2(x,y) - 5H_1(x,y) - 2 \tag{13}$$

a polynomial realized in terms of Hermite polynomials the corresponding $P(x)$ in $F[x]$ is

$$P(x) = 2x^4 + 3x^2 - 5x - 2\,. \tag{14}$$

We get therefore

$$\begin{aligned} &q(x) = \frac{2}{3}x^2 - \frac{2}{9}x + \frac{59}{27}\,, &&\hat{q}\Big(x + 2y\frac{\partial}{\partial x}\Big) = \frac{2}{3}\hat{M}^2 - \frac{2}{9}\hat{M} + \frac{59}{27}\\ &f(x) = 3x^2 + x - 5\,, &&\varphi(x,y) = 3H_2(x,y) + H_1(x,y) - 5\\ &r(x) = -\frac{224}{27}x + \frac{241}{27}\,, &&\rho(x,y) = -\frac{224}{27}H_1(x,y) + \frac{241}{27}\,. \end{aligned} \tag{15}$$

We must however underline that, unlike the case of ordinary polynomials, the decomposition (12) in not “unique” and owing to the commutativity of $q(x)$ and $f(x)$, can also be written as:

$$\begin{aligned} &\pi(x,y) = \hat{f}\Big(x + 2y\frac{\partial}{\partial x}\Big)\sigma(x,y) + \rho(x,y)\\ &\sigma(x,y) = e^{y\frac{\partial^2}{\partial x^2}}q(x)\,. \end{aligned} \tag{16}$$

The remainder theorem of ordinary polynomials writes

$$P(x) = q(x)(x-K) + P(K) \tag{17}$$

where $P(K)$ denotes the polynomial $P(x)$ calculated in K.

It is evident that since

$$e^{y\frac{\partial^2}{\partial x^2}}P(K) = P(K) \tag{18}$$

and since $H_1(x,y) = x$, the generalization of the remainder theorem writes

$$\pi(x,y) = \hat{q}\Big(x + 2y\frac{\partial}{\partial x}\Big)(x-K) + P(K)\,. \tag{19}$$

The extension of the fundamental theorem of algebra is also obvious and reads:

$$\hat{\pi}(x,y) = \prod_{i=1}^{m}(\hat{M} - \alpha_i) \tag{20}$$

where α_i are complex roots of the polynomial $P(x)$ and should not be confused with the zeros of $\pi(x,y)$.

3 – Concluding remarks

Before closing the paper let us note that since

$$e^{-y\frac{\partial^2}{\partial x^2}}[H_n(x,y)] = x^n \tag{21}$$

if

$$\sigma(x,y) = \sum_{n=0}^{m} a_n H_n(x,y) \tag{22}$$

then

$$P(x) = e^{-y\frac{\partial^2}{\partial x^2}}\sigma(x,y)\,. \tag{23}$$

The results discussed in the present note can be extended to any quasi-monomial, this is indeed the case of Laguerre polynomials, which have been proved to be quasi-monomial under the action of integral operators [5]. This more general aspect of the problem and the usefulness of the results of the present note for the study of the zeros of generalized polynomials will be discussed elsewhere.

Possible extensions to polynomials of many variables and many indices can also be done and division algorithm and remainder theorems can be extended to Hermite-type polynomials with many variables and many indices. This type of extension does not require any conceptual difficulties, the only problems being connected with computational difficulties which will be analyzed in a forthcoming paper.

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